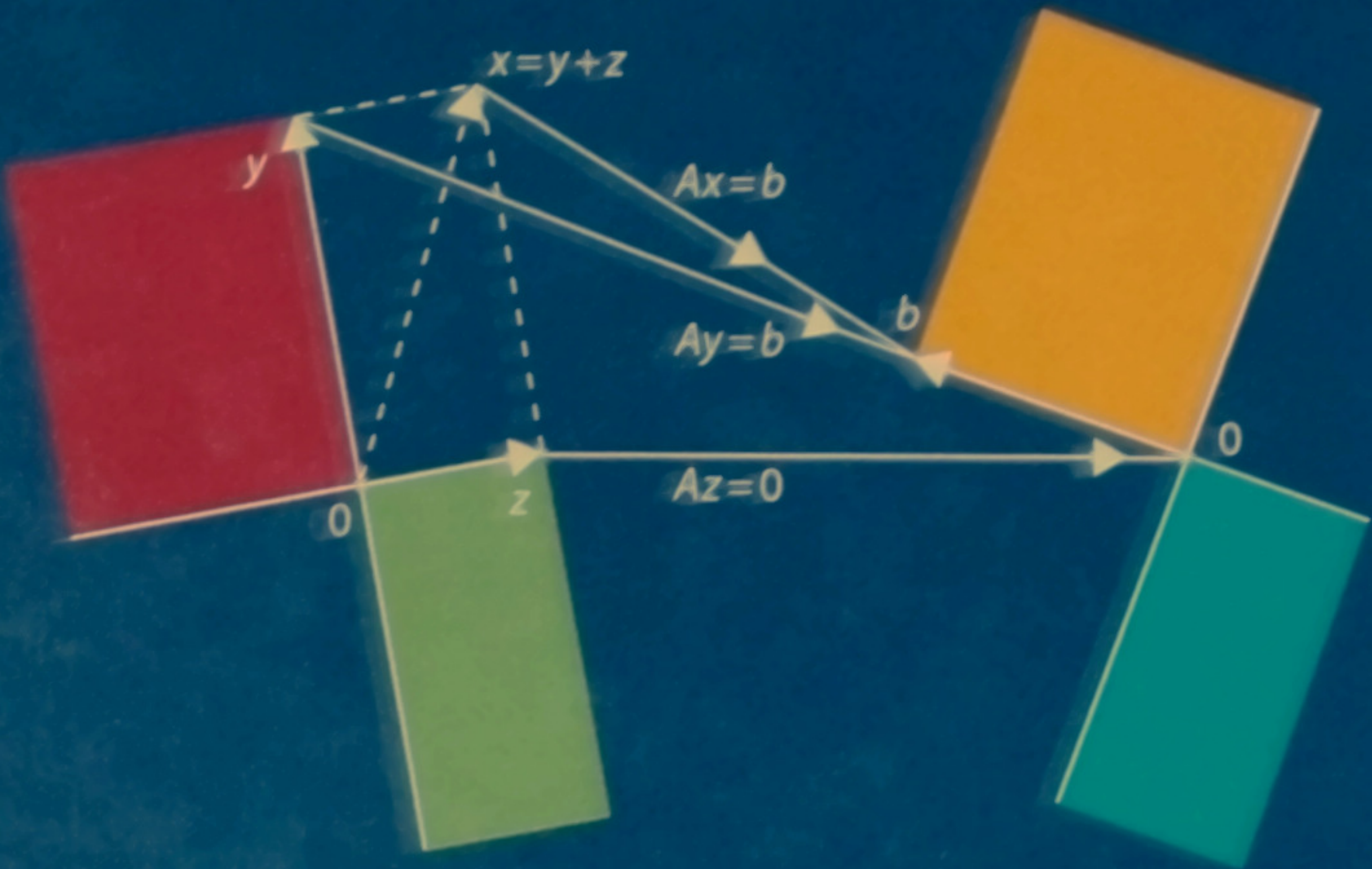
Introduction to

LINEAR ALGEBRA

FOURTH EDITION



GILBERT STRANG

INTRODUCTION
TO
LINEAR ALGEBRA

Fourth Edition

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The website for this book is **math.mit.edu/linearalgebra**.
A Solutions Manual is available to instructors by email from the publisher.

Course material including syllabus and Teaching Codes and exams and also videotaped lectures are available on the teaching website: **web.mit.edu/18.06**

Linear Algebra is included in MIT's OpenCourseWare site **ocw.mit.edu**.
This provides video lectures of the full linear algebra course 18.06.
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The front cover captures a central idea of linear algebra.
$Ax = b$ is solvable when b is in the (orange) column space of A.
One particular solution y is in the (red) row space: $Ay = b$.
Add any vector z from the (green) nullspace of A: $Az = 0$.
The complete solution is $x = y + z$. Then $Ax = Ay + Az = b$.

The cover design was the inspiration of a creative collaboration:
Lois Sellers (**birchdesignassociates.com**) and Gail Corbett.

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Preface

I will be happy with this preface if three important points come through clearly:

1. The beauty and variety of linear algebra, and its extreme usefulness
2. The goals of this book, and the new features in this Fourth Edition
3. The steady support from our linear algebra websites and the video lectures

May I begin with notes about two websites that are constantly used, and the new one.

ocw.mit.edu Messages come from thousands of students and faculty about linear algebra on this OpenCourseWare site. The 18.06 course includes video lectures of a complete semester of classes. Those lectures offer an independent review of the whole subject based on this textbook—the professor's time stays free and the student's time can be 3 a.m. (The reader doesn't have to be in a class at all.) A million viewers around the world have seen these videos (*amazing*). I hope you find them helpful.

web.mit.edu/18.06 This site has homeworks and exams (with solutions) for the current course as it is taught, and as far back as 1996. There are also review questions, Java demos, Teaching Codes, and short essays (*and the video lectures*). My goal is to make this book as useful as possible, with all the course material we can provide.

math.mit.edu/linearalgebra The newest website is devoted specifically to this Fourth Edition. It will be a permanent record of ideas and codes and good problems and solutions. Several sections of the book are directly available online, plus notes on teaching linear algebra. The content is growing quickly and contributions are welcome from everyone.

The Fourth Edition

Thousands of readers know earlier editions of *Introduction to Linear Algebra*. The new cover shows the **Four Fundamental Subspaces**—the row space and nullspace are on the left side, the column space and the nullspace of A^{T} are on the right. It is not usual to put the central ideas of the subject on display like this! You will meet those four spaces in Chapter 3, and you will understand why that picture is so central to linear algebra.

Those were named the Four Fundamental Subspaces in my first book, and they start from a matrix A. Each row of A is a vector in n-dimensional space. When the matrix has m rows, each column is a vector in m-dimensional space. The crucial operation in linear algebra is taking ***linear combinations of vectors***. (That idea starts on page 1 of the book and never stops.) *When we take all linear combinations of the column vectors, we get the column space*. If this space includes the vector b, we can solve the equation $Ax = b$.

I have to stop here or you won't read the book. May I call special attention to the new Section 1.3 in which these ideas come early—with two specific examples. You are not expected to catch every detail of vector spaces in one day! But you will see the first matrices in the book, and a picture of their column spaces, and even an *inverse matrix*. You will be learning the language of linear algebra in the best and most efficient way: by using it.

Every section of the basic course now ends with ***Challenge Problems***. They follow a large collection of review problems, which ask you to use the ideas in that section—the dimension of the column space, a basis for that space, the rank and inverse and determinant and eigenvalues of A. Many problems look for computations by hand on a small matrix, and they have been highly praised. The new Challenge Problems go a step further, and sometimes they go deeper. Let me give four examples:

Section 2.1: Which row exchanges of a Sudoku matrix produce another Sudoku matrix?

Section 2.4: From the shapes of A, B, C, is it faster to compute AB times C or A times BC?

Background: The great fact about multiplying matrices is that *AB times C gives the same answer as A times BC*. This simple statement is the reason behind the rule for matrix multiplication. If AB is square and C is a vector, it's faster to do BC first. Then multiply by A to produce ABC. The question asks about other shapes of A, B, and C.

Section 3.4: If $Ax = b$ and $Cx = b$ have the same solutions for every b, is $A = C$?

Section 4.1: What conditions on the four vectors r, n, c, ℓ allow them to be bases for the row space, the nullspace, the column space, and the left nullspace of a 2 by 2 matrix?

The Start of the Course

The equation $Ax = b$ uses the language of linear combinations right away. The vector Ax is *a combination of the columns of A*. The equation is asking for ***a combination that produces b***. The solution vector x comes at three levels and all are important:

1. ***Direct solution*** to find x by forward elimination and back substitution.
2. ***Matrix solution*** using the inverse of A: $x = A^{-1}b$ (if A has an inverse).
3. ***Vector space solution*** $x = y + z$ as shown on the cover of the book:

   Particular solution (to $Ay = b$) plus ***nullspace solution*** (to $Az = 0$)

Direct elimination is the most frequently used algorithm in scientific computing, and the idea is not hard. Simplify the matrix A so it becomes triangular—then all solutions come quickly. I don't spend forever on practicing elimination, it will get learned.

The speed of every new supercomputer is tested on $Ax = b$: it's pure linear algebra. IBM and Los Alamos announced a new world record of 10^{15} operations per second in 2008.

That *petaflop speed* was reached by solving many equations in parallel. High performance computers avoid operating on single numbers, they feed on whole submatrices.

The processors in the Roadrunner are based on the Cell Engine in PlayStation 3. What can I say, video games are now the largest market for the fastest computations.

Even a supercomputer doesn't want the inverse matrix: too slow. Inverses give the simplest formula $x = A^{-1}b$ but not the top speed. And everyone must know that determinants are even slower—there is no way a linear algebra course should begin with formulas for the determinant of an n by n matrix. Those formulas have a place, but not first place.

Structure of the Textbook

Already in this preface, you can see the style of the book and its goal. That goal is serious, to explain this beautiful and useful part of mathematics. You will see how the applications of linear algebra reinforce the key ideas. I hope every teacher will learn something new; familiar ideas can be seen in a new way. The book moves gradually and steadily from *numbers* to *vectors* to *subspaces*—each level comes naturally and everyone can get it.

Here are ten points about the organization of this book:

1. Chapter 1 starts with vectors and dot products. If the class has met them before, focus quickly on linear combinations. The new Section 1.3 provides three independent vectors whose combinations fill all of 3-dimensional space, and three dependent vectors in a plane. ***Those two examples are the beginning of linear algebra.***

2. Chapter 2 shows the row picture and the column picture of $Ax = b$. The heart of linear algebra is in that connection between the rows of A and the columns: the same numbers but very different pictures. Then begins the algebra of matrices: an elimination matrix E multiplies A to produce a zero. The goal here is to capture the whole process—start with A and end with an ***upper triangular*** U.

   Elimination is seen in the beautiful form $A = LU$. The ***lower triangular*** L holds all the forward elimination steps, and U is the matrix for back substitution.

3. Chapter 3 is linear algebra at the best level: ***subspaces***. The column space contains all linear combinations of the columns. The crucial question is: ***How many of those columns are needed***? The answer tells us the dimension of the column space, and the key information about A. We reach the Fundamental Theorem of Linear Algebra.

4. Chapter 4 has m equations and only n unknowns. It is almost sure that $Ax = b$ has no solution. We cannot throw out equations that are close but not perfectly exact. When we solve by ***least squares***, the key will be the matrix $A^{\mathrm{T}}A$. This wonderful matrix $A^{\mathrm{T}}A$ appears everywhere in applied mathematics, when A is rectangular.

5. ***Determinants*** in Chapter 5 give formulas for all that has come before—inverses, pivots, volumes in n-dimensional space, and more. We don't need those formulas to compute! They slow us down. But $\det A = 0$ tells when a matrix is singular, and that test is the key to eigenvalues.

6. ***Section* 6.1 *introduces eigenvalues for* 2 *by* 2 *matrices*.** Many courses want to see eigenvalues early. It is completely reasonable to come here directly from Chapter 3, because the determinant is easy for a 2 by 2 matrix. *The key equation is* $Ax = \lambda x$.

   Eigenvalues and eigenvectors are an astonishing way to understand a square matrix. They are not for $Ax = b$, they are for dynamic equations like $du/dt = Au$. The idea is always the same: *follow the eigenvectors*. In those special directions, A acts like a single number (the eigenvalue λ) and the problem is one-dimensional.

   Chapter 6 is full of applications. One highlight is ***diagonalizing a symmetric matrix***. Another highlight—not so well known but more important every day—is the diagonalization of *any matrix*. This needs two sets of eigenvectors, not one, and they come (of course!) from $A^{\mathrm{T}}A$ and AA^{T}. This Singular Value Decomposition often marks the end of the basic course and the start of a second course.

7. Chapter 7 explains the ***linear transformation*** approach—it is linear algebra without coordinates, the ideas without computations. Chapter 9 is the opposite—all about how $Ax = b$ and $Ax = \lambda x$ are really solved. Then Chapter 10 moves from real numbers and vectors to complex vectors and matrices. The Fourier matrix F is the most important complex matrix we will ever see. And the ***Fast Fourier Transform*** (multiplying quickly by F and F^{-1}) is a revolutionary algorithm.

8. Chapter 8 is full of applications, more than any single course could need:

   8.1 *Matrices in Engineering*—differential equations replaced by matrix equations

   8.2 *Graphs and Networks*—leading to the edge-node matrix for Kirchhoff's Laws

   8.3 *Markov Matrices*—as in Google's *PageRank* algorithm

   8.4 *Linear Programming*—a new requirement $x \geq 0$ and minimization of the cost

   8.5 *Fourier Series*—linear algebra for functions and digital signal processing

   8.6 *Matrices in Statistics and Probability*—$Ax = b$ is weighted by average errors

   8.7 *Computer Graphics*—matrices move and rotate and compress images.

9. Every section in the basic course ends with a ***Review of the Key Ideas***.

10. How should computing be included in a linear algebra course? It can open a new understanding of matrices—every class will find a balance. I chose the language of MATLAB as a direct way to describe linear algebra: eig(ones(4)) will produce the eigenvalues 4, 0, 0, 0 of the 4 by 4 all-ones matrix. *Go to **netlib.org** for codes.*

    You can freely choose a different system. More and more software is open source.

The new website ***math.mit.edu/linearalgebra*** provides further ideas about teaching and learning. Please contribute! Good problems are welcome by email: ***gs@math.mit.edu***. Send new applications too, linear algebra is an incredibly useful subject.

The Variety of Linear Algebra

Calculus is mostly about one special operation (the derivative) and its inverse (the integral). Of course I admit that calculus could be important .... But so many applications of mathematics are discrete rather than continuous, digital rather than analog. The century of data has begun! You will find a light-hearted essay called "Too Much Calculus" on my website. ***The truth is that vectors and matrices have become the language to know.***

Part of that language is the wonderful variety of matrices. Let me give three examples:

Symmetric matrix	***Orthogonal matrix***	***Triangular matrix***
$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A key goal is learning to "read" a matrix. You need to see the meaning in the numbers. This is really the essence of mathematics—patterns and their meaning.

May I end with this thought for professors. You might feel that the direction is right, and wonder if your students are ready. ***Just give them a chance***! Literally thousands of students have written to me, frequently with suggestions and surprisingly often with thanks. They know this course has a purpose, because the professor and the book are on their side. Linear algebra is a fantastic subject, enjoy it.

Help With This Book

I can't even name all the friends who helped me, beyond thanking Brett Coonley at MIT and Valutone in Mumbai and SIAM in Philadelphia for years of constant and dedicated support. The greatest encouragement of all is the feeling that you are doing something worthwhile with your life. Hundreds of generous readers have sent ideas and examples and corrections (and favorite matrices!) that appear in this book. *Thank you all.*

Background of the Author

This is my eighth textbook on linear algebra, and I have not written about myself before. I hesitate to do it now. It is the mathematics that is important, and the reader. The next paragraphs add something personal as a way to say that textbooks are written by people.

I was born in Chicago and went to school in Washington and Cincinnati and St. Louis. My college was MIT (and my linear algebra course was *extremely abstract*). After that came Oxford and UCLA, then back to MIT for a very long time. I don't know how many thousands of students have taken 18.06 (more than a million when you include the videos on ***ocw.mit.edu***). The time for a fresh approach was right, because this fantastic subject was only revealed to math majors—**we needed to open linear algebra to the world.**

Those years of teaching led to the Haimo Prize from the Mathematical Association of America. For encouraging education worldwide, the International Congress of Industrial and Applied Mathematics awarded me the first Su Buchin Prize. I am extremely grateful, more than I could possibly say. What I hope most is that you will like linear algebra.

Chapter 1

Introduction to Vectors

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $v + w$. We multiply them by numbers c and d to get cv and dw. Combining those two operations (adding cv to dw) gives the ***linear combination*** $cv + dw$.

$$\textbf{Linear combination} \qquad cv + dw = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c + 2d \\ c + 3d \end{bmatrix}$$

$$\textbf{Example} \quad v + w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ is the combination with } c = d = 1$$

Linear combinations are all-important in this subject! Sometimes we want one particular combination, the specific choice $c = 2$ and $d = 1$ that produces $cv + dw = (4, 5)$. Other times we want *all the combinations* of v and w (coming from all c and d).

The vectors cv lie along a line. When w is not on that line, **the combinations** $cv + dw$ **fill the whole two-dimensional plane**. (I have to say "two-dimensional" because linear algebra allows higher-dimensional planes.) Starting from four vectors u, v, w, z in four-dimensional space, their combinations $cu + dv + ew + fz$ are likely to fill the space—but not always. The vectors and their combinations could even lie on one line.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n-dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n-dimensional space). The first steps are the operations in Sections 1.1 and 1.2. Then Section 1.3 outlines three fundamental ideas.

1.1 ***Vector addition*** $v + w$ ***and linear combinations*** $cv + dw$.

1.2 ***The dot product*** $v \cdot w$ ***of two vectors and the length*** $\|v\| = \sqrt{v \cdot v}$.

1.3 ***Matrices*** A, ***linear equations*** $Ax = b$, ***solutions*** $x = A^{-1}b$.

1.1 Vectors and Linear Combinations

"You can't add apples and oranges." In a strange way, this is the reason for vectors. We have two separate numbers v_1 and v_2. That pair produces a ***two-dimensional vector*** $\boldsymbol{v}$:

$$\textbf{Column vector} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \begin{aligned} v_1 &= \text{first component} \\ v_2 &= \text{second component} \end{aligned}$$

We write $\boldsymbol{v}$ as a ***column***, not as a row. The main point so far is to have a single letter $\boldsymbol{v}$ (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in *lightface italic*).

Even if we don't add v_1 to v_2, we do *add vectors*. The first components of $\boldsymbol{v}$ and $\boldsymbol{w}$ stay separate from the second components:

$$\begin{matrix}\textbf{VECTOR}\\ \textbf{ADDITION}\end{matrix} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

You see the reason. We want to add apples to apples. Subtraction of vectors follows the same idea: *The components of* $\boldsymbol{v} - \boldsymbol{w}$ *are* $v_1 - w_1$ *and* $v_2 - w_2$.

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c. There are two ways to double a vector. One way is to add $\boldsymbol{v} + \boldsymbol{v}$. The other way (the usual way) is to multiply each component by 2:

$$\begin{matrix}\textbf{SCALAR}\\ \textbf{MULTIPLICATION}\end{matrix} \qquad 2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad \text{and} \quad -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\boldsymbol{v}$ are cv_1 and cv_2. The number c is called a "scalar".

Notice that the sum of $-\boldsymbol{v}$ and $\boldsymbol{v}$ is the zero vector. This is $\mathbf{0}$, which is not the same as the number zero! The vector $\mathbf{0}$ has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v}$—***adding vectors and multiplying by scalars***.

The order of addition makes no difference: $\boldsymbol{v} + \boldsymbol{w}$ equals $\boldsymbol{w} + \boldsymbol{v}$. Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. Check also by an example:

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix} \qquad \boldsymbol{w} + \boldsymbol{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{8} \end{bmatrix}.$$

Linear Combinations

Combining addition with scalar multiplication, we now form "*linear combinations*" of $\boldsymbol{v}$ and $\boldsymbol{w}$. Multiply $\boldsymbol{v}$ by c and multiply $\boldsymbol{w}$ by d; then add $c\boldsymbol{v} + d\boldsymbol{w}$.

DEFINITION ***The sum of $c\boldsymbol{v}$ and $d\boldsymbol{w}$ is a*** *linear combination* ***of $\boldsymbol{v}$ and $\boldsymbol{w}$.***

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\boldsymbol{v}$:

$$\begin{aligned} 1\boldsymbol{v} + 1\boldsymbol{w} &= \text{sum of vectors in Figure 1.1a} \\ 1\boldsymbol{v} - 1\boldsymbol{w} &= \text{difference of vectors in Figure 1.1b} \\ 0\boldsymbol{v} + 0\boldsymbol{w} &= \textit{zero vector} \\ c\boldsymbol{v} + 0\boldsymbol{w} &= \text{vector } c\boldsymbol{v} \text{ in the direction of } \boldsymbol{v} \end{aligned}$$

The zero vector is always a possible combination (its coefficients are zero). Every time we see a "space" of vectors, that zero vector will be included. This big view, taking *all* the combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$, is linear algebra at work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). That vector $\boldsymbol{v}$ is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are $4, 2$. This point is another representation of the vector—so we have three ways to describe $\boldsymbol{v}$:

Represent vector $\boldsymbol{v}$ Two numbers Arrow from $(0, 0)$ Point in the plane

We add using the numbers. We visualize $\boldsymbol{v} + \boldsymbol{w}$ using arrows:

Vector addition (head to tail) ***At the end of $\boldsymbol{v}$, place the start of $\boldsymbol{w}$.***

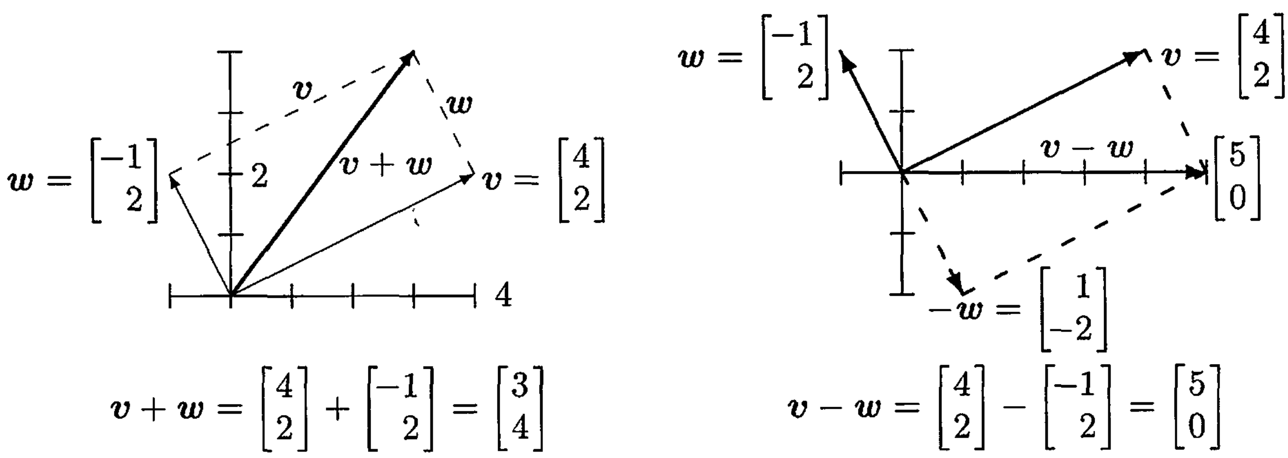


Figure 1.1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

We travel along $\boldsymbol{v}$ and then along $\boldsymbol{w}$. Or we take the diagonal shortcut along $\boldsymbol{v} + \boldsymbol{w}$. We could also go along $\boldsymbol{w}$ and then $\boldsymbol{v}$. In other words, $\boldsymbol{w} + \boldsymbol{v}$ gives the same answer as $\boldsymbol{v} + \boldsymbol{w}$.

These are different ways along the parallelogram (in this example it is a rectangle). The sum is the diagonal vector $v + w$.

The zero vector $\mathbf{0} = (0, 0)$ is too short to draw a decent arrow, but you know that $v + \mathbf{0} = v$. For $2v$ we double the length of the arrow. We reverse w to get $-w$. This reversing gives the subtraction on the right side of Figure 1.1.

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of v are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2), when it starts from $(0, 0)$. Now we allow vectors to have three components (v_1, v_2, v_3).

The xy plane is replaced by three-dimensional space. Here are typical vectors (still column vectors but with three components):

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad v + w = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}.$$

The vector v corresponds to an arrow in 3-space. Usually the arrow starts at the "origin", where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3. There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends***.

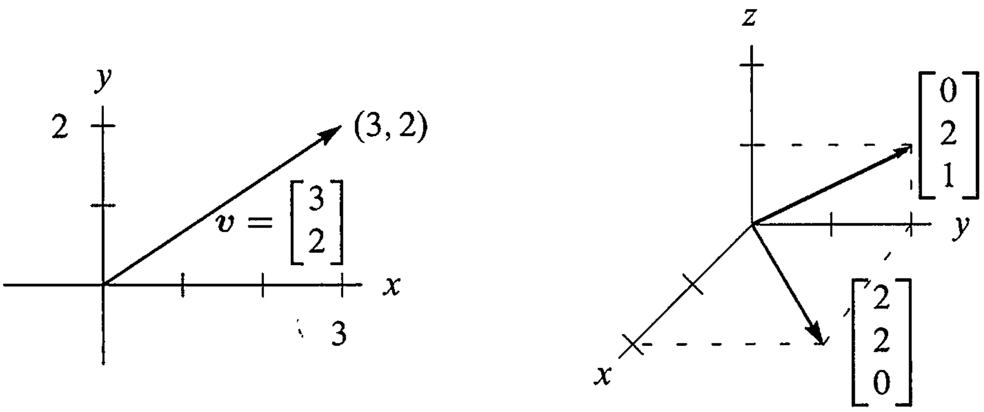


Figure 1.2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

From now on $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ ***is also written as*** $v = (1, 1, -1)$.

The reason for the row form (in parentheses) is to save space. But $\boldsymbol{v} = (1, 1, -1)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[\,1 \quad 1 \quad -1\,]$ is absolutely different, even though it has the same three components. That row vector is the "transpose" of the column $\boldsymbol{v}$.

In three dimensions, $\boldsymbol{v} + \boldsymbol{w}$ is still found a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When $\boldsymbol{w}$ starts at the end of $\boldsymbol{v}$, the third side is $\boldsymbol{v} + \boldsymbol{w}$. The other way around the parallelogram is $\boldsymbol{w} + \boldsymbol{v}$. Question: Do the four sides all lie in the same plane? *Yes.* And the sum $\boldsymbol{v} + \boldsymbol{w} - \boldsymbol{v} - \boldsymbol{w}$ goes completely around to produce the ____ vector.

A typical linear combination of three vectors in three dimensions is $\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w}$:

Linear combination
Multiply by $1, 4, -2$
Then add

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$$

The Important Questions

For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$. For two vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v}$. For three vectors, the combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$. Will you take the big step from *one* combination to *all* combinations? Every c and d and e are allowed. Suppose the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in three-dimensional space:

1. What is the picture of *all* combinations $c\boldsymbol{u}$?
2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$?
3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

The answers depend on the particular vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

1. The combinations $c\boldsymbol{u}$ fill a ***line***.
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane***.
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

The zero vector $(0, 0, 0)$ is on the line because c can be zero. It is on the plane because c and d can be zero. The line of vectors $c\boldsymbol{u}$ is infinitely long (forward and backward). It is the plane of all $c\boldsymbol{u} + d\boldsymbol{v}$ (combining two vectors in three-dimensional space) that I especially ask you to think about.

Adding all $c\boldsymbol{u}$ on one line to all $d\boldsymbol{v}$ on the other line fills in the plane in Figure 1.3.

When we include a third vector $\boldsymbol{w}$, the multiples $e\boldsymbol{w}$ give a third line. Suppose that third line is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$. Then combining all $e\boldsymbol{w}$ with all $c\boldsymbol{u} + d\boldsymbol{v}$ fills up the whole three-dimensional space.

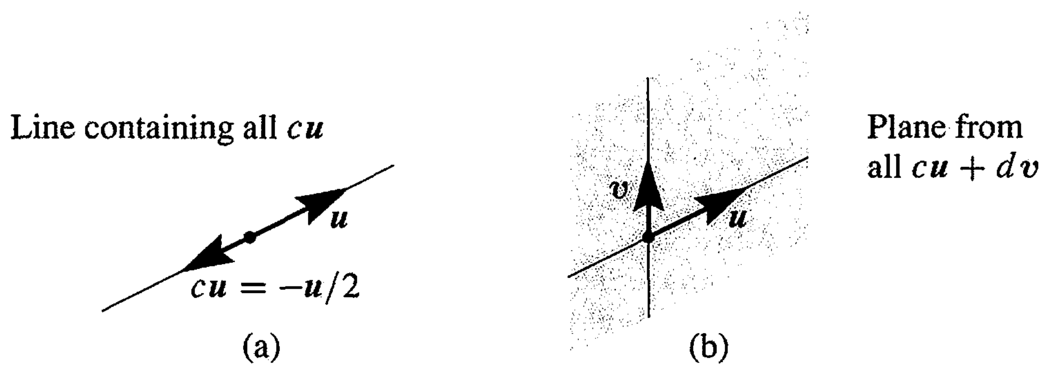


Figure 1.3: (a) Line through u. (b) The plane containing the lines through u and v.

This is the typical situation! **Line**, then **plane**, then **space**. But other possibilities exist. When w happens to be $cu + dv$, the third vector is in the plane of the first two. The combinations of u, v, w will not go outside that uv plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

■ REVIEW OF THE KEY IDEAS ■

1. A vector v in two-dimensional space has two components v_1 and v_2.
2. $v + w = (v_1 + w_1,\ v_2 + w_2)$ and $cv = (cv_1, cv_2)$ are found a component at a time.
3. A linear combination of three vectors u and v and w is $cu + dv + ew$.
4. Take *all* linear combinations of u, or u and v, or u, v, w. In three dimensions, those combinations typically fill a line, then a plane, and the whole space $\mathbf{R}^3$.

■ WORKED EXAMPLES ■

1.1 A The linear combinations of $v = (1, 1, 0)$ and $w = (0, 1, 1)$ fill a plane. *Describe that plane.* Find a vector that is *not* a combination of v and w.

Solution **The combinations $cv + dw$ fill a plane in $\mathbf{R}^3$.** The vectors in that plane allow any c and d. The plane of Figure 1.3 fills in between the "u-line" and the "v-line".

$$\text{Combinations} \quad cv + dw = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix} \text{ fill a plane.}$$

Four particular vectors in that plane are $(0, 0, 0)$ and $(2, 3, 1)$ and $(5, 7, 2)$ and $(\pi, 2\pi, \pi)$. The second component $c + d$ is always the sum of the first and third components. *The vector* $(1, 2, 3)$ *is not in the plane, because* $2 \neq 1 + 3$.

Another description of this plane through $(0,0,0)$ is to know that $\boldsymbol{n} = (1,-1,1)$ is **perpendicular** to the plane. Section 1.2 will confirm that 90° angle by testing dot products: $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{n} = 0$.

1.1 B For $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (0,1)$, describe all points $c\boldsymbol{v}$ with (1) *whole numbers* c (2) *nonnegative* $c \geq 0$. Then add all vectors $d\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\boldsymbol{w}$.

Solution

(1) The vectors $c\boldsymbol{v} = (c, 0)$ with whole numbers c are **equally spaced points** along the x axis (the direction of $\boldsymbol{v}$). They include $(-2,0), (-1,0), (0,0), (1,0), (2,0)$.

(2) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a ***half-line***. It is the *positive* x axis. This half-line starts at $(0,0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$.

(1′) Adding all vectors $d\boldsymbol{w} = (0, d)$ puts a vertical line through those points $c\boldsymbol{v}$. We have infinitely many ***parallel lines*** from (*whole number* c, *any number* d).

(2′) Adding all vectors $d\boldsymbol{w}$ puts a vertical line through every $c\boldsymbol{v}$ on the half-line. Now we have a ***half-plane***. It is the right half of the xy plane (any $x \geq 0$, any height y).

1.1 C Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution In applying mathematics, many problems have two parts:

1 *Modeling part* Express the problem by a set of equations.

2 *Computational part* Solve those equations by a fast and accurate algorithm.

Here we are only asked for the first part (the equations). Chapter 2 is devoted to the second part (the algorithm). Our example fits into a fundamental model for linear algebra:

$$\text{Find } c_1, \ldots, c_n \quad \text{so that} \quad c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b}.$$

For $n = 2$ we could find a formula for the $\boldsymbol{c}$'s. The "elimination method" in Chapter 2 succeeds far beyond $n = 100$. For n greater than 1 million, see Chapter 9. Here $n = 2$:

Vector equation
$$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The required equations for c and d just come from the two components separately:

Two scalar equations
$$\begin{aligned} 2c - d &= 1 \\ -c + 2d &= 0 \end{aligned}$$

You could think of those as two lines that cross at the solution $c = \frac{2}{3}, d = \frac{1}{3}$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

1 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)}\ \begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } \begin{bmatrix}3\\6\\9\end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix}1\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\3\end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix}2\\0\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\2\\2\end{bmatrix} \text{ and } \begin{bmatrix}2\\2\\3\end{bmatrix}$$

2 Draw $\boldsymbol{v} = \begin{bmatrix}4\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}-2\\2\end{bmatrix}$ and $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ in a single xy plane.

3 If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix}5\\1\end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix}1\\5\end{bmatrix}$, compute and draw $\boldsymbol{v}$ and $\boldsymbol{w}$.

4 From $\boldsymbol{v} = \begin{bmatrix}2\\1\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}1\\2\end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

5 Compute $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$. How do you know $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in a plane?

$$\textbf{In a plane} \qquad \boldsymbol{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix}2\\-3\\-1\end{bmatrix}.$$

6 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to ____. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, -6)$.

7 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix}2\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

8 The parallelogram in Figure 1.1 has diagonal $\boldsymbol{v} + \boldsymbol{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

9 If three corners of a parallelogram are $(1, 1)$, $(4, 2)$, and $(1, 3)$, what are all three of the possible fourth corners? Draw two of them.

Problems 10–14 are about special vectors on cubes and clocks in Figure 1.4.

10 Which point of the cube is $\boldsymbol{i} + \boldsymbol{j}$? Which point is the vector sum of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$? Describe all points (x, y, z) in the cube.

11 Four corners of the cube are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are ____.

12 How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? A typical corner is $(0, 0, 1, 0)$. A typical edge goes to $(0, 1, 0, 0)$.

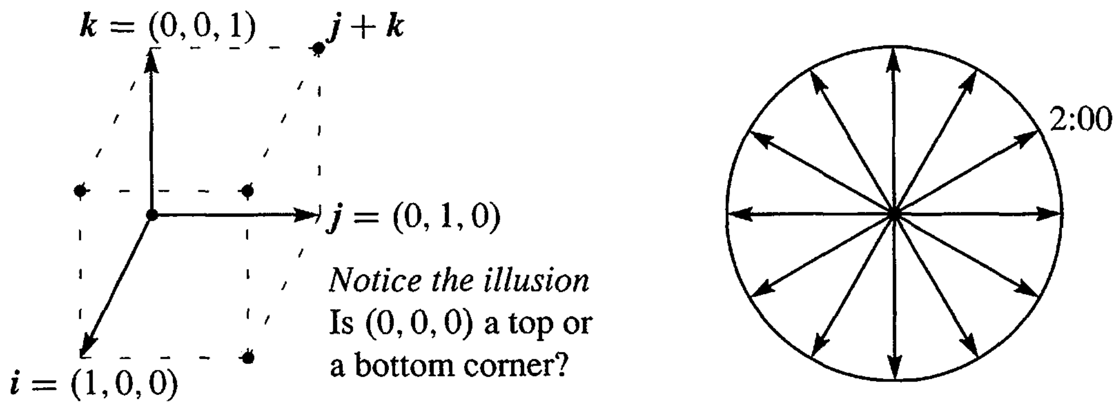


Figure 1.4: Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and twelve clock vectors.

13 (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) What are the components of that 2:00 vector $\boldsymbol{v} = (\cos\theta, \sin\theta)$?

14 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0,0)$ at the center. The vector to 12:00 is doubled to $(0,2)$. Add the new twelve vectors.

Problems 15–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (Figure 1.5a).

15 Figure 1.5a shows $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$.

16 Mark the point $-\boldsymbol{v} + 2\boldsymbol{w}$ and any other combination $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.

17 Locate $\frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{2}{3}\boldsymbol{v} + \frac{2}{3}\boldsymbol{w}$. The combinations $c\boldsymbol{v} + c\boldsymbol{w}$ fill out what line?

18 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

19 Restricted only by $c \ge 0$ and $d \ge 0$ draw the "cone" of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

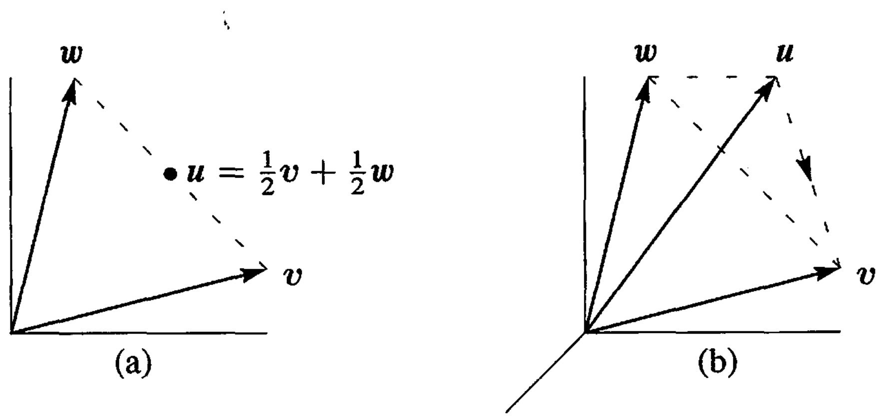


Figure 1.5: Problems **15–19** in a plane Problems **20–25** in 3-dimensional space

Problems 20–25 deal with u, v, w in three-dimensional space (see Figure 1.5b).

20 Locate $\frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w$ and $\frac{1}{2}u + \frac{1}{2}w$ in Figure 1.5b. Challenge problem: Under what restrictions on c, d, e, will the combinations $cu + dv + ew$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.

21 The three sides of the dashed triangle are $v - u$ and $w - v$ and $u - w$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

22 Shade in the pyramid of combinations $cu + dv + ew$ with $c \geq 0, d \geq 0, e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(u + v + w)$ as inside or outside this pyramid.

23 If you look at *all* combinations of those u, v, and w, is there any vector that can't be produced from $cu + dv + ew$? Different answer if u, v, w are all in _____.

24 Which vectors are combinations of u and v, and *also* combinations of v and w?

25 Draw vectors u, v, w so that their combinations $cu + dv + ew$ fill only a line. Find vectors u, v, w so that their combinations $cu + dv + ew$ fill only a plane.

26 What combination $c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

27 *Review Question.* In xyz space, where is the plane of all linear combinations of $i = (1, 0, 0)$ and $i + j = (1, 1, 0)$?

Challenge Problems

28 Find vectors v and w so that $v + w = (4, 5, 6)$ and $v - w = (2, 5, 8)$. This is a question with _____ unknown numbers, and an equal number of equations to find those numbers.

29 Find two different combinations of the three vectors $u = (1, 3)$ and $v = (2, 7)$ and $w = (1, 5)$ that produce $b = (0, 1)$. Slightly delicate question: If I take any three vectors u, v, w in the plane, will there always be two different combinations that produce $b = (0, 1)$?

30 The linear combinations of $v = (a, b)$ and $w = (c, d)$ fill the plane unless _____. Find four vectors u, v, w, z with four components each so that their combinations $cu + dv + ew + fz$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

31 Write down three equations for c, d, e so that $cu + dv + ew = b$. Can you somehow find c, d, and e?

$$u = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1.2 Lengths and Dot Products

The first section backed off from multiplying vectors. Now we go forward to define the *"dot product"* of $\boldsymbol{v}$ and $\boldsymbol{w}$. This multiplication involves the separate products v_1w_1 and v_2w_2, but it doesn't stop there. Those two numbers are added to produce the single number $\boldsymbol{v}\cdot\boldsymbol{w}$. *This is the geometry section (lengths and angles).*

DEFINITION The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v}\cdot\boldsymbol{w}$:

$$\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2. \tag{1}$$

Example 1 The vectors $\boldsymbol{v} = (4,2)$ and $\boldsymbol{w} = (-1,2)$ have a *zero* dot product:

Dot product is zero
Perpendicular vectors
$$\begin{bmatrix}4\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\2\end{bmatrix} = -4+4 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular.* The angle between them is 90°. When we drew them in Figure 1.1, we saw a rectangle (not just any parallelogram). The clearest example of perpendicular vectors is $\boldsymbol{i} = (1,0)$ along the x axis and $\boldsymbol{j} = (0,1)$ up the y axis. Again the dot product is $\boldsymbol{i}\cdot\boldsymbol{j} = 0+0 = 0$. Those vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ form a right angle.

The dot product of $\boldsymbol{v} = (1,2)$ and $\boldsymbol{w} = (3,1)$ is 5. Soon $\boldsymbol{v}\cdot\boldsymbol{w}$ will reveal the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (not 90°). Please check that $\boldsymbol{w}\cdot\boldsymbol{v}$ is also 5.

The dot product $\boldsymbol{w}\cdot\boldsymbol{v}$ equals $\boldsymbol{v}\cdot\boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

Example 2 Put a weight of 4 at the point $x = -1$ (left of zero) and a weight of 2 at the point $x = 2$ (right of zero). The x axis will balance on the center point (like a see-saw). The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4,2)$. The vector of distances from the center is $(v_1, v_2) = (-1,2)$. The weights times the distances, w_1v_1 and w_2v_2, give the "moments". The equation for the see-saw to balance is $w_1v_1 + w_2v_2 = 0$.

Example 3 Dot products enter in economics and business. We have three goods to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. *Selling q_1 units at the price p_1 brings in q_1p_1.* The total income (quantities q times prices p) is ***the dot product $\boldsymbol{q}\cdot\boldsymbol{p}$ in three dimensions***:

$$\textbf{\textit{Income}} = (q_1,q_2,q_3)\cdot(p_1,p_2,p_3) = q_1p_1 + q_2p_2 + q_3p_3 = \textbf{\textit{dot product.}}$$

A zero dot product means that "the books balance". Total sales equal total purchases if $\boldsymbol{q}\cdot\boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). A supermarket with thousands of goods goes quickly into high dimensions.

Small note: Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

Main point To compute $\boldsymbol{v} \cdot \boldsymbol{w}$, multiply each v_i times w_i. Then add $\Sigma v_i w_i$.

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself.* In this case $\boldsymbol{v}$ equals $\boldsymbol{w}$. When the vector is $\boldsymbol{v} = (1,2,3)$, the dot product with itself is $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 = 14$:

Dot product $\boldsymbol{v} \cdot \boldsymbol{v}$
Length squared

$$\|\boldsymbol{v}\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = \mathbf{14}.$$

Instead of a 90° angle between vectors we have 0°. The answer is not zero because $\boldsymbol{v}$ is not perpendicular to itself. The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ gives the *length of* $\boldsymbol{v}$ *squared.*

DEFINITION The ***length*** $\|\boldsymbol{v}\|$ of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$:

Length = norm($\boldsymbol{v}$) $\qquad$ **length** $= \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}.$

In two dimensions the length is $\sqrt{v_1^2 + v_2^2}$. In three dimensions it is $\sqrt{v_1^2 + v_2^2 + v_3^2}$. By the calculation above, the length of $\boldsymbol{v} = (1,2,3)$ is $\|\boldsymbol{v}\| = \sqrt{14}$.

Here $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ is just the ordinary length of the arrow that represents the vector. In two dimensions, the arrow is in a plane. If the components are 1 and 2, the arrow is the third side of a right triangle (Figure 1.6). The Pythagoras formula $a^2 + b^2 = c^2$, which connects the three sides, is $1^2 + 2^2 = \|\boldsymbol{v}\|^2$.

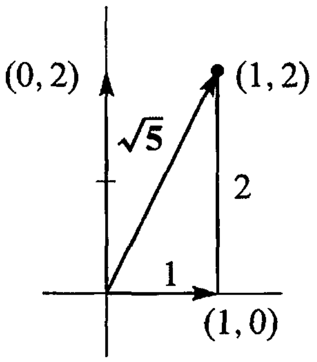
For the length of $\boldsymbol{v} = (1,2,3)$, we used the right triangle formula twice. The vector $(1,2,0)$ in the base has length $\sqrt{5}$. This base vector is perpendicular to $(0,0,3)$ that goes straight up. So the diagonal of the box has length $\|\boldsymbol{v}\| = \sqrt{5+9} = \sqrt{14}$.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus the vector $(1,1,1,1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. The diagonal in n dimensions has length $\sqrt{n}$.

The word "**unit**" is always indicating that some measurement equals "one". The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we define the idea of a "unit vector".

DEFINITION ***A unit vector $\boldsymbol{u}$ is a vector whose length equals one.*** Then $\boldsymbol{u} \cdot \boldsymbol{u} = \mathbf{1}$.

An example in four dimensions is $\boldsymbol{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $\boldsymbol{u} \cdot \boldsymbol{u}$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $\boldsymbol{v} = (1,1,1,1)$ by its length $\|\boldsymbol{v}\| = 2$ to get this unit vector.



$$\begin{aligned} v \cdot v &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$

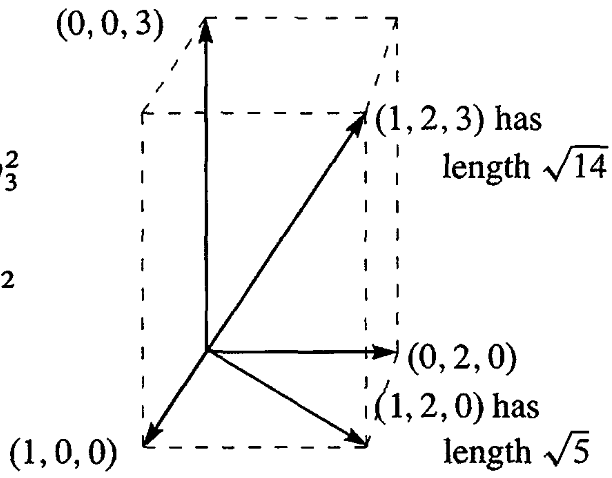


Figure 1.6: The length $\sqrt{v \cdot v}$ of two-dimensional and three-dimensional vectors.

Example 4 The standard unit vectors along the x and y axes are written $\boldsymbol{i}$ and $\boldsymbol{j}$. In the xy plane, the unit vector that makes an angle "theta" with the x axis is $(\cos\theta, \sin\theta)$:

$$\textbf{Unit vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

When $\theta = 0$, the horizontal vector $\boldsymbol{u}$ is $\boldsymbol{i}$. When $\theta = 90^\circ$ (or $\frac{\pi}{2}$ radians), the vertical vector is $\boldsymbol{j}$. At any angle, the components $\cos\theta$ and $\sin\theta$ produce $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ because $\cos^2\theta + \sin^2\theta = 1$. These vectors reach out to the unit circle in Figure 1.7. Thus $\cos\theta$ and $\sin\theta$ are simply the coordinates of that point at angle θ on the unit circle.

Since $(2, 2, 1)$ has length 3, the vector $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has length 1. Check that $\boldsymbol{u} \cdot \boldsymbol{u} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$. For a unit vector, **divide any nonzero v by its length $\|v\|$**.

Unit vector $\quad \boldsymbol{u} = v/\|v\|$ **is a unit vector in the same direction as v.**

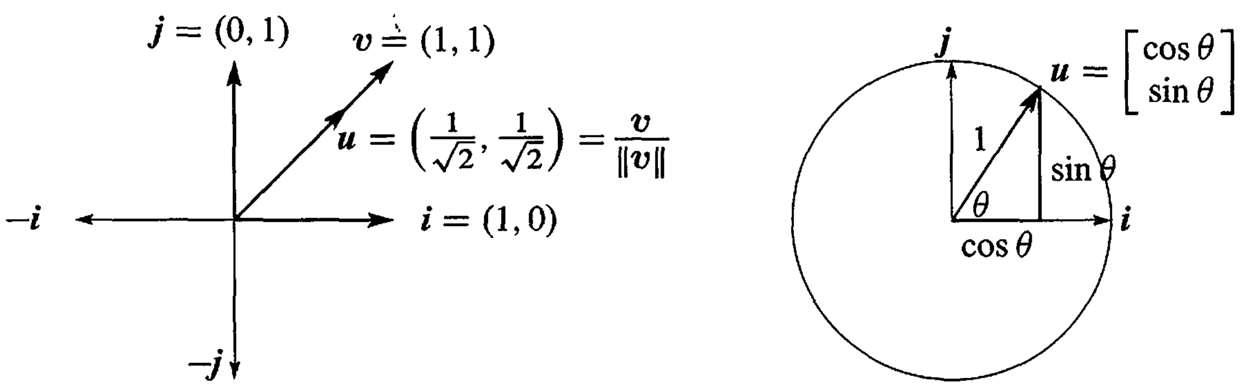


Figure 1.7: The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle 45° (left) divides $v = (1, 1)$ by its length $\|v\| = \sqrt{2}$. The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ.

The Angle Between Two Vectors

We stated that perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is 90°. To explain this, we have to connect angles to dot products. Then we show how $\boldsymbol{v} \cdot \boldsymbol{w}$ finds the angle between any two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

Right angles *The dot product is $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ when $\boldsymbol{v}$ is perpendicular to $\boldsymbol{w}$.*

Proof When $\boldsymbol{v}$ and $\boldsymbol{w}$ are perpendicular, they form two sides of a right triangle. The third side is $\boldsymbol{v} - \boldsymbol{w}$ (the hypotenuse going across in Figure 1.8). The *Pythagoras Law* for the sides of a right triangle is $a^2 + b^2 = c^2$:

$$\textbf{Perpendicular vectors} \quad \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2 \tag{2}$$

Writing out the formulas for those lengths in two dimensions, this equation is

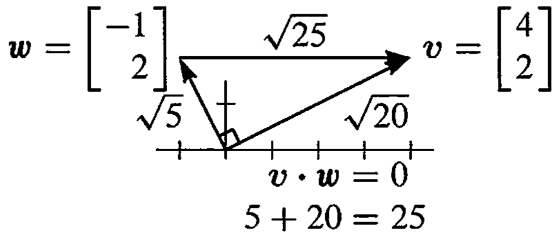
$$\textbf{Pythagoras} \qquad \left(v_1^2 + v_2^2\right) + \left(w_1^2 + w_2^2\right) = (v_1 - w_1)^2 + (v_2 - w_2)^2. \tag{3}$$

The right side begins with $v_1^2 - 2v_1w_1 + w_1^2$. Then v_1^2 and w_1^2 are on both sides of the equation and they cancel, leaving $-2v_1w_1$. Also v_2^2 and w_2^2 cancel, leaving $-2v_2w_2$. (In three dimensions there would be $-2v_3w_3$.) Now divide by -2:

$$0 = -2v_1w_1 - 2v_2w_2 \quad \text{which leads to} \quad \boldsymbol{v_1 w_1 + v_2 w_2 = 0}. \tag{4}$$

Conclusion Right angles produce $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. The dot product is zero when the angle is $\theta = 90°$. Then $\cos\theta = 0$. The zero vector $\boldsymbol{v} = \mathbf{0}$ is perpendicular to every vector $\boldsymbol{w}$ because $\mathbf{0} \cdot \boldsymbol{w}$ is always zero.

Now suppose $\boldsymbol{v} \cdot \boldsymbol{w}$ is **not zero.** It may be positive, it may be negative. The sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ immediately tells whether we are below or above a right angle. The angle is less than 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive. The angle is above 90° when $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative. The right side of Figure 1.8 shows a typical vector $\boldsymbol{v} = (3, 1)$. The angle with $\boldsymbol{w} = (1, 3)$ is less than 90° because $\boldsymbol{v} \cdot \boldsymbol{w} = 6$ is positive.



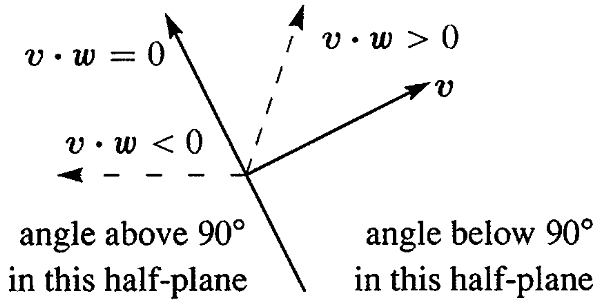


Figure 1.8: Perpendicular vectors have $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. Then $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2$.

The borderline is where vectors are perpendicular to $\boldsymbol{v}$. On that dividing line between plus and minus, $(1,-3)$ is perpendicular to $(3,1)$. The dot product is zero.

The dot product reveals the exact angle θ. This is not necessary for linear algebra—you could stop here! Once we have matrices, we won't come back to θ. But while we are on the subject of angles, this is the place for the formula.

Start with **unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$**. The sign of $\boldsymbol{u}\cdot\boldsymbol{U}$ tells whether $\theta < 90^\circ$ or $\theta > 90^\circ$. Because the vectors have length 1, we learn more than that. ***The dot product $\boldsymbol{u}\cdot\boldsymbol{U}$ is the cosine of θ.*** This is true in any number of dimensions.

Unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ at angle θ have $\boldsymbol{u}\cdot\boldsymbol{U} = \cos\theta$. **Certainly** $|\boldsymbol{u}\cdot\boldsymbol{U}| \le 1$.

Remember that $\cos\theta$ is never greater than 1. It is never less than -1. ***The dot product of unit vectors is between -1 and 1.***

Figure 1.9 shows this clearly when the vectors are $\boldsymbol{u} = (\cos\theta, \sin\theta)$ and $\boldsymbol{i} = (1,0)$. The dot product is $\boldsymbol{u}\cdot\boldsymbol{i} = \cos\theta$. That is the cosine of the angle between them.

After rotation through any angle α, these are still unit vectors. The vector $\boldsymbol{i} = (1,0)$ rotates to $(\cos\alpha, \sin\alpha)$. The vector $\boldsymbol{u}$ rotates to $(\cos\beta, \sin\beta)$ with $\beta = \alpha + \theta$. Their dot product is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$. From trigonometry this is the same as $\cos(\beta-\alpha)$. But $\beta - \alpha$ is the angle θ, so the dot product is $\cos\theta$.

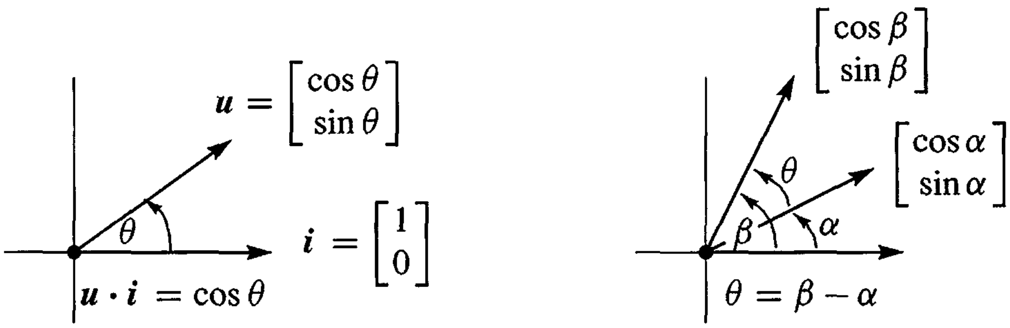


Figure 1.9: The dot product of unit vectors is the cosine of the angle θ.

Problem 24 proves $|\boldsymbol{u}\cdot\boldsymbol{U}| \le 1$ directly, without mentioning angles. The inequality and the cosine formula $\boldsymbol{u}\cdot\boldsymbol{U} = \cos\theta$ are always true for unit vectors.

What if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not unit vectors? Divide by their lengths to get $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ and $\boldsymbol{U} = \boldsymbol{w}/\|\boldsymbol{w}\|$. Then the dot product of those unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ gives $\cos\theta$.

COSINE FORMULA If $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors then $\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta.$

Whatever the angle, this dot product of $v/\|v\|$ with $w/\|w\|$ never exceeds one. That is the ***"Schwarz inequality"*** $|v \cdot w| \leq \|v\|\,\|w\|$ for dot products—or more correctly the Cauchy-Schwarz-Buniakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics).

Since $|\cos\theta|$ never exceeds 1, the cosine formula gives two great inequalities:

SCHWARZ INEQUALITY	$\lvert v \cdot w\rvert \leq \|v\|\,\|w\|$
TRIANGLE INEQUALITY	$\|v + w\| \leq \|v\| + \|w\|$

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

The angle is below 90° because $v \cdot w = 4$ is positive. By the Schwarz inequality, $v \cdot w = 4$ is less than $\|v\|\,\|w\| = 5$. Side $3 = \|v + w\|$ is less than side 1 + side 2, by the triangle inequality. For $v + w = (3,3)$ that says $\sqrt{18} < \sqrt{5} + \sqrt{5}$. Square this to get $18 < 20$.

Example 6 The dot product of $v = (a,b)$ and $w = (b,a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality in this case says that $2ab \leq a^2 + b^2$.

This is more famous if we write $x = a^2$ and $y = b^2$. The "geometric mean" $\sqrt{xy}$ is not larger than the "arithmetic mean" = average $\frac{1}{2}(x + y)$.

$$\begin{matrix}\textbf{Geometric} \\ \textbf{mean}\end{matrix} \leq \begin{matrix}\textbf{Arithmetic} \\ \textbf{mean}\end{matrix} \qquad ab \leq \frac{a^2 + b^2}{2} \quad \text{becomes} \quad \sqrt{xy} \leq \frac{x + y}{2}.$$

Example 5 had $a = 2$ and $b = 1$. So $x = 4$ and $y = 1$. The geometric mean $\sqrt{xy} = 2$ is below the arithmetic mean $\frac{1}{2}(1 + 4) = 2.5$.

Notes on Computing

Write the components of v as $v(1), \ldots, v(N)$ and similarly for w. In FORTRAN, the sum $v + w$ requires a loop to add components separately. The dot product also uses a loop to add the separate $v(j)w(j)$. Here are VPLUSW and VDOTW:

FORTRAN

```
   DO 10 J = 1,N
10 VPLUSW(J) = v(J) + w(J)
```

```
   DO 10 J = 1,N
10 VDOTW = VDOTW + V(J) * W(J)
```

MATLAB *and also* PYTHON *work directly with whole vectors, not their components.* No loop is needed. When v and w have been defined, $v + w$ is immediately understood.

Input v and w as rows—the prime $'$ transposes them to columns. $2v + 3w$ uses $*$ for multiplication by 2 and 3. The result will be printed unless the line ends in a semicolon.

MATLAB $\quad v = [2 \ \ 3 \ \ 4]' \ ; \quad w = [1 \ \ 1 \ \ 1]' \ ; \quad u = 2 * v + 3 * w$

The dot product $v \cdot w$ is usually seen as ***a row times a column (with no dot)***:

$$\text{Instead of} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{we more often see} \quad [\,1 \ \ 2\,] \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{or} \quad v' * w$$

The length of v is known to MATLAB as norm (v). We could define it ourselves as sqrt $(v' * v)$, using the square root function—also known. The cosine we have to define ourselves! The angle (in radians) comes from the *arc cosine* (acos) function:

Cosine formula	cosine = $v' * w/(\text{norm}\,(v) * \text{norm}\,(w))$
Angle formula	angle = acos (cosine)

An M-file would create a new function **cosine** (v, w) for future use. The M-files created especially for this book are listed at the end. R and PYTHON are open source software.

■ REVIEW OF THE KEY IDEAS ■

1. The dot product $v \cdot w$ multiplies each component v_i by w_i and adds all $v_i w_i$.
2. The length $\|v\|$ of a vector is the square root of $v \cdot v$.
3. $u = v/\|v\|$ is a ***unit vector***. Its length is 1.
4. The dot product is $v \cdot w = 0$ when vectors v and w are perpendicular.
5. The cosine of θ (the angle between any nonzero v and w) never exceeds 1:

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} \qquad \textbf{\textit{Schwarz inequality}} \quad |v \cdot w| \leq \|v\|\,\|w\|.$$

Problem 21 will produce the ***triangle inequality*** $\|v + w\| \leq \|v\| + \|w\|$.

■ WORKED EXAMPLES ■

1.2 A For the vectors $v = (3, 4)$ and $w = (4, 3)$ test the Schwarz inequality on $v \cdot w$ and the triangle inequality on $\|v + w\|$. Find $\cos\theta$ for the angle between v and w. When will we have *equality* $|v \cdot w| = \|v\|\,\|w\|$ and $\|v + w\| = \|v\| + \|w\|$?

Solution The dot product is $v \cdot w = (3)(4) + (4)(3) = 24$. The length of v is $\|v\| = \sqrt{9+16} = 5$ and also $\|w\| = 5$. The sum $v + w = (7,7)$ has length $7\sqrt{2} < 10$.

Schwarz inequality	$\lvert v \cdot w \rvert \leq \|v\|\,\|w\|$	is $24 < 25$.
Triangle inequality	$\|v + w\| \leq \|v\| + \|w\|$	is $7\sqrt{2} < 5 + 5$.
Cosine of angle	$\cos\theta = \frac{24}{25}$	Thin angle from $v = (3,4)$ to $w = (4,3)$

Suppose one vector is a multiple of the other as in $w = cv$. Then the angle is $0°$ or $180°$. In this case $|\cos\theta| = 1$ and $|v \cdot w|$ *equals* $\|v\|\,\|w\|$. If the angle is $0°$, as in $w = 2v$, then $\|v + w\| = \|v\| + \|w\|$. The triangle is completely flat.

1.2 B Find a unit vector u in the direction of $v = (3,4)$. Find a unit vector U that is perpendicular to u. How many possibilities for U?

Solution For a unit vector u, divide v by its length $\|v\| = 5$. For a perpendicular vector V we can choose $(-4,3)$ since the dot product $v \cdot V$ is $(3)(-4) + (4)(3) = 0$. For a *unit* vector U, divide V by its length $\|V\|$:

$$u = \frac{v}{\|v\|} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad U = \frac{V}{\|V\|} = \left(-\frac{4}{5}, \frac{3}{5}\right) \qquad u \cdot U = 0$$

The only other perpendicular unit vector would be $-U = (\frac{4}{5}, -\frac{3}{5})$.

1.2 C Find a vector $x = (c,d)$ that has dot products $x \cdot r = 1$ and $x \cdot s = 0$ with the given vectors $r = (2,-1)$ and $s = (-1,2)$.

How is this question related to Example **1.1 C**, which solved $cv + dw = b = (1,0)$?

Solution Those two dot products give linear equations for c and d. Then $x = (c,d)$.

$$\begin{array}{lrl} x \cdot r = 1 & 2c - \ d = 1 & \textbf{The same equations as} \\ x \cdot s = 0 & -\ c + 2d = 0 & \textbf{in Worked Example 1.1 C} \end{array}$$

The second equation makes x perpendicular to $s = (-1,2)$. So I can see the geometry: Go in the perpendicular direction $(2,1)$. When you reach $x = \frac{1}{3}(2,1)$, the dot product with $r = (2,-1)$ has the required value $x \cdot r = 1$.

Comment on n equations for $x = (x_1, \ldots, x_n)$ in n-dimensional space

Section 1.1 would start with column vectors $v_1, \ldots, v_n$. The goal is to combine them to produce a required vector $x_1 v_1 + \cdots + x_n v_n = b$. This section would start from vectors $r_1, \ldots, r_n$. Now the goal is to find x with the required dot products $x \cdot r_i = b_i$.

Soon the v's will be the columns of a matrix A, and the r's will be the rows of A. Then the (one and only) problem will be to solve $Ax = b$.

Problem Set 1.2

1 Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v}+\boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u}=\begin{bmatrix}-.6\\ .8\end{bmatrix} \qquad \boldsymbol{v}=\begin{bmatrix}3\\4\end{bmatrix} \qquad \boldsymbol{w}=\begin{bmatrix}8\\6\end{bmatrix}.$$

2 Compute the lengths $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$ of those vectors. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

3 Find unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ in Problem 1, and the cosine of the angle θ. Choose vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ that make $0°$, $90°$, and $180°$ angles with $\boldsymbol{w}$.

4 For any *unit* vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, find the dot products (actual numbers) of

(a) $\boldsymbol{v}$ and $-\boldsymbol{v}$ (b) $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ (c) $\boldsymbol{v}-2\boldsymbol{w}$ and $\boldsymbol{v}+2\boldsymbol{w}$

5 Find unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the directions of $\boldsymbol{v} = (3,1)$ and $\boldsymbol{w} = (2,1,2)$. Find unit vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ that are perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 (a) Describe every vector $\boldsymbol{w} = (w_1, w_2)$ that is perpendicular to $\boldsymbol{v} = (2,-1)$.

(b) The vectors that are perpendicular to $\boldsymbol{V} = (1,1,1)$ lie on a ____.

(c) The vectors that are perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a ____.

7 Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\boldsymbol{v}=\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}1\\0\end{bmatrix}$ (b) $\boldsymbol{v}=\begin{bmatrix}2\\2\\-1\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}2\\-1\\2\end{bmatrix}$

(c) $\boldsymbol{v}=\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}$ (d) $\boldsymbol{v}=\begin{bmatrix}3\\1\end{bmatrix}$ and $\boldsymbol{w}=\begin{bmatrix}-1\\-2\end{bmatrix}$.

8 True or false (give a reason if true or a counterexample if false):

(a) If $\boldsymbol{u}$ is perpendicular (in three dimensions) to $\boldsymbol{v}$ and $\boldsymbol{w}$, those vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

(b) If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}+2\boldsymbol{w}$.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular unit vectors then $\|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{2}$.

9 The slopes of the arrows from $(0,0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1. **Suppose the product v_2w_2/v_1w_1 of those slopes is -1.** Show that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the vectors are perpendicular.

10 Draw arrows from $(0,0)$ to the points $\boldsymbol{v} = (1,2)$ and $\boldsymbol{w} = (-2,1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the arrows are ____.

11 If $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative, what does this say about the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$? Draw a 3-dimensional vector $\boldsymbol{v}$ (an arrow), and show where to find all $\boldsymbol{w}$'s with $\boldsymbol{v} \cdot \boldsymbol{w} < 0$.

12 With $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (1, 5)$ choose a number c so that $\boldsymbol{w} - c\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$. Then find the formula that gives this number c for any nonzero $\boldsymbol{v}$ and $\boldsymbol{w}$. (*Note*: $c\boldsymbol{v}$ is the "projection" of $\boldsymbol{w}$ onto $\boldsymbol{v}$.)

13 Find two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(1, 0, 1)$ and to each other.

14 Find nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that are perpendicular to $(1, 1, 1, 1)$ and to each other.

15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x + y) =$ ____. This would come in Example 6 from the Schwarz inequality for $\boldsymbol{v} = (\sqrt{2}, \sqrt{8})$ and $\boldsymbol{w} = (\sqrt{8}, \sqrt{2})$. Find $\cos\theta$ for this $\boldsymbol{v}$ and $\boldsymbol{w}$.

16 **How long is the vector $\boldsymbol{v} = (1, 1, \ldots, 1)$ in 9 dimensions**? Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$ and a unit vector $\boldsymbol{w}$ that is perpendicular to $\boldsymbol{v}$.

17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ along the axes? Check the formula $\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides $\boldsymbol{v} = (4, 2)$ and $\boldsymbol{w} = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = (\text{length of } \boldsymbol{v} + \boldsymbol{w})^2.$$

19 (Rules for dot products) These equations are simple but useful:

(1) $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ (2) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ (3) $(c\boldsymbol{v}) \cdot \boldsymbol{w} = c(\boldsymbol{v} \cdot \boldsymbol{w})$

Use (2) with $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ to prove $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$.

20 The "Law of Cosines" comes from $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$:

$$\textbf{Cosine Law} \qquad \|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2.$$

If $\theta < 90^\circ$ show that $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$ (the third side).

21 The ***triangle inequality*** says: (length of $\boldsymbol{v} + \boldsymbol{w}$) $\leq$ (length of $\boldsymbol{v}$) + (length of $\boldsymbol{w}$).

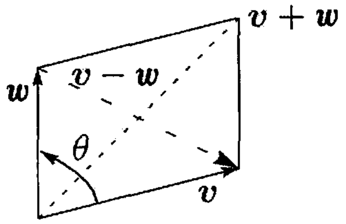
Problem 19 found $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2$. Use the Schwarz inequality $\boldsymbol{v} \cdot \boldsymbol{w} \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ to show that $\|$**side 3**$\|$ can not exceed $\|$**side 1**$\|$ + $\|$**side 2**$\|$:

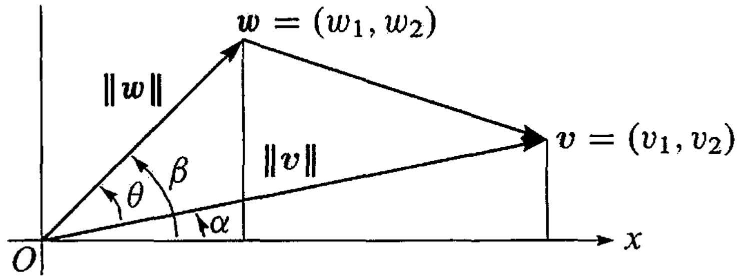
$$\textbf{Triangle inequality} \qquad \|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \quad \textbf{or} \quad \|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

22 The Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those two sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.





23 The figure shows that $\cos\alpha = v_1/\|v\|$ and $\sin\alpha = v_2/\|v\|$. Similarly $\cos\beta$ is ____ and $\sin\beta$ is ____. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

24 One-line proof of the Schwarz inequality $|\boldsymbol{u}\cdot\boldsymbol{U}| \le 1$ for unit vectors:

$$|\boldsymbol{u}\cdot\boldsymbol{U}| \le |u_1|\,|U_1| + |u_2|\,|U_2| \le \frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = \frac{1+1}{2} = 1.$$

Put $(u_1, u_2) = (.6, .8)$ and $(U_1, U_2) = (.8, .6)$ in that whole line and find $\cos\theta$.

25 Why is $|\cos\theta|$ never greater than 1 in the first place?

26 If $\boldsymbol{v} = (1, 2)$ draw all vectors $\boldsymbol{w} = (x, y)$ in the xy plane with $\boldsymbol{v}\cdot\boldsymbol{w} = x + 2y = 5$. Which is the shortest $\boldsymbol{w}$?

27 (*Recommended*) If $||\boldsymbol{v}|| = 5$ and $||\boldsymbol{w}|| = 3$, what are the smallest and largest values of $||\boldsymbol{v} - \boldsymbol{w}||$? What are the smallest and largest values of $\boldsymbol{v}\cdot\boldsymbol{w}$?

Challenge Problems

28 Can three vectors in the xy plane have $\boldsymbol{u}\cdot\boldsymbol{v} < 0$ and $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ and $\boldsymbol{u}\cdot\boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would certainly be impossible ...).

29 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\boldsymbol{v} = (x, y, z)$ and the vector $\boldsymbol{w} = (z, x, y)$. Challenge question: Explain why $\boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ is always $-\frac{1}{2}$.

30 How could you prove $\sqrt[3]{xyz} \le \frac{1}{3}(x+y+z)$ (geometric mean $\le$ arithmetic mean)?

31 Find four perpendicular unit vectors with all components equal to $\frac{1}{2}$ or $-\frac{1}{2}$.

32 Using v = randn(3, 1) in MATLAB, create a random unit vector $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$. Using V = randn(3, 30) create 30 more random unit vectors $\boldsymbol{U}_j$. What is the average size of the dot products $|\boldsymbol{u}\cdot\boldsymbol{U}_j|$? In calculus, the average $\int_0^\pi |\cos\theta|d\theta/\pi = 2/\pi$.

1.3 Matrices

This section is based on two carefully chosen examples. They both start with three vectors. I will take their combinations using *matrices*. The three vectors in the first example are $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$:

$$\textbf{First example} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Their linear combinations in three-dimensional space are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$:

$$\textbf{Combinations} \qquad c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (1)$$

Now something important: *Rewrite that combination using a matrix.* The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ go into the columns of the matrix A. That matrix "multiplies" a vector:

$$\begin{array}{l}\textbf{Same combination}\\ \textbf{is now } \boldsymbol{A} \textbf{ times } \boldsymbol{x}\end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \qquad (2)$$

The numbers c, d, e are the components of a vector $\boldsymbol{x}$. The matrix A times the vector $\boldsymbol{x}$ is the same as the combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ of the three columns:

$$\textbf{Matrix times vector} \qquad A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}. \qquad (3)$$

This is more than a definition of $A\boldsymbol{x}$, because the rewriting brings a crucial change in viewpoint. At first, the numbers c, d, e were multiplying the vectors. Now the matrix is multiplying those numbers. **The matrix A acts on the vector $\boldsymbol{x}$.** The result $A\boldsymbol{x}$ is a combination $\boldsymbol{b}$ of the columns of A.

To see that action, I will write x_1, x_2, x_3 instead of c, d, e. I will write b_1, b_2, b_3 for the components of $A\boldsymbol{x}$. With new letters we see

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}. \qquad (4)$$

The input is $\boldsymbol{x}$ and the output is $\boldsymbol{b} = A\boldsymbol{x}$. This A is a "**difference matrix**" because $\boldsymbol{b}$ contains differences of the input vector $\boldsymbol{x}$. The top difference is $x_1 - x_0 = x_1 - 0$.

Here is an example to show differences of numbers (squares in $\boldsymbol{x}$, odd numbers in $\boldsymbol{b}$):

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \text{squares} \qquad A\boldsymbol{x} = \begin{bmatrix} 1-0 \\ 4-1 \\ 9-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \boldsymbol{b}\,. \tag{5}$$

That pattern would continue for a 4 by 4 difference matrix. The next square would be $x_4 = 16$. The next difference would be $x_4 - x_3 = 16 - 9 = 7$ (this is the next odd number). The matrix finds all the differences at once.

Important Note. You may already have learned about multiplying $A\boldsymbol{x}$, a matrix times a vector. Probably it was explained differently, using the rows instead of the columns. The usual way takes the dot product of each row with $\boldsymbol{x}$:

Dot products with rows

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1,x_2,x_3) \\ (-1,1,0) \cdot (x_1,x_2,x_3) \\ (0,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}.$$

Those dot products are the same x_1 and $x_2 - x_1$ and $x_3 - x_2$ that we wrote in equation (4). The new way is to work with $A\boldsymbol{x}$ *a column at a time*. Linear combinations are the key to linear algebra, and the output $A\boldsymbol{x}$ is a linear combination of the **columns** of A.

With numbers, you can multiply $A\boldsymbol{x}$ either way (I admit to using rows). With letters, columns are the good way. Chapter 2 will repeat these rules of matrix multiplication, and explain the underlying ideas. There we will multiply matrices both ways.

Linear Equations

One more change in viewpoint is crucial. Up to now, the numbers x_1, x_2, x_3 were known (called c, d, e at first). The right hand side $\boldsymbol{b}$ was not known. We found that vector of differences by multiplying $A\boldsymbol{x}$. **Now we think of $\boldsymbol{b}$ as known and we look for $\boldsymbol{x}$.**

Old question: Compute the linear combination $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$ to find $\boldsymbol{b}$.
New question: Which combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ produces a particular vector $\boldsymbol{b}$?

This is the inverse problem—to find the input $\boldsymbol{x}$ that gives the desired output $\boldsymbol{b} = A\boldsymbol{x}$. You have seen this before, as a system of linear equations for x_1, x_2, x_3. The right hand sides of the equations are b_1, b_2, b_3. We can solve that system to find x_1, x_2, x_3:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{aligned} x_1 \qquad\quad &= b_1 \\ -x_1 + x_2 \quad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \qquad \textbf{Solution} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned} \tag{6}$$

Let me admit right away—most linear systems are not so easy to solve. In this example, the first equation decided $x_1 = b_1$. Then the second equation produced $x_2 = b_1 + b_2$. *The equations could be solved in order* (top to bottom) *because the matrix A was selected to be* **lower triangular**.

Look at two specific choices 0, 0, 0 and 1, 3, 5 of the right sides b_1, b_2, b_3:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ gives } \boldsymbol{x} = \begin{bmatrix} 1 \\ 1+3 \\ 1+3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The first solution (all zeros) is more important than it looks. In words: *If the output is $\boldsymbol{b} = \boldsymbol{0}$, then the input must be $\boldsymbol{x} = \boldsymbol{0}$.* That statement is true for this matrix A. It is not true for all matrices. Our second example will show (for a different matrix C) how we can have $C\boldsymbol{x} = \boldsymbol{0}$ when $C \neq 0$ and $\boldsymbol{x} \neq \boldsymbol{0}$.

This matrix A is "**invertible**". From $\boldsymbol{b}$ we can recover $\boldsymbol{x}$.

The Inverse Matrix

Let me repeat the solution $\boldsymbol{x}$ in equation (6). A sum matrix will appear!

$$A\boldsymbol{x} = \boldsymbol{b} \text{ is solved by } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \quad (7)$$

If the differences of the x's are the b's, the sums of the b's are the x's. That was true for the odd numbers $\boldsymbol{b} = (1, 3, 5)$ and the squares $\boldsymbol{x} = (1, 4, 9)$. It is true for all vectors. **The sum matrix S in equation (7) is the inverse of the difference matrix A.**

Example: The differences of $\boldsymbol{x} = (1, 2, 3)$ are $\boldsymbol{b} = (1, 1, 1)$. So $\boldsymbol{b} = A\boldsymbol{x}$ and $\boldsymbol{x} = S\boldsymbol{b}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} \text{ and } S\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$$

Equation (7) for the solution vector $\boldsymbol{x} = (x_1, x_2, x_3)$ tells us two important facts:

1. For every $\boldsymbol{b}$ there is one solution to $A\boldsymbol{x} = \boldsymbol{b}$. **2.** A matrix S produces $\boldsymbol{x} = S\boldsymbol{b}$.

The next chapters ask about other equations $A\boldsymbol{x} = \boldsymbol{b}$. Is there a solution? How is it computed? In linear algebra, the notation for the "inverse matrix" is A^{-1}:

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is solved by} \quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = S\boldsymbol{b}.$$

Note on calculus. Let me connect these special matrices A and S to calculus. The vector $\boldsymbol{x}$ changes to a function $x(t)$. The differences $A\boldsymbol{x}$ become the *derivative* $dx/dt = b(t)$. In the inverse direction, the sum $S\boldsymbol{b}$ becomes the *integral* of $b(t)$. The Fundamental Theorem of Calculus says that ***integration S is the inverse of differentiation A***.

$$\boldsymbol{A\boldsymbol{x} = \boldsymbol{b}} \textbf{\textit{ and }} \boldsymbol{\boldsymbol{x} = S\boldsymbol{b}} \qquad \frac{dx}{dt} = b \text{ and } x(t) = \int_0^t b. \quad (8)$$

The derivative of distance traveled (x) is the velocity (b). The integral of $b(t)$ is the distance $x(t)$. Instead of adding $+C$, I measured the distance from $x(0) = 0$. In the same way, the differences started at $x_0 = 0$. This zero start makes the pattern complete, when we write $x_1 - x_0$ for the first component of Ax (we just wrote x_1).

Notice another analogy with calculus. The differences of squares $0, 1, 4, 9$ are odd numbers $1, 3, 5$. The derivative of $x(t) = t^2$ is $2t$. A perfect analogy would have produced the even numbers $b = 2, 4, 6$ at times $t = 1, 2, 3$. But differences are not the same as derivatives, and our matrix A produces not $2t$ but $2t - 1$ (these one-sided "backward differences" are centered at $t - \frac{1}{2}$):

$$x(t) - x(t-1) = t^2 - (t-1)^2 = t^2 - (t^2 - 2t + 1) = 2t - 1. \tag{9}$$

The Problem Set will follow up to show that "forward differences" produce $2t + 1$. A better choice (not always seen in calculus courses) is a **centered difference** that uses $x(t+1) - x(t-1)$. Divide Δx by the distance Δt from $t - 1$ to $t + 1$, which is 2:

$$\textbf{Centered difference of } x(t) = t^2 \qquad \frac{(t+1)^2 - (t-1)^2}{2} = 2t \quad \text{exactly.} \tag{10}$$

Difference matrices are great. Centered is best. Our second example is *not invertible.*

Cyclic Differences

This example keeps the same columns $\boldsymbol{u}$ and $\boldsymbol{v}$ but changes $\boldsymbol{w}$ to a new vector $\boldsymbol{w}^*$:

$$\textbf{Second example} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w}^* = \begin{bmatrix} \mathbf{-1} \\ 0 \\ 1 \end{bmatrix}.$$

Now the linear combinations of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}^*$ lead to a **cyclic difference matrix** C:

$$\textbf{Cyclic} \qquad C\boldsymbol{x} = \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}. \tag{11}$$

This matrix C is not triangular. It is not so simple to solve for $\boldsymbol{x}$ when we are given $\boldsymbol{b}$. Actually it is impossible to find *the* solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the three equations either have **infinitely many solutions** or else **no solution**:

$$\begin{array}{l} C\boldsymbol{x} = \mathbf{0} \\ \textbf{Infinitely} \\ \textbf{many } \boldsymbol{x} \end{array} \qquad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{is solved by all vectors} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}. \tag{12}$$

Every constant vector (c, c, c) has zero differences when we go cyclically. This undetermined constant c is like the $+C$ that we add to integrals. The cyclic differences have $x_1 - x_3$ in the first component, instead of starting from $x_0 = 0$.

The other very likely possibility for $C\boldsymbol{x} = \boldsymbol{b}$ is **no solution** at all:

$$C\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \begin{array}{l} \text{Left sides add to 0} \\ \text{Right sides add to 9} \\ \textit{No solution } x_1, x_2, x_3 \end{array} \tag{13}$$

Look at this example geometrically. No combination of $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}^*$ will produce the vector $\boldsymbol{b} = (1, 3, 5)$. The combinations don't fill the whole three-dimensional space. The right sides must have $b_1 + b_2 + b_3 = 0$ to allow a solution to $C\boldsymbol{x} = \boldsymbol{b}$, because the left sides $x_1 - x_3$, $x_2 - x_1$, and $x_3 - x_2$ always add to zero.

Put that in different words. **All linear combinations** $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}^* = \boldsymbol{b}$ **lie on the plane given by** $b_1 + b_2 + b_3 = 0$. This subject is suddenly connecting algebra with geometry. Linear combinations can fill all of space, or only a plane. We need a picture to show the crucial difference between $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ (the first example) and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$.

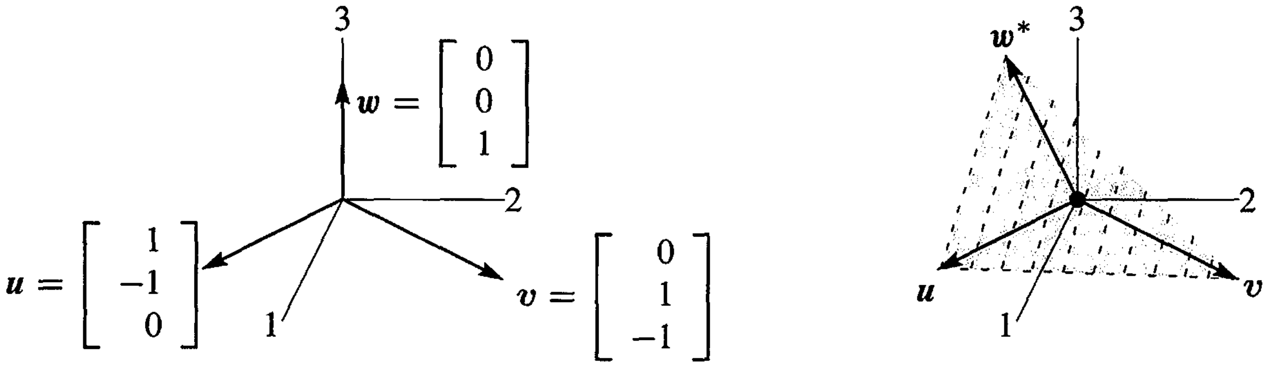


Figure 1.10: Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane.

Independence and Dependence

Figure 1.10 shows those column vectors, first of the matrix A and then of C. The first two columns $\boldsymbol{u}$ and $\boldsymbol{v}$ are the same in both pictures. If we only look at the combinations of those two vectors, we will get a two-dimensional plane. **The key question is whether the third vector is in that plane:**

Independence $\quad \boldsymbol{w}$ is not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.
Dependence $\quad \boldsymbol{w}^*$ *is* in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$.

The important point is that the new vector $\boldsymbol{w}^*$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^* = 0 \qquad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\boldsymbol{u} - \boldsymbol{v}. \tag{14}$$

All three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ have components adding to zero. Then all their combinations will have $b_1 + b_2 + b_3 = 0$ (as we saw above, by adding the three equations). This is the equation for the plane containing all combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$. By including $\boldsymbol{w}^*$ we get *no new vectors* because $\boldsymbol{w}^*$ is already on that plane.

The original $\boldsymbol{w} = (0, 0, 1)$ is not on the plane: $0 + 0 + 1 \neq 0$. The combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ fill the whole three-dimensional space. We know this already, because the solution $\boldsymbol{x} = S\boldsymbol{b}$ in equation (6) gave the right combination to produce any $\boldsymbol{b}$.

The two matrices A and C, with third columns $\boldsymbol{w}$ and $\boldsymbol{w}^*$, allowed me to mention two key words of linear algebra: independence and dependence. The first half of the course will develop these ideas much further—I am happy if you see them early in the two examples:

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are **independent**. No combination except $0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{0}$ gives $\boldsymbol{b} = \boldsymbol{0}$.

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ are **dependent**. Other combinations (specifically $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^*$) give $\boldsymbol{b} = \boldsymbol{0}$.

You can picture this in three dimensions. The three vectors lie in a plane or they don't. Chapter 2 has n vectors in n-dimensional space. *Independence or dependence* is the key point. The vectors go into the columns of an n by n matrix:

Independent columns: $A\boldsymbol{x} = \boldsymbol{0}$ has one solution. A is an **invertible matrix.**

Dependent columns: $A\boldsymbol{x} = \boldsymbol{0}$ has many solutions. A is a **singular matrix.**

Eventually we will have n vectors in m-dimensional space. The matrix A with those n columns is now *rectangular* (m by n). Understanding $A\boldsymbol{x} = \boldsymbol{b}$ is the problem of Chapter 3.

■ REVIEW OF THE KEY IDEAS ■

1. **Matrix times vector**: $A\boldsymbol{x}$ = combination of the columns of A.
2. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, when A is an invertible matrix.
3. The difference matrix A is inverted by the sum matrix $S = A^{-1}$.
4. The cyclic matrix C has no inverse. Its three columns lie in the same plane. Those dependent columns add to the zero vector. $C\boldsymbol{x} = \boldsymbol{0}$ has many solutions.
5. This section is looking ahead to key ideas, not fully explained yet.

■ WORKED EXAMPLES ■

1.3 A Change the southwest entry a_{31} of A (row 3, column 1) to $a_{31} = \mathbf{1}$:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \mathbf{1} & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ \mathbf{x_1} - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution $\boldsymbol{x}$ for any $\boldsymbol{b}$. From $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ read off the inverse matrix A^{-1}.

Solution Solve the (linear triangular) system $Ax = b$ from top to bottom:

$$\begin{array}{ll} \text{first} & x_1 = b_1 \\ \text{then} & x_2 = b_1 + b_2 \\ \text{then} & x_3 = \quad\;\; b_2 + b_3 \end{array} \qquad \text{This says that } x = A^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This is good practice to see the columns of the inverse matrix multiplying b_1, b_2, and b_3. The first column of A^{-1} is the solution for $b = (1,0,0)$. The second column is the solution for $b = (0,1,0)$. The third column x of A^{-1} is the solution for $Ax = b = (0,0,1)$.

The three columns of A are still independent. They don't lie in a plane. The combinations of those three columns, using the right weights x_1, x_2, x_3, can produce any three-dimensional vector $b = (b_1, b_2, b_3)$. Those weights come from $x = A^{-1}b$.

1.3 B This E is an **elimination matrix**. E has a subtraction, E^{-1} has an addition.

$$Ex = b \qquad \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix}$$

The first equation is $x_1 = b_1$. The second equation is $x_2 - \ell x_1 = b_2$. The inverse will *add* $\ell x_1 = \ell b_1$, because the elimination matrix *subtracted* ℓx_1:

$$x = E^{-1}b \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ \ell b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$$

1.3 C Change C from a cyclic difference to a **centered difference** producing $x_3 - x_1$:

$$Cx = b \qquad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ 0 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{15}$$

Show that $Cx = b$ can only be solved when $b_1 + b_3 = 0$. That is a plane of vectors b in three-dimensional space. Each column of C is in the plane, the matrix has no inverse. So this plane contains all combinations of those columns (which are all the vectors Cx).

Solution The first component of $b = Cx$ is x_2, and the last component of b is $-x_2$. So we always have $b_1 + b_3 = 0$, for every choice of x.

If you draw the column vectors in C, the first and third columns fall on the same line. In fact (column 1) $=$ $-$(column 3). So the three columns will lie in a plane, and C is *not* an invertible matrix. We cannot solve $Cx = b$ unless $b_1 + b_3 = 0$.

I included the zeros so you could see that this matrix produces "centered differences". Row i of Cx is x_{i+1} (*right of center*) minus x_{i-1} (*left of center*). Here is the 4 by 4 centered difference matrix:

$$Cx = b \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ x_4 - x_2 \\ 0 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{16}$$

Surprisingly this matrix is now invertible! The first and last rows give x_2 and x_3. Then the middle rows give x_1 and x_4. It is possible to write down the inverse matrix C^{-1}. But 5 by 5 will be singular (*not invertible*) again ...

Problem Set 1.3

1 Find the linear combination $2s_1 + 3s_2 + 4s_3 = \boldsymbol{b}$. Then write $\boldsymbol{b}$ as a matrix-vector multiplication $S\boldsymbol{x}$. Compute the dot products (row of S) $\cdot\, \boldsymbol{x}$:

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ go into the columns of } S.$$

2 Solve these equations $S\boldsymbol{y} = \boldsymbol{b}$ with s_1, s_2, s_3 in the columns of S:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}.$$

The sum of the first n odd numbers is _____.

3 Solve these three equations for y_1, y_2, y_3 in terms of B_1, B_2, B_3:

$$S\boldsymbol{y} = \boldsymbol{B} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Write the solution $\boldsymbol{y}$ as a matrix $A = S^{-1}$ times the vector $\boldsymbol{B}$. Are the columns of S independent or dependent?

4 Find a combination $x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3$ that gives the zero vector:

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \boldsymbol{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \boldsymbol{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a _____. The matrix W with those columns is *not invertible.*

5 The rows of that matrix W produce three vectors (I write them as columns):

$$\boldsymbol{r}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \quad \boldsymbol{r}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \quad \boldsymbol{r}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane. There must be many combinations with $y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + y_3\boldsymbol{r}_3 = \boldsymbol{0}$. Find two sets of y's.

6 Which values of c give dependent columns (combination equals zero)?

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

7 If the columns combine into $A\boldsymbol{x} = \mathbf{0}$ then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = 0$:

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{By rows} \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

8 Moving to a 4 by 4 difference equation $A\boldsymbol{x} = \boldsymbol{b}$, find the four components x_1, x_2, x_3, x_4. Then write this solution as $\boldsymbol{x} = S\boldsymbol{b}$ to find the inverse matrix $S = A^{-1}$:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \boldsymbol{b}.$$

9 What is the *cyclic* 4 by 4 difference matrix C? It will have 1 and -1 in each row. Find all solutions $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ to $C\boldsymbol{x} = \mathbf{0}$. The four columns of C lie in a "three-dimensional hyperplane" inside four-dimensional space.

10 A *forward* difference matrix Δ is *upper* triangular:

$$\Delta \boldsymbol{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix in $\boldsymbol{z} = \Delta^{-1}\boldsymbol{b}$?

11 Show that the forward differences $(t+1)^2 - t^2$ are $2t+1$ = *odd numbers*. As in calculus, the difference $(t+1)^n - t^n$ will begin with the derivative of t^n, which is _____.

12 The last lines of the Worked Example say that the 4 by 4 centered difference matrix in (16) *is* invertible. Solve $C\boldsymbol{x} = (b_1, b_2, b_3, b_4)$ to find its inverse in $\boldsymbol{x} = C^{-1}\boldsymbol{b}$.

Challenge Problems

13 The very last words say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $C\boldsymbol{x} = \boldsymbol{b}$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)

14 If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important; two columns are falling on one line. You could use numbers first to see how a, b, c, d are related. The question will lead to:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent columns when it has dependent rows.

Chapter 2

Solving Linear Equations

2.1 Vectors and Linear Equations

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers—we never see x times y. Our first linear system is certainly not big. But you will see how far it leads:

$$\begin{array}{lrcrcr} \textbf{Two equations} & x & - & 2y & = & 1 \\ \textbf{Two unknowns} & 3x & + & 2y & = & 11 \end{array} \qquad (1)$$

We begin *a row at a time*. The first equation $\boldsymbol{x - 2y = 1}$ produces a straight line in the xy plane. The point $x = 1, y = 0$ is on the line because it solves that equation. The point $x = 3, y = 1$ is also on the line because $3 - 2 = 1$. If we choose $x = 101$ we find $y = 50$.

The slope of this particular line is $\frac{1}{2}$, because y increases by 1 when x changes by 2. But slopes are important in calculus and this is linear algebra!

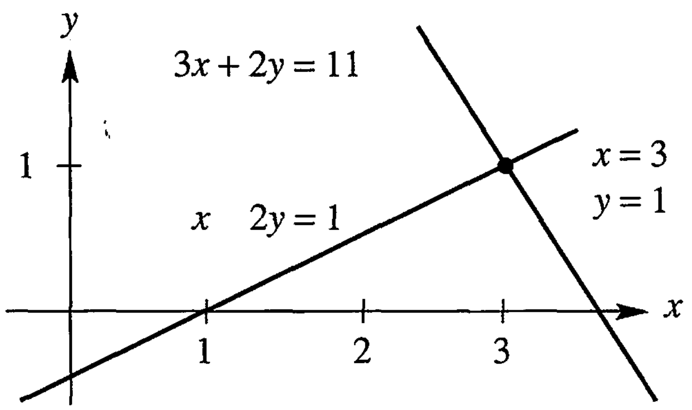


Figure 2.1: *Row picture*: The point (3, 1) where the lines meet is the solution.

Figure 2.1 shows that line $x - 2y = 1$. The second line in this "row picture" comes from the second equation $3x + 2y = 11$. You can't miss the intersection point where the two lines meet. *The point* $x = 3$, $y = 1$ *lies on both lines.* That point solves both equations at once. This is the solution to our system of linear equations.

ROWS ***The row picture shows two lines meeting at a single point (the solution).***

Turn now to the column picture. I want to recognize the same linear system as a "vector equation". Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get a vector equation:

Combination equals b $$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}=\boldsymbol{b}. \tag{2}$$

This has two column vectors on the left side. The problem is *to find the combination of those vectors that equals the vector on the right.* We are multiplying the first column by x and the second column by y, and adding. With the right choices $x = 3$ and $y = 1$ (the same numbers as before), this produces 3(***column 1***) $+ 1$(***column 2***) $= \boldsymbol{b}$.

COLUMNS ***The column picture combines the column vectors on the left side to produce the vector b on the right side.***

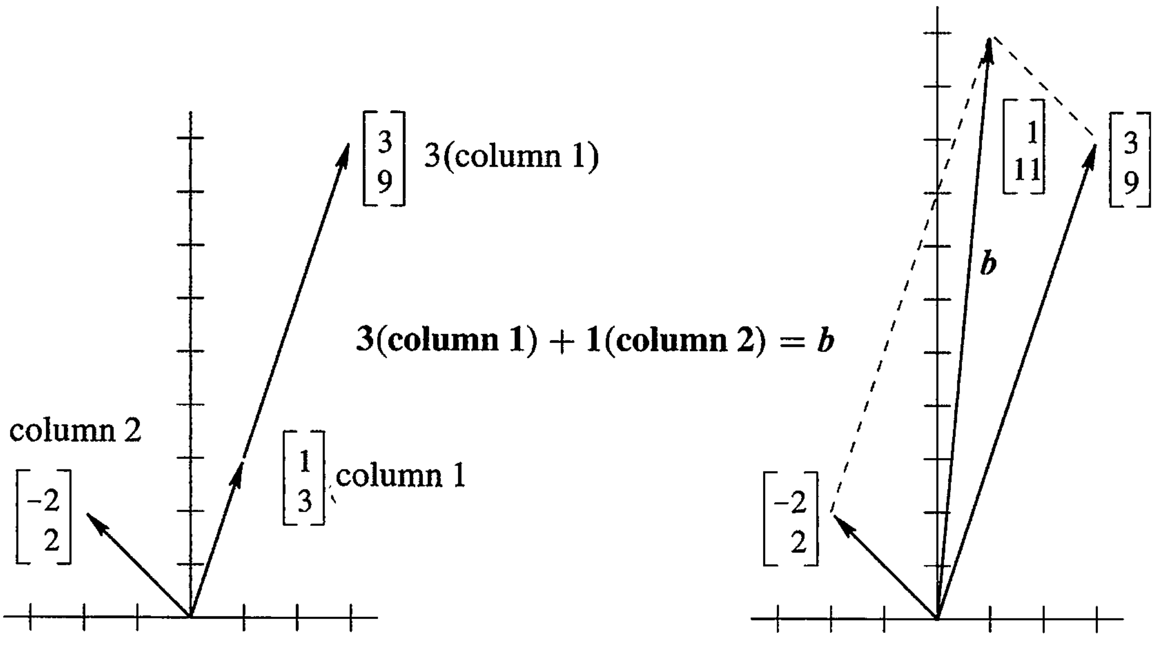


Figure 2.2: *Column picture*: A combination of columns produces the right side (1,11).

Figure 2.2 is the "column picture" of two equations in two unknowns. The first part shows the two separate columns, and that first column multiplied by 3. This multiplication by a *scalar* (a number) is one of the two basic operations in linear algebra:

Scalar multiplication $$3\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3\\9\end{bmatrix}.$$

If the components of a vector $\boldsymbol{v}$ are v_1 and v_2, then $c\boldsymbol{v}$ has components cv_1 and cv_2.

The other basic operation is *vector addition*. We add the first components and the second components separately. The vector sum is $(1, 11)$ as desired:

$$\textbf{Vector addition} \qquad \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The right side of Figure 2.2 shows this addition. The sum along the diagonal is the vector $\boldsymbol{b} = (1, 11)$ on the right side of the linear equations.

To repeat: The left side of the vector equation is a ***linear combination*** of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That step is crucially important, because it contains both of the basic operations:

$$\textbf{Linear combination} \qquad 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Of course the solution $x = 3, y = 1$ is the same as in the row picture. I don't know which picture you prefer! I suspect that the two intersecting lines are more familiar at first. You may like the row picture better, but only for one day. My own preference is to combine column vectors. It is a lot easier to see a combination of four vectors in four-dimensional space, than to visualize how four hyperplanes might possibly meet at a point. (*Even one hyperplane is hard enough...*)

The ***coefficient matrix*** on the left side of the equations is the 2 by 2 matrix A:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}.$$

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $A\boldsymbol{x} = \boldsymbol{b}$:

$$\textbf{Matrix equation} \qquad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The row picture deals with the two rows of A. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into $\boldsymbol{x}$. Here is matrix-vector multiplication:

$$\begin{matrix}\textbf{Dot products with rows} \\ \textbf{Combination of columns}\end{matrix} \qquad A\boldsymbol{x} = \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Looking ahead This chapter is going to solve n equations in n unknowns (for any n). I am not going at top speed, because smaller systems allow examples and pictures and a complete understanding. You are free to go faster, as long as **matrix multiplication and inversion** become clear. Those two ideas will be the keys to invertible matrices.

I can list four steps to understanding elimination using matrices.

1. Elimination goes from A to a triangular U by a sequence of matrix steps E_{ij}.

2. The inverse matrices E_{ij}^{-1} in reverse order bring U back to the original A.

3. In matrix language that reverse order is $A = LU =$ (lower triangle) (upper triangle).

4. Elimination succeeds if A is invertible. (It may need row exchanges.).

The most-used algorithm in computational science takes those steps (MATLAB calls it **lu**). But linear algebra goes beyond square invertible matrices! For m by n matrices, $A\boldsymbol{x} = \mathbf{0}$ may have many solutions. Those solutions will go into a **vector space**. The **rank** of A leads to the **dimension** of that vector space.

All this comes in Chapter 3, and I don't want to hurry. But I must get there.

Three Equations in Three Unknowns

The three unknowns are x, y, z. We have three linear equations:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array} \tag{3}$$

We look for numbers x, y, z that solve all three equations at once. Those desired numbers might or might not exist. For this system, they do exist. When the number of unknowns matches the number of equations, there is *usually* one solution. Before solving the problem, we visualize it both ways:

ROW ***The row picture shows three planes meeting at a single point.***

COLUMN ***The column picture combines three columns to produce*** $(6, 4, 2)$.

In the row picture, each equation produces a *plane* in three-dimensional space. The first plane in Figure 2.3 comes from the first equation $x + 2y + 3z = 6$. That plane crosses the x and y and z axes at the points $(6, 0, 0)$ and $(0, 3, 0)$ and $(0, 0, 2)$. Those three points solve the equation and they determine the whole plane.

The vector $(x, y, z) = (0, 0, 0)$ does not solve $x + 2y + 3z = 6$. Therefore that plane does not contain the origin. The plane $x + 2y + 3z = 0$ does pass through the origin, and it is parallel to $x + 2y + 3z = 6$. When the right side increases to 6, the parallel plane moves away from the origin.

The second plane is given by the second equation $2x + 5y + 2z = 4$. *It intersects the first plane in a line* $\boldsymbol{L}$. The usual result of two equations in three unknowns is a line $\boldsymbol{L}$ of solutions. (Not if the equations were $x + 2y + 3z = 6$ and $x + 2y + 3z = 0$.)

The third equation gives a third plane. It cuts the line $\boldsymbol{L}$ at a single point. That point lies on all three planes and it solves all three equations. It is harder to draw this triple intersection point than to imagine it. The three planes meet at the solution (which we haven't found yet). **The column form will now show immediately why** $z = 2$.

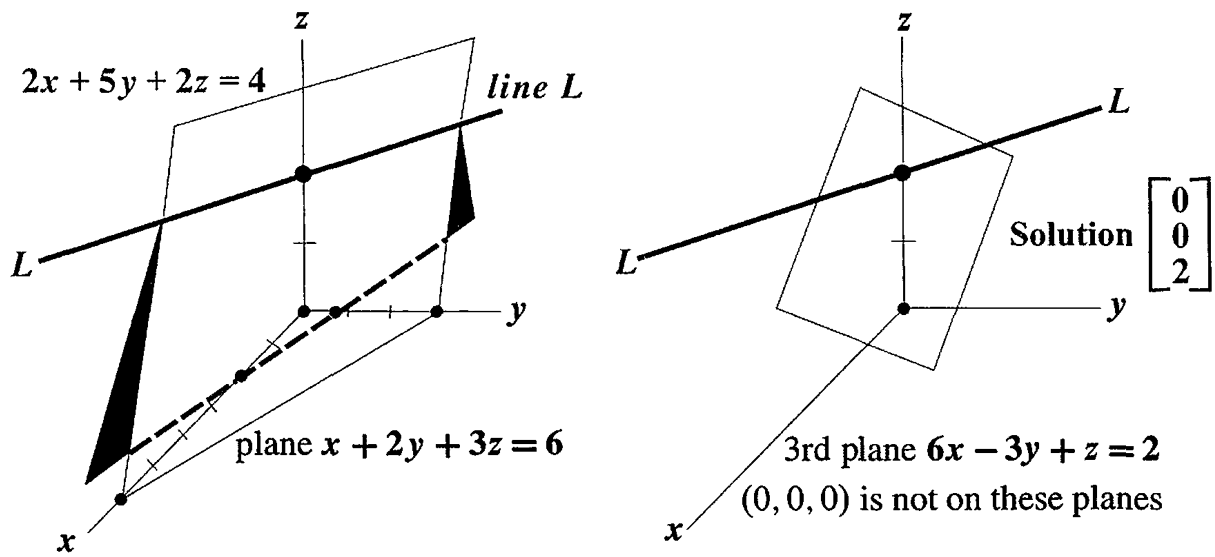


Figure 2.3: *Row picture*: Two planes meet at a line, three planes at a point.

The column picture starts with the vector form of the equations $Ax = b$:

$$\textbf{Combine columns} \qquad x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}. \tag{4}$$

The unknowns are the coefficients x, y, z. We want to multiply the three column vectors by the correct numbers x, y, z to produce $b = (6, 4, 2)$.

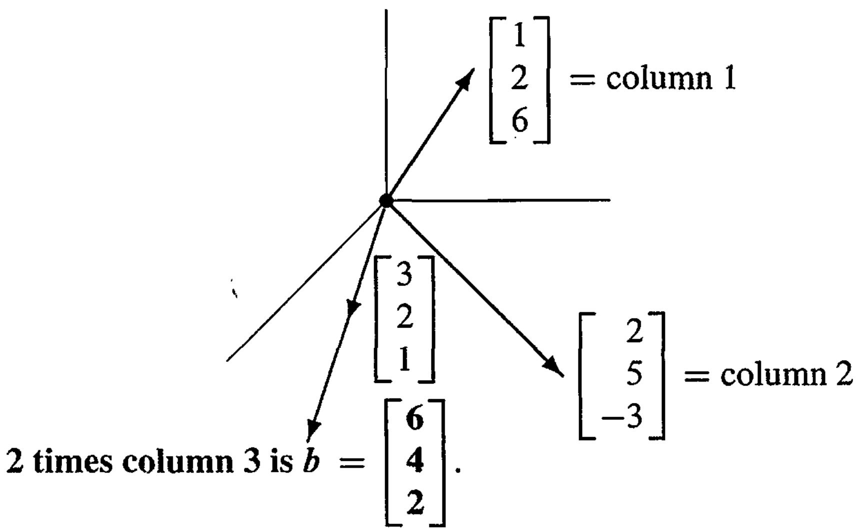


Figure 2.4: *Column picture*: $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = b$.

Figure 2.4 shows this column picture. Linear combinations of those columns can produce any vector b! The combination that produces $b = (6, 4, 2)$ is just 2 times the third column. *The coefficients we need are* $x = 0$, $y = 0$, *and* $z = 2$.

The three planes in the row picture meet at that same solution point $(0,0,2)$:

Correct combination $(x,y,z) = (\mathbf{0},\mathbf{0},\mathbf{2})$

$$\mathbf{0}\begin{bmatrix}1\\2\\6\end{bmatrix}+\mathbf{0}\begin{bmatrix}2\\5\\-3\end{bmatrix}+\mathbf{2}\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}.$$

The Matrix Form of the Equations

We have three rows in the row picture and three columns in the column picture (plus the right side). The three rows and three columns contain nine numbers. *These nine numbers fill a* 3 *by* 3 *matrix* A:

$$\textit{The "coefficient matrix" in } A\boldsymbol{x} = \boldsymbol{b} \textit{ is} \qquad A = \begin{bmatrix}1&2&3\\2&5&2\\6&-3&1\end{bmatrix}.$$

The capital letter A stands for all nine coefficients (in this square array). The letter $\boldsymbol{b}$ denotes the column vector with components $6,4,2$. The unknown $\boldsymbol{x}$ is also a column vector, with components x,y,z. (We use boldface because it is a vector, $\boldsymbol{x}$ because it is unknown.) By rows the equations were (3), by columns they were (4), and by matrices they are (5):

$$\textit{Matrix equation } A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{bmatrix}1&2&3\\2&5&2\\6&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}. \tag{5}$$

Basic question: What does it mean to "multiply A times $\boldsymbol{x}$"? We can multiply by rows or by columns. Either way, $A\boldsymbol{x} = \boldsymbol{b}$ must be a correct representation of the three equations. You do the same nine multiplications either way.

Multiplication by rows $\quad A\boldsymbol{x}$ comes from *dot products*, each row times the column $\boldsymbol{x}$:

$$A\boldsymbol{x} = \begin{bmatrix}(\textit{row 1})\cdot\boldsymbol{x}\\(\textit{row 2})\cdot\boldsymbol{x}\\(\textit{row 3})\cdot\boldsymbol{x}\end{bmatrix}. \tag{6}$$

Multiplication by columns $\quad A\boldsymbol{x}$ is a ***combination of column vectors***:

$$A\boldsymbol{x} = x\,(\textit{column 1}) + y\,(\textit{column 2}) + z\,(\textit{column 3}). \tag{7}$$

When we substitute the solution $\boldsymbol{x} = (0,0,2)$, the multiplication $A\boldsymbol{x}$ produces $\boldsymbol{b}$:

$$\begin{bmatrix}1&2&3\\2&5&2\\6&-3&1\end{bmatrix}\begin{bmatrix}0\\0\\2\end{bmatrix} = \textit{2 times column 3} = \begin{bmatrix}6\\4\\2\end{bmatrix}.$$

The dot product from the first row is $(1,2,3)\cdot(0,0,2) = 6$. The other rows give dot products 4 and 2. ***This book sees $A\boldsymbol{x}$ as a combination of the columns of A.***

Example 1 Here are 3 by 3 matrices A and I = identity, with three 1's and six 0's:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad Ix = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

If you are a row person, the dot product of $(1, 0, 0)$ with $(4, 5, 6)$ is 4. If you are a column person, the linear combination Ax is 4 times the first column $(1, 1, 1)$. In that matrix A, the second and third columns are zero vectors.

The other matrix I is special. It has ones on the "main diagonal". *Whatever vector this matrix multiplies, that vector is not changed.* This is like multiplication by 1, but for matrices and vectors. The exceptional matrix in this example is the 3 by 3 ***identity matrix***:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{always yields the multiplication} \quad Ix = x.$$

Matrix Notation

The first row of a 2 by 2 matrix contains a_{11} and a_{12}. The second row contains a_{21} and a_{22}. The first index gives the row number, so that a_{ij} is an entry in row i. The second index j gives the column number. But those subscripts are not very convenient on a keyboard! Instead of a_{ij} we type $A(i, j)$. ***The entry*** $a_{57} = A(5, 7)$ ***would be in row 5, column 7.***

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}.$$

For an m by n matrix, the row index i goes from 1 to m. The column index j stops at n. There are mn entries $a_{ij} = A(i, j)$. A square matrix of order n has n^2 entries.

Multiplication in MATLAB

I want to express A and x and their product Ax using MATLAB commands. This is a first step in learning that language. I begin by defining the matrix A and the vector x. This vector is a 3 by 1 matrix, with three rows and one column. Enter matrices a row at a time, and use a semicolon to signal the end of a row:

$$A = [1 \;\; 2 \;\; 3; \;\; 2 \;\; 5 \;\; 2; \;\; 6 \; -3 \;\; 1]$$
$$x = [0;\, 0;\, 2]$$

Here are three ways to multiply Ax in MATLAB. In reality, $A * x$ is the good way to do it. MATLAB is a high level language, and it works with matrices:

$$\textit{\textbf{Matrix multiplication}} \quad b = A * x$$

We can also pick out the first row of A (as a smaller matrix!). The notation for that 1 by 3 submatrix is $A(1,:)$. **Here the colon symbol keeps all columns of row 1**:

$$\textbf{\textit{Row at a time}} \quad \boldsymbol{b} = [\,A(1,:) * \boldsymbol{x}\ ;\ A(2,:) * \boldsymbol{x}\ ;\ A(3,:) * \boldsymbol{x}\,]$$

Each entry is a dot product, row times column, 1 by 3 matrix times 3 by 1 matrix.

The other way to multiply uses the columns of A. The first column is the 3 by 1 submatrix $A(:,1)$. Now the colon symbol : is keeping all rows of column 1. This column multiplies $x(1)$ and the other columns multiply $x(2)$ and $x(3)$:

$$\textbf{\textit{Column at a time}} \quad \boldsymbol{b} = A(:,1) * x(1) + A(:,2) * x(2) + A(:,3) * x(3)$$

I think that matrices are stored by columns. Then multiplying a column at a time will be a little faster. So $A * \boldsymbol{x}$ is actually executed by columns.

You can see the same choice in a FORTRAN-type structure, which operates on single entries of A and $\boldsymbol{x}$. This lower level language needs an outer and inner "DO loop". When the outer loop uses the row number I, multiplication is a row at a time. The inner loop $J = 1,3$ goes along each row I.

When the outer loop uses J, multiplication is a column at a time. I will do that in MATLAB (which really needs two more lines "end" and "end" to close "for i" and "for j").

FORTRAN by rows	MATLAB **by columns**
DO 10 $I = 1,3$	for $j = 1:3$
DO 10 $J = 1,3$	for $i = 1:3$
10 $B(I) = B(I) + A(I,J) * X(J)$	$b(i) = b(i) + A(i,j) * x(j)$

Notice that MATLAB is sensitive to upper case versus lower case (capital letters and small letters). If the matrix is A then its entries are not $a(i,j)$: not recognized.

I think you will prefer the higher level $A * \boldsymbol{x}$. FORTRAN won't appear again in this book. *Maple* and *Mathematica* and graphing calculators also operate at the higher level. Multiplication is $A.\ x$ in *Mathematica.* It is **multiply**(A, x); or equally **evalm**$(A\& * x)$; in *Maple*. Those languages allow symbolic entries $a, b, x, \ldots$ and not only real numbers. Like MATLAB's Symbolic Toolbox, they give the symbolic answer.

■ REVIEW OF THE KEY IDEAS ■

1. The basic operations on vectors are multiplication $c\boldsymbol{v}$ and vector addition $\boldsymbol{v} + \boldsymbol{w}$.
2. Together those operations give linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$.
3. Matrix-vector multiplication $A\boldsymbol{x}$ can be computed by dot products, a row at a time. But $A\boldsymbol{x}$ should be understood as a combination of the columns of A.
4. Column picture: $A\boldsymbol{x} = \boldsymbol{b}$ asks for a combination of columns to produce $\boldsymbol{b}$.
5. Row picture: Each equation in $A\boldsymbol{x} = \boldsymbol{b}$ gives a line ($n = 2$) or a plane ($n = 3$) or a "hyperplane" ($n > 3$). They intersect at the solution or solutions, if any.

■ WORKED EXAMPLES ■

2.1 A Describe the column picture of these three equations $Ax = b$. Solve by careful inspection of the columns (instead of elimination):

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \\ 3x + 5y + 6z &= -5 \end{aligned} \qquad \text{which is} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}.$$

Solution The column picture asks for a linear combination that produces $\boldsymbol{b}$ from the three columns of A. In this example $\boldsymbol{b}$ is *minus the second column.* So the solution is $x = 0, y = -1, z = 0$. To show that $(0, -1, 0)$ is the *only* solution we have to know that "A is invertible" and "the columns are independent" and "the determinant isn't zero."

Those words are not yet defined but the test comes from elimination: We need (and for this matrix we find) a full set of three nonzero pivots.

Suppose the right side changes to $\boldsymbol{b} = (4, 4, 8) =$ sum of the first two columns. Then the good combination has $x = 1, y = 1, z = 0$. The solution becomes $\boldsymbol{x} = (1, 1, 0)$.

2.1 B This system has *no solution.* The planes in the row picture don't meet at a point. ***No combination of the three columns produces b. How to show this?***

$$\begin{aligned} x + 3y + 5z &= 4 \\ x + 2y - 3z &= 5 \\ 2x + 5y + 2z &= 8 \end{aligned} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & -3 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} = \boldsymbol{b}$$

(1) Multiply the equations by $1, 1, -1$ and add to get $0 = 1$. *No solution.* Are any two of the planes parallel? What are the equations of planes parallel to $x + 3y + 5z = 4$?

(2) Take the dot product of each column of A (and also $\boldsymbol{b}$) with $\boldsymbol{y} = (1, 1, -1)$. How do those dot products show that the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution?

(3) Find three right side vectors $\boldsymbol{b}^*$ and $\boldsymbol{b}^{**}$ and $\boldsymbol{b}^{***}$ that *do* allow solutions.

Solution

(1) Multiplying the equations by $1, 1, -1$ and adding gives $0 = 1$:

$$\begin{array}{rl} x + 3y + 5z = 4 & \\ x + 2y - 3z = 5 & \\ -[2x + 5y + 2z = 8] & \\ \hline 0x + 0y + 0z = 1 & \textit{No Solution} \end{array}$$

The planes don't meet at a point, even though no two planes are parallel. For a plane parallel to $x + 3y + 5z = 4$, change the "4". The parallel plane $x + 3y + 5z = 0$ goes through the origin $(0, 0, 0)$. And the equation multiplied by any nonzero constant still gives the same plane, as in $2x + 6y + 10z = 8$.

(2) The dot product of each column of A with $\boldsymbol{y} = (1, 1, -1)$ is *zero*. On the right side, $\boldsymbol{y} \cdot \boldsymbol{b} = (1, 1, -1) \cdot (4, 5, 8) = 1$ is *not zero*. So a solution is impossible.

(3) There is a solution when $\boldsymbol{b}$ is a combination of the columns. These three choices of $\boldsymbol{b}$ have solutions $\boldsymbol{x}^* = (1, 0, 0)$ and $\boldsymbol{x}^{**} = (1, 1, 1)$ and $\boldsymbol{x}^{***} = (0, 0, 0)$:

$$\boldsymbol{b}^* = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \text{first column} \qquad \boldsymbol{b}^{**} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \text{sum of columns} \qquad \boldsymbol{b}^{***} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem Set 2.1

Problems 1–8 are about the row and column pictures of $A\boldsymbol{x} = \boldsymbol{b}$.

1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $\boldsymbol{x} = (x, y, z) = (2, 3, 4)$:

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= 3 \\ 0x + 0y + 1z &= 4 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Draw the vectors in the column picture. Two times column 1 plus three times column 2 plus four times column 3 equals the right side $\boldsymbol{b}$.

2 If the equations in Problem 1 are multiplied by $2, 3, 4$ they become $DX = \boldsymbol{B}$:

$$\begin{aligned} 2x + 0y + 0z &= 4 \\ 0x + 3y + 0z &= 9 \\ 0x + 0y + 4z &= 16 \end{aligned} \qquad \text{or} \qquad DX = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = \boldsymbol{B}$$

Why is the row picture the same? Is the solution $\boldsymbol{X}$ the same as $\boldsymbol{x}$? What is changed in the column picture—the columns or the right combination to give $\boldsymbol{B}$?

3 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

5 The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line **L**). Find three solutions on **L**.

6 Move the third plane in Problem 5 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't _____ that line.

7 In Problem 5 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is _____. Find two combinations of the columns that give $\boldsymbol{b} = (2, 3, 5)$. This is only possible for $\boldsymbol{b} = (4, 6, c)$ if $c =$ _____.

8 Normally 4 "planes" in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produce $\boldsymbol{b}$. What combination of $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$ produces $\boldsymbol{b} = (3,3,3,2)$? What 4 equations for x, y, z, t are you solving?

Problems 9–14 are about multiplying matrices and vectors.

9 Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

$$\text{(a)} \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

10 Compute each $A\boldsymbol{x}$ in Problem 9 as a combination of the columns:

$$\text{9(a) becomes} \quad A\boldsymbol{x} = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

How many separate multiplications for $A\boldsymbol{x}$, when the matrix is "3 by 3"?

11 Find the two components of $A\boldsymbol{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

12 Multiply A times $\boldsymbol{x}$ to find three components of $A\boldsymbol{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

13 (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.

(b) The planes from the m equations $A\boldsymbol{x} = \boldsymbol{b}$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.

14 Write $2x+3y+z+5t = 8$ as a matrix A (how many rows?) multiplying the column vector $\boldsymbol{x} = (x, y, z, t)$ to produce $\boldsymbol{b}$. The solutions $\boldsymbol{x}$ fill a plane or "hyperplane" in 4-dimensional space. *The plane is 3-dimensional with no 4D volume.*

Problems 15–22 ask for matrices that act in special ways on vectors.

15 (a) What is the 2 by 2 identity matrix? I times $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.

(b) What is the 2 by 2 exchange matrix? P times $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]$.

16 (a) What 2 by 2 matrix R rotates every vector by 90°? R times $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]$.

(b) What 2 by 2 matrix R^2 rotates every vector by 180°?

17 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

18 What 2 by 2 matrix E subtracts the first component from the second component? What 3 by 3 matrix does the same?

$$E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

19 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z+x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z-x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1}, the two results are (_____) and (_____).

20 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (_____) and (_____).

21 What 2 by 2 matrix R rotates every vector through 45°? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

22 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication $A\boldsymbol{x}$. The matrix A has one row. The solutions to $A\boldsymbol{x} = \boldsymbol{0}$ lie on a _____ perpendicular to the vector _____. The columns of A are only in _____ -dimensional space.

23 In MATLAB notation, write the commands that define this matrix A and the column vectors $\boldsymbol{x}$ and $\boldsymbol{b}$. What command would test whether or not $A\boldsymbol{x} = \boldsymbol{b}$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

24 The MATLAB commands `A = eye(3)` and `v = [3:5]'` produce the 3 by 3 identity matrix and the column vector $(3, 4, 5)$. What are the outputs from `A*v` and `v'*v`? (Computer not needed!) If you ask for `v*A`, what happens?

25 If you multiply the 4 by 4 all-ones matrix A = ones(4) and the column v = ones(4,1), what is A∗v? (Computer not needed.) If you multiply B = eye(4) + ones(4) times w = zeros(4,1) + 2∗ones(4,1), what is B∗w?

Questions 26–28 review the row and column pictures in 2, 3, and 4 dimensions.

26 Draw the row and column pictures for the equations $x - 2y = 0, x + y = 6$.

27 For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.

28 For four linear equations in two unknowns x and y, the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.

29 Start with the vector $\boldsymbol{u}_0 = (1, 0)$. Multiply again and again by the same "Markov matrix" $A = [.8\ .3;\ .2\ .7]$. The next three vectors are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$:

$$\boldsymbol{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = \underline{\qquad} \qquad \boldsymbol{u}_3 = A\boldsymbol{u}_2 = \underline{\qquad}.$$

What property do you notice for all four vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

Challenge Problems

30 Continue Problem 29 from $\boldsymbol{u}_0 = (1, 0)$ to $\boldsymbol{u}_7$, and also from $\boldsymbol{v}_0 = (0, 1)$ to $\boldsymbol{v}_7$. What do you notice about $\boldsymbol{u}_7$ and $\boldsymbol{v}_7$? Here are two MATLAB codes, with while and for. They plot $\boldsymbol{u}_0$ to $\boldsymbol{u}_7$ and $\boldsymbol{v}_0$ to $\boldsymbol{v}_7$. You can use other languages:

```
u = [1 ; 0]; A = [.8 .3 ; .2 .7];
x = u; k = [0 : 7];
while size(x,2) <= 7
  u = A*u; x = [x u];
end
plot(k, x)
```

```
v = [0 ; 1]; A = [.8 .3 ; .2 .7];
x = v; k = [0 : 7];
for j = 1 : 7
  v = A*v; x = [x v];
end
plot(k, x)
```

The $\boldsymbol{u}$'s and $\boldsymbol{v}$'s are approaching a steady state vector $\boldsymbol{s}$. Guess that vector and check that $A\boldsymbol{s} = \boldsymbol{s}$. If you start with $\boldsymbol{s}$, you stay with $\boldsymbol{s}$.

31 Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if a 4 by 4 magic matrix has entries $1, \ldots, 16$?

32 Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are the first two columns of a 3 by 3 matrix A. Which third columns $\boldsymbol{w}$ would make this matrix singular? Describe a typical column picture of $A\boldsymbol{x} = \boldsymbol{b}$ in that singular case, and a typical row picture (for a random $\boldsymbol{b}$).

33 **Multiplying by A is a "linear transformation".** Those important words mean:

If $\boldsymbol{w}$ is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

It is this "***linearity***" $A\boldsymbol{w} = cA\boldsymbol{u} + dA\boldsymbol{v}$ that gives us the name *linear algebra*.

Problem: If $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\boldsymbol{u}$ and $A\boldsymbol{v}$ are the columns of A.

Combine $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. **If $\boldsymbol{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ how is $A\boldsymbol{w}$ connected to $A\boldsymbol{u}$ and $A\boldsymbol{v}$?**

34 Start from the four equations $-x_{i+1} + 2x_i - x_{i-1} = i$ (for $i = 1, 2, 3, 4$ with $x_0 = x_5 = 0$). Write those equations in their matrix form $A\boldsymbol{x} = \boldsymbol{b}$. Can you solve them for x_1, x_2, x_3, x_4?

35 A 9 by 9 ***Sudoku matrix*** S has the numbers $1, \ldots, 9$ in every row and column, and in every 3 by 3 block. For the all-ones vector $\boldsymbol{x} = (1, \ldots, 1)$, what is $S\boldsymbol{x}$?

A better question is: **Which row exchanges will produce another Sudoku matrix?** Also, which exchanges of block rows give another Sudoku matrix?

Section 2.7 will look at all possible permutations (reorderings) of the rows. I can see 6 orders for the first 3 rows, all giving Sudoku matrices. Also 6 permutations of the next 3 rows, and of the last 3 rows. And 6 block permutations of the block rows?

2.2 The Idea of Elimination

This chapter explains a systematic way to solve linear equations. The method is called ***"elimination"***, and you can see it immediately in our 2 by 2 example. Before elimination, x and y appear in both equations. After elimination, the first unknown x has disappeared from the second equation $8y = 8$:

$$\textbf{Before} \quad \begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \textbf{After} \quad \begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned} \qquad \begin{aligned} &(\textit{multiply equation } 1 \textit{ by } 3) \\ &(\textit{subtract to eliminate } 3x) \end{aligned}$$

The new equation $8y = 8$ instantly gives $y = 1$. Substituting $y = 1$ back into the first equation leaves $x - 2 = 1$. Therefore $x = 3$ and the solution $(x, y) = (3, 1)$ is complete.

Elimination produces an ***upper triangular system***—this is the goal. The nonzero coefficients $1, -2, 8$ form a triangle. That system is solved from the bottom upwards—first $y = 1$ and then $x = 3$. This quick process is called ***back substitution***. It is used for upper triangular systems of any size, after elimination gives a triangle.

Important point: The original equations have the same solution $x = 3$ and $y = 1$. Figure 2.5 shows each system as a pair of lines, intersecting at the solution point $(3, 1)$. After elimination, the lines still meet at the same point. Every step worked with correct equations.

How did we get from the first pair of lines to the second pair? We subtracted 3 times the first equation from the second equation. The step that eliminates x from equation 2 is the fundamental operation in this chapter. We use it so often that we look at it closely:

To eliminate x: Subtract a multiple of equation 1 from equation 2.

Three times $x - 2y = 1$ gives $3x - 6y = 3$. When this is subtracted from $3x + 2y = 11$, the right side becomes 8. The main point is that $3x$ cancels $3x$. What remains on the left side is $2y - (-6y)$ or $8y$, and x is eliminated. **The system became triangular.**

Ask yourself how that multiplier $\ell = 3$ was found. The first equation contains $\mathbf{1}x$. ***So the first pivot was* 1** (the coefficient of x). The second equation contains $\mathbf{3}x$, **so the multiplier was 3**. Then subtraction $3x - 3x$ produced the zero and the triangle.

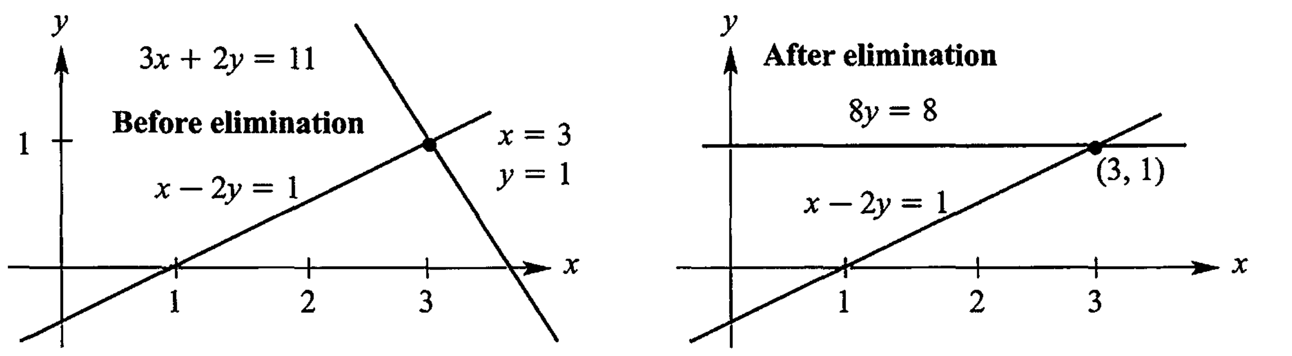


Figure 2.5: Eliminating x makes the second line horizontal. Then $8y = 8$ gives $y = 1$.

You will see the multiplier rule if I change the first equation to $4x - 8y = 4$. (Same straight line but the first pivot becomes 4.) The correct multiplier is now $\ell = \frac{3}{4}$. *To find the multiplier, divide the coefficient " 3" to be eliminated by the pivot " 4"*:

$$\begin{array}{lll} 4x - 8y = 4 & \textbf{Multiply equation 1 by } \tfrac{3}{4} & 4x - 8y = 4 \\ 3x + 2y = 11 & \textbf{Subtract from equation 2} & \phantom{4x - {}} 8y = 8. \end{array}$$

The final system is triangular and the last equation still gives $y = 1$. Back substitution produces $4x - 8 = 4$ and $4x = 12$ and $x = 3$. We changed the numbers but not the lines or the solution. ***Divide by the pivot to find that multiplier*** $\boldsymbol{\ell = \frac{3}{4}}$:

Pivot	=	***first nonzero in the row that does the elimination***
Multiplier	=	***(entry to eliminate) divided by (pivot)*** $= \frac{3}{4}$.

The new second equation starts with the second pivot, which is 8. We would use it to eliminate y from the third equation if there were one. *To solve n equations we want n pivots.* ***The pivots are on the diagonal of the triangle after elimination.***

You could have solved those equations for x and y without reading this book. It is an extremely humble problem, but we stay with it a little longer. Even for a 2 by 2 system, elimination might break down. By understanding the possible breakdown (when we can't find a full set of pivots), you will understand the whole process of elimination.

Breakdown of Elimination

Normally, elimination produces the pivots that take us to the solution. But failure is possible. At some point, the method might ask us to *divide by zero.* We can't do it. The process has to stop. There might be a way to adjust and continue—or failure may be unavoidable.

Example 1 fails with ***no solution to*** $0y = 8$. Example 2 fails with ***too many solutions to*** $0y = 0$. Example 3 succeeds by exchanging the equations.

Example 1 ***Permanent failure with no solution.*** Elimination makes this clear:

$$\begin{array}{rcr} x - 2y = 1 & \text{Subtract 3 times} & x - 2y = 1 \\ 3x - 6y = 11 & \text{eqn. 1 from eqn. 2} & \mathbf{0}y = 8. \end{array}$$

There is *no* solution to $0y = 8$. Normally we divide the right side 8 by the second pivot, but *this system has no second pivot.* ***(Zero is never allowed as a pivot!)*** The row and column pictures in Figure 2.6 show why failure was unavoidable. If there is no solution, elimination will discover that fact by reaching an equation like $0y = 8$.

The row picture of failure shows parallel lines—which never meet. A solution must lie on both lines. With no meeting point, the equations have no solution.

The column picture shows the two columns $(1, 3)$ and $(-2, -6)$ in the same direction. *All combinations of the columns lie along a line.* But the column from the right side is in a different direction $(1, 11)$. No combination of the columns can produce this right side—therefore no solution.

When we change the right side to $(1, 3)$, failure shows as a whole line of solution points. Instead of no solution, next comes Example 2 with infinitely many.

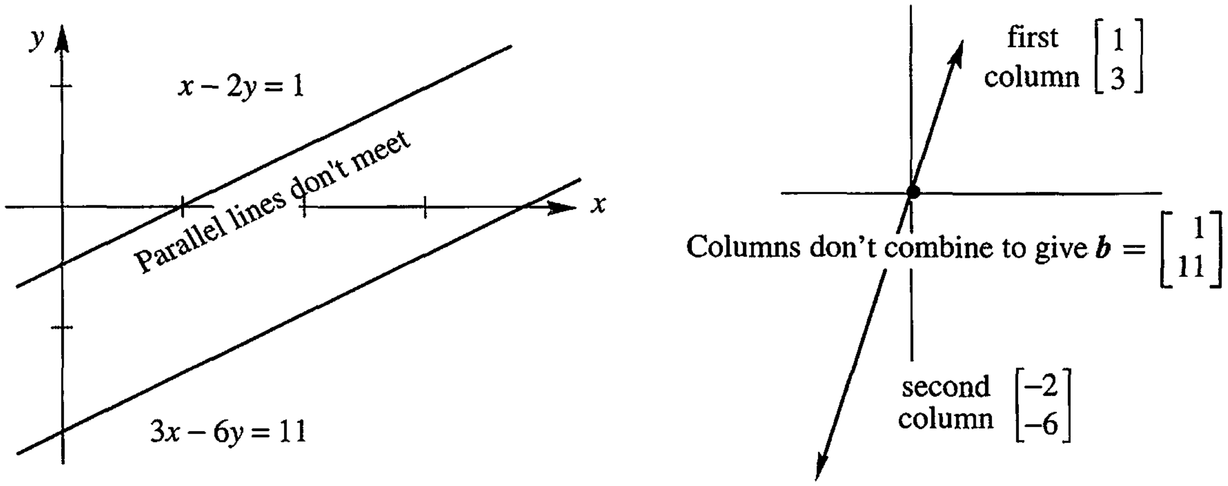


Figure 2.6: Row picture and column picture for Example 1: ***no solution.***

Example 2 ***Failure with infinitely many solutions. Change*** $\boldsymbol{b} = (1, 11)$ ***to*** $(1, 3)$.

$x - 2y = 1$	Subtract 3 times	$x - 2y = 1$	Still only
$3x - 6y = 3$	eqn. 1 from eqn. 2	$\mathbf{0}y = \mathbf{0}.$	**one pivot.**

Every y satisfies $0y = 0$. There is really only one equation $x - 2y = 1$. The unknown y is *"free"*. After y is freely chosen, x is determined as $x = 1 + 2y$.

In the row picture, the parallel lines have become the same line. Every point on that line satisfies both equations. We have a whole line of solutions in Figure 2.7.

In the column picture, $\boldsymbol{b} = (1, 3)$ is now the same as column 1. So we can choose $x = 1$ and $y = 0$. We can also choose $x = 0$ and $y = -\frac{1}{2}$; column 2 times $-\frac{1}{2}$ equals $\boldsymbol{b}$. Every (x, y) that solves the row problem also solves the column problem.

Failure For n equations we do not get n pivots

Elimination leads to an equation $\mathbf{0} \neq \mathbf{0}$ (no solution) or $\mathbf{0} = \mathbf{0}$ (many solutions)

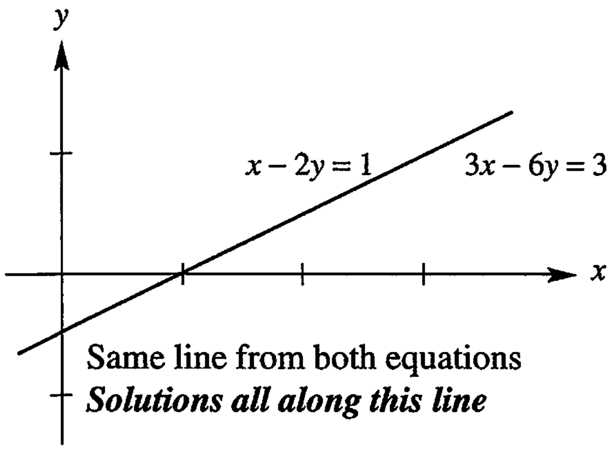
Success comes with n **pivots. But we may have to exchange the** n **equations.**

Elimination can go wrong in a third way—but this time it can be fixed. *Suppose the first pivot position contains zero.* We refuse to allow zero as a pivot. When the first equation has no term involving x, we can exchange it with an equation below:

Example 3 ***Temporary failure (zero in pivot). A row exchange produces two pivots:***

Permutation	$0x + 2y = 4$	Exchange the	$3x - 2y = 5$
	$3x - 2y = 5$	two equations	$2y = 4.$

The new system is already triangular. This small example is ready for back substitution. The last equation gives $y = 2$, and then the first equation gives $x = 3$. The row picture is normal (two intersecting lines). The column picture is also normal (column vectors not in the same direction). The pivots 3 and 2 are normal—but a ***row exchange*** was required.



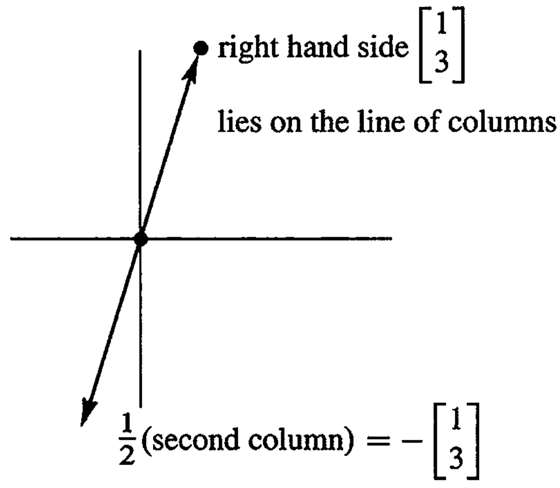


Figure 2.7: Row and column pictures for Example 2: *infinitely many solutions.*

Examples 1 and 2 are ***singular***—there is no second pivot. Example 3 is ***nonsingular***—there is a full set of pivots and exactly one solution. Singular equations have no solution or infinitely many solutions. Pivots must be nonzero because we have to divide by them.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems. Three by three is enough to see the pattern. For now the matrices are square—an equal number of rows and columns. Here is a 3 by 3 system, specially constructed so that all steps lead to whole numbers and not fractions:

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \tag{1}$$

What are the steps? The first pivot is the boldface **2** (upper left). Below that pivot we want to eliminate the 4. *The first multiplier is the ratio* $4/2 = 2$. Multiply the pivot equation by $\ell_{21} = 2$ and subtract. Subtraction removes the $4x$ from the second equation:

Step 1 Subtract 2 times equation 1 from equation 2. This leaves $y + z = 4$.

We also eliminate $-2x$ from equation 3—still using the first pivot. The quick way is to add equation 1 to equation 3. Then $2x$ cancels $-2x$. We do exactly that, but the rule in this book is to *subtract rather than add.* The systematic pattern has multiplier $\ell_{31} = -2/2 = -1$. Subtracting -1 times an equation is the same as adding:

Step 2 Subtract -1 times equation 1 from equation 3. This leaves $y + 5z = 12$.

The two new equations involve only y and z. The second pivot (in boldface) is 1:

$$\textbf{x is eliminated} \qquad \begin{aligned} \mathbf{1}y + 1z &= 4 \\ 1y + 5z &= 12 \end{aligned}$$

We have reached a 2 *by* 2 *system.* The final step eliminates y to make it 1 by 1:

Step 3 Subtract equation 2_{new} from 3_{new}. The multiplier is $1/1 = 1$. Then $4z = 8$.

The original $A\boldsymbol{x} = \boldsymbol{b}$ has been converted into an upper triangular $U\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{array}{r} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{array} \quad \begin{array}{c} A\boldsymbol{x} = \boldsymbol{b} \\ \text{has become} \\ U\boldsymbol{x} = \boldsymbol{c} \end{array} \quad \begin{array}{r} 2x + 4y - 2z = 2 \\ 1y + 1z = 4 \\ 4z = 8. \end{array} \tag{2}$$

The goal is achieved—forward elimination is complete from A to U. ***Notice the pivots* 2, 1, 4 *along the diagonal of* U.** The pivots 1 and 4 were hidden in the original system. Elimination brought them out. $U\boldsymbol{x} = \boldsymbol{c}$ is ready for ***back substitution***, which is quick:

$$(4z = 8 \quad \text{gives} \quad z = 2) \qquad (y + z = 4 \quad \text{gives} \quad y = 2) \qquad (\text{equation 1 gives} \quad x = -1)$$

The solution is $(x, y, z) = (-1, 2, 2)$. The row picture has three planes from three equations. All the planes go through this solution. The original planes are sloping, but the last plane $4z = 8$ after elimination is horizontal.

The column picture shows a combination $A\boldsymbol{x}$ of column vectors producing the right side $\boldsymbol{b}$. The coefficients in that combination are $-1, 2, 2$ (the solution):

$$A\boldsymbol{x} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \quad \text{equals} \quad \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}. \tag{3}$$

The numbers x, y, z multiply columns 1, 2, 3 in $A\boldsymbol{x} = \boldsymbol{b}$ and also in the triangular $U\boldsymbol{x} = \boldsymbol{c}$.

For a 4 by 4 problem, or an n by n problem, elimination proceeds the same way. Here is the whole idea, column by column from A to U, when elimination succeeds.

Column 1. ***Use the first equation to create zeros below the first pivot.***

Column 2. ***Use the new equation 2 to create zeros below the second pivot.***

Columns 3 to n. ***Keep going to find all n pivots and the triangular U.***

$$\text{After column 2 we have} \quad \begin{bmatrix} \boldsymbol{x} & x & x & x \\ 0 & \boldsymbol{x} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}. \quad \text{We want} \quad \begin{bmatrix} \boldsymbol{x} & x & x & x \\ & \boldsymbol{x} & x & x \\ & & \boldsymbol{x} & x \\ & & & \boldsymbol{x} \end{bmatrix}. \tag{4}$$

The result of forward elimination is an upper triangular system. It is nonsingular if there is a full set of n pivots (never zero!). *Question*: Which x on the left could be changed to boldface $\boldsymbol{x}$ because the pivot is known? Here is a final example to show the original $A\boldsymbol{x} = \boldsymbol{b}$, the triangular system $U\boldsymbol{x} = \boldsymbol{c}$, and the solution (x, y, z) from back substitution:

$$\begin{array}{rl} x + y + z = 6 & \\ x + 2y + 2z = 9 & \textbf{Forward} \\ x + 2y + 3z = 10 & \textbf{Forward} \end{array} \qquad \begin{array}{r} x + y + z = 6 \\ y + z = 3 \\ z = 1 \end{array} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{array}{l} \textbf{Back} \\ \textbf{Back} \\ \end{array}$$

All multipliers are 1. All pivots are 1. All planes meet at the solution $(3, 2, 1)$. The columns of A combine with 3, 2, 1 to give $\boldsymbol{b} = (6, 9, 10)$. The triangle shows $U\boldsymbol{x} = \boldsymbol{c} = (6, 3, 1)$.

■ REVIEW OF THE KEY IDEAS ■

1. A linear system ($A\boldsymbol{x} = \boldsymbol{b}$) becomes upper triangular ($U\boldsymbol{x} = \boldsymbol{c}$) after elimination.
2. We subtract ℓ_{ij} times equation j from equation i, to make the (i, j) entry zero.
3. The multiplier is $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. Pivots can not be zero!
4. A zero in the pivot position can be repaired if there is a nonzero below it.
5. The upper triangular system is solved by back substitution (starting at the bottom).
6. When breakdown is permanent, the system has no solution or infinitely many.

■ WORKED EXAMPLES ■

2.2 A When elimination is applied to this matrix A, what are the first and second pivots? What is the multiplier ℓ_{21} in the first step (ℓ_{21} times row 1 is *subtracted* from row 2)?

A has a ***first difference*** in row 1 and a ***second difference*** $-1, 2, -1$ in row 2.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

What entry in the 2, 2 position (instead of 2) would force an exchange of rows 2 and 3? Why is the lower left multiplier $\ell_{31} = 0$, subtracting zero times row 1 from row 3?
If you change the corner entry from $a_{33} = 2$ ***to*** $a_{33} = 1$***, why does elimination fail***?

Solution The first pivot is 1. The multiplier ℓ_{21} is $-1/1 = -1$. When -1 times row 1 is subtracted (so row 1 is added to row 2), the second pivot is revealed as 1.

If we reduce the middle entry "2" to "1", that would force a row exchange. (Zero will appear in the second pivot position.) The multiplier ℓ_{31} is zero because $a_{31} = 0$. A zero at the start of a row needs no elimination. This A is a "*band matrix*".

The last pivot is 1. So if the original corner entry a_{33} is reduced by 1 (to $a_{33} = 1$), elimination would produce 0. **No third pivot, elimination fails.**

2.2 B Suppose A is already a ***triangular matrix*** (upper triangular or lower triangular). *Where do you see its pivots*? When does $A\boldsymbol{x} = \boldsymbol{b}$ have exactly one solution for every $\boldsymbol{b}$?

Solution The pivots of a triangular matrix are already set along the main diagonal. *Elimination succeeds when all those numbers are nonzero.* Use ***back*** substitution when A is upper triangular, go ***forward*** when A is lower triangular.

2.2 C Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in the last equation.

$$\begin{array}{rll} \textbf{Success} & x+y+z=7 & \quad x+y+z=7 \\ \textbf{then} & x+y-z=5 & \quad x+y-z=5 \\ \textbf{Failure} & x-y+z=3 & \quad -x-y+z=3 \end{array}$$

Solution For the first system, subtract equation 1 from equations 2 and 3 (the multipliers are $\ell_{21}=1$ and $\ell_{31}=1$). The 2, 2 entry becomes zero, so exchange equations:

$$\textbf{Success} \quad \begin{aligned} x+y+z &= 7 \\ \mathbf{0}y-2z &= -2 \\ -2y+0z &= -4 \end{aligned} \quad \text{exchanges into} \quad \begin{aligned} x+y+z &= 7 \\ -\mathbf{2}y+0z &= -4 \\ -2z &= -2 \end{aligned}$$

Then back substitution gives $z=1$ and $y=2$ and $x=4$. The pivots are $1,-2,-2$.

For the second system, subtract equation 1 from equation 2 as before. Add equation 1 to equation 3. This leaves zero in the 2, 2 entry *and also below*:

$$\textbf{Failure} \quad \begin{aligned} x+y+z &= 7 \\ \mathbf{0}y-2z &= -2 \\ \mathbf{0}y+2z &= 10 \end{aligned} \quad \begin{array}{l} \text{There is } \textit{no pivot} \text{ in column 2 (it was } - \text{ column 1)} \\ \text{A further elimination step gives } 0z = 8 \\ \text{The three planes don't meet} \end{array}$$

Plane 1 meets plane 2 in a line. Plane 1 meets plane 3 in a parallel line. *No solution.*

If we change the "3" in the original third equation to "-5" then elimination would lead to $0=0$. There are infinitely many solutions! *The three planes now meet along a whole line.*

Changing 3 to -5 moved the third plane to meet the other two. The second equation gives $z=1$. Then the first equation leaves $x+y=6$. **No pivot in column 2 makes y free** (it can have any value). Then $x=6-y$.

Problem Set 2.2

Problems 1–10 are about elimination on 2 by 2 systems.

1 What multiple ℓ_{21} of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x+3y &= 1 \\ 10x+9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

2 Solve the triangular system of Problem 1 by back substitution, y before x. Verify that x times $(2,10)$ plus y times $(3,9)$ equals $(1,11)$. If the right side changes to $(4,44)$, what is the new solution?

3 What multiple of equation 1 should be *subtracted* from equation 2?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution?

4 What multiple ℓ of equation 1 should be subtracted from equation 2 to remove c?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$: singular.

5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

Singular system $$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

7 For which numbers a does elimination break down (1) permanently (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

8 For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

9 What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture for $\boldsymbol{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

10 In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ _____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ _____.

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

11 (Recommended) A system of linear equations can't have exactly two solutions. *Why*?

(a) If (x, y, z) and (X, Y, Z) are two solutions, what is another solution?

(b) If 25 planes meet at two points, where else do they meet?

12 Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back substitution for z, y, x.

13 Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract _____ times row _____ from row _____.

14 Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

15 Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \phantom{{}-z} &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

16 (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

17 If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{array}{llll} \textbf{Equal} & 2x - y + z = 0 & 2x + 2y + z = 0 & \textbf{Equal} \\ \textbf{rows} & 2x - y + z = 0 & 4x + 4y + z = 0 & \textbf{columns} \\ & 4x + y + z = 2 & 6x + 6y + z = 2. & \end{array}$$

18 Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0, 0, 0)$?

19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

20 Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a ____ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

21 Find the pivots and the solution for both systems ($A\boldsymbol{x} = \boldsymbol{b}$ and $K\boldsymbol{x} = \boldsymbol{b}$):

$$\begin{aligned} 2x + y \qquad\qquad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned} \qquad\qquad \begin{aligned} 2x - y \qquad\qquad &= 0 \\ -x + 2y - z \qquad &= 0 \\ - y + 2z - t &= 0 \\ - z + 2t &= 5. \end{aligned}$$

22 If you extend Problem 21 following the $1, 2, 1$ pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot? K is my favorite matrix.

23 If elimination leads to $x + y = 1$ and $2y = 3$, find three possible original problems.

24 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

25 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

26 Look for a matrix that has row sums 4 and 8, and column sums 2 and s:

$$\text{Matrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{matrix}$$

The four equations are solvable only if $s =$ ____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} = (a, b, c, d)$ and make A triangular by elimination.

27 Elimination in the usual order gives what matrix U and what solution to this "lower triangular" system? We are really solving by *forward substitution*:

$$\begin{aligned} 3x \qquad\qquad &= 3 \\ 6x + 2y \qquad &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

28 Create a MATLAB command A(2, :) = ... for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.

Challenge Problems

29 Find experimentally the average 1st and 2nd and 3rd pivot sizes from MATLAB 's $[L, U] = \mathbf{lu}(\mathbf{rand}(3))$. The average size $\mathbf{abs}(U(1,1))$ is above $\frac{1}{2}$ because **lu** picks the largest available pivot in column 1. Here $A = \mathbf{rand}(3)$ has random entries between 0 and 1.

30 If the last corner entry is $A(5,5) = 11$ and the last pivot of A is $U(5,5) = 4$, what different entry $A(5,5)$ would have made A singular?

31 Suppose elimination takes A to U without row exchanges. Then row j of U is a combination of which rows of A? If $A\boldsymbol{x} = \mathbf{0}$, is $U\boldsymbol{x} = \mathbf{0}$? If $A\boldsymbol{x} = \boldsymbol{b}$, is $U\boldsymbol{x} = \boldsymbol{b}$? If A starts out lower triangular, what is the upper triangular U?

32 Start with 100 equations $A\boldsymbol{x} = \mathbf{0}$ for 100 unknowns $\boldsymbol{x} = (x_1, \ldots, x_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is _____.

(b) Singular systems $A\boldsymbol{x} = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is _____.

(c) Invent a 100 by 100 singular matrix with no zero entries.

(d) For your matrix, describe in words the row picture and the column picture of $A\boldsymbol{x} = \mathbf{0}$. Not necessary to draw 100-dimensional space.

2.3 Elimination Using Matrices

We now combine two ideas—elimination and matrices. The goal is to express all the steps of elimination (and the final result) in the clearest possible way. In a 3 by 3 example, elimination could be described in words. For larger systems, a long list of steps would be hopeless. You will see how to subtract a multiple of row j from row i—*using a matrix* E.

The 3 by 3 example in the previous section has the beautifully short form $Ax = b$:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \quad \text{is the same as} \quad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \tag{1}$$

The nine numbers on the left go into the matrix A. That matrix not only sits beside x, it *multiplies* x. The rule for "A times x" is exactly chosen to yield the three equations.

Review of A times x. A matrix times a vector gives a vector. The matrix is square when the number of equations (three) matches the number of unknowns (three). Our matrix is 3 by 3. A general square matrix is n by n. Then the vector x is in n-dimensional space.

$$\textit{The unknown in } \mathbf{R}^3 \textit{ is} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \textit{and the solution is} \quad x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Key point: $Ax = b$ represents the row form and also the column form of the equations.

$$\textbf{Column form} \qquad Ax = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = b.$$

This rule for Ax is used so often that we express it once more for emphasis.

Ax is a combination of the columns of A. Components of x multiply those columns:

$$Ax = x_1 \text{ times (column 1)} + \cdots + x_n \text{ times (column } n\text{)}.$$

When we compute the components of Ax, we use the row form of matrix multiplication. The ith component is a dot product with row i of A, which is $[a_{i1} \;\; a_{i2} \;\; \ldots \;\; a_{in}]$. The short formula for that dot product with x uses "sigma notation".

Components of Ax are dot products with rows of A.

The ith component of Ax is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$. This is $\displaystyle\sum_{j=1}^{n} a_{ij}x_j$.

The sigma symbol $\sum$ is an instruction to add.[1] Start with $j = 1$ and stop with $j = n$. Start the sum with $a_{i1}x_1$ and stop with $a_{in}x_n$. That produces (row i) $\cdot\, x$.

[1] Einstein shortened this even more by omitting the $\sum$. The repeated j in $a_{ij}x_j$ automatically meant addition. He also wrote the sum as $a_i^j x_j$. Not being Einstein, we include the $\sum$.

One point to repeat about matrix notation: The entry in row 1, column 1 (the top left corner) is a_{11}. The entry in row 1, column 3 is a_{13}. The entry in row 3, column 1 is a_{31}. (Row number comes before column number.) The word "entry" for a matrix corresponds to "component" for a vector. General rule: $a_{ij} = A(i,j)$ ***is in row*** i***, column*** j.

Example 1 This matrix has $a_{ij} = 2i + j$. Then $a_{11} = 3$. Also $a_{12} = 4$ and $a_{21} = 5$. Here is Ax with numbers and letters:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + 4\cdot 1 \\ 5\cdot 2 + 6\cdot 1 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

The first component of Ax is $6 + 4 = 10$. ***A row times a column gives a dot product.***

The Matrix Form of One Elimination Step

$Ax = b$ is a convenient form for the original equation. What about the elimination steps? The first step in this example subtracts 2 times the first equation from the second equation. On the right side, 2 times the first component of b is subtracted from the second component:

$$\textbf{First step} \qquad b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \quad \text{changes to} \quad b_{\text{new}} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

We want to do that subtraction with a matrix! The same result $b_{\text{new}} = Eb$ is achieved when we multiply an "elimination matrix" E times b. It subtracts $2b_1$ from b_2:

$$\textit{The elimination matrix is} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplication by E subtracts 2 times row 1 from row 2. Rows 1 and 3 stay the same:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

The first and third rows of E are rows from the identity matrix I. The new second component is the number 4 that appeared after the elimination step. This is $b_2 - 2b_1$.

It is easy to describe the "elementary matrices" or "elimination matrices" like this E. Start with the identity matrix I. *Change one of its zeros to the multiplier* $-\ell$:

The ***identity matrix*** has 1's on the diagonal and otherwise 0's. Then $Ib = b$ for all b. The ***elementary matrix or elimination matrix*** E_{ij} that subtracts a multiple ℓ of row j from row i has the extra nonzero entry $-\ell$ in the i, j position (still diagonal 1's).

Example 2 The matrix E_{31} has $-\ell$ in the 3, 1 position:

$$\textbf{Identity} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{Elimination} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix}.$$

When you multiply I times $\boldsymbol{b}$, you get $\boldsymbol{b}$. But E_{31} subtracts ℓ times the first component from the third component. With $\ell = 4$ this example gives $9 - 4 = 5$:

$$I\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \quad \text{and} \quad E\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 3 \\ \mathbf{9} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \mathbf{5} \end{bmatrix}.$$

What about the left side of $A\boldsymbol{x} = \boldsymbol{b}$? Both sides are multiplied by E_{31}. ***The purpose of E_{31} is to produce a zero in the (3, 1) position of the matrix.***

The notation fits this purpose. Start with A. Apply E's to produce zeros below the pivots (the first E is E_{21}). End with a triangular U. We now look in detail at those steps.

First a small point. The vector $\boldsymbol{x}$ stays the same. The solution is not changed by elimination. (That may be more than a small point.) It is the coefficient matrix that is changed. When we start with $A\boldsymbol{x} = \boldsymbol{b}$ and multiply by E, the result is $EA\boldsymbol{x} = E\boldsymbol{b}$. The new matrix EA is the result of *multiplying E times A*.

Confession The *elimination matrices E_{ij}* are great examples, but you won't see them later. They show how a matrix acts on rows. By taking several elimination steps, we will see how to *multiply matrices* (and the order of the E's becomes important). ***Products and inverses*** are especially clear for E's. It is those two ideas that the book will now use.

Matrix Multiplication

The big question is: ***How do we multiply two matrices?*** When the first matrix is E, we already know what to expect for EA. This particular E subtracts 2 times row 1 from row 2 of this matrix A and any matrix. The multiplier is $\ell = 2$:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ -2 & -3 & 7 \end{bmatrix} \quad (\textit{with the zero}). \qquad (2)$$

This step does not change rows 1 and 3 of A. Those rows are unchanged in EA—only row 2 is different. *Twice the first row has been subtracted from the second row.* Matrix multiplication agrees with elimination—and the new system of equations is $EA\boldsymbol{x} = E\boldsymbol{b}$.

$EA\boldsymbol{x}$ is simple but it involves a subtle idea. Start with $A\boldsymbol{x} = \boldsymbol{b}$. Multiplying both sides by E gives $E(A\boldsymbol{x}) = E\boldsymbol{b}$. With matrix multiplication, this is also $(EA)\boldsymbol{x} = E\boldsymbol{b}$. **The first was E times $A\boldsymbol{x}$, the second is EA times $\boldsymbol{x}$. They are the same.** Parentheses are not needed. We just write $EA\boldsymbol{x}$.

That rule extends to a matrix C with several column vectors like $C = [\boldsymbol{c}_1\ \boldsymbol{c}_2\ \boldsymbol{c}_3]$. When multiplying EAC, you can do AC first or EA first. This is the point of an "associative law" like $3 \times (4 \times 5) = (3 \times 4) \times 5$. Multiply 3 times 20, or multiply 12 times 5. Both answers are 60. That law seems so clear that it is hard to imagine it could be false.

The "commutative law" $3 \times 4 = 4 \times 3$ looks even more obvious. But EA is usually different from AE. When E multiplies on the right, it acts on the ***columns*** of A.

Associative law is true	$A(BC) = (AB)C$
Commutative law is false	Often $AB \neq BA$

There is another requirement on matrix multiplication. Suppose B has only one column (this column is $\boldsymbol{b}$). The matrix-matrix law for EB should agree with the matrix-vector law for $E\boldsymbol{b}$. Even more, we should be able to *multiply matrices EB a column at a time*:

If B has several columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$, then the columns of EB are $E\boldsymbol{b}_1, E\boldsymbol{b}_2, E\boldsymbol{b}_3$.

Matrix multiplication $\quad AB = A\,[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3] = [A\boldsymbol{b}_1\ A\boldsymbol{b}_2\ A\boldsymbol{b}_3]. \qquad (3)$

This holds true for the matrix multiplication in (2). If you multiply column 3 of A by E, you correctly get column 3 of EA:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \qquad E(\text{column } j \text{ of } A) = \text{column } j \text{ of } EA.$$

This requirement deals with columns, while elimination is applied to rows. **The next section describes each entry of every product AB.** The beauty of matrix multiplication is that all three approaches (*rows, columns, whole matrices*) come out right.

The Matrix P_{ij} for a Row Exchange

To subtract row j from row i we use E_{ij}. To exchange or "permute" those rows we use another matrix P_{ij} (a **permutation matrix**). A row exchange is needed when zero is in the pivot position. Lower down, that pivot column may contain a nonzero. By exchanging the two rows, we have a pivot and elimination goes forward.

What matrix P_{23} exchanges row 2 with row 3? We can find it by exchanging rows of the identity matrix I:

$$\textbf{Permutation matrix} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is a ***row exchange matrix***. Multiplying by P_{23} exchanges components 2 and 3 of any column vector. Therefore it also exchanges rows 2 and 3 of any matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{3} \\ \mathbf{5} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{5} \\ \mathbf{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}.$$

On the right, P_{23} is doing what it was created for. With zero in the second pivot position and "6" below it, the exchange puts 6 into the pivot.

Matrices *act*. They don't just sit there. We will soon meet other permutation matrices, which can change the order of several rows. Rows $1, 2, 3$ can be moved to $3, 1, 2$. Our P_{23} is one particular permutation matrix—it exchanges rows 2 and 3.

Row Exchange Matrix P_{ij} is the identity matrix with rows i and j reversed. When this **"permutation matrix"** P_{ij} multiplies a matrix, it exchanges rows i and j.

$$\textbf{\textit{To exchange equations 1 and 3 multiply by}} \quad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Usually row exchanges are not required. The odds are good that elimination uses only the E_{ij}. But the P_{ij} are ready if needed, to move a pivot up to the diagonal.

The Augmented Matrix

This book eventually goes far beyond elimination. Matrices have all kinds of practical applications, in which they are multiplied. Our best starting point was a square E times a square A, because we met this in elimination—and we know what answer to expect for EA. The next step is to allow a *rectangular matrix*. It still comes from our original equations, but now it includes the right side $\boldsymbol{b}$.

Key idea: Elimination does the same row operations to A and to $\boldsymbol{b}$. ***We can include b as an extra column and follow it through elimination.*** The matrix A is enlarged or "augmented" by the extra column $\boldsymbol{b}$:

$$\textbf{\textit{Augmented matrix}} \quad \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 4 & 9 & -3 & \mathbf{8} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix}.$$

Elimination acts on whole rows of this matrix. The left side and right side are both multiplied by E, to subtract 2 times equation 1 from equation 2. With $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ those steps happen together:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 4 & 9 & -3 & \mathbf{8} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 0 & 1 & 1 & \mathbf{4} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix}.$$

The new second row contains $0, 1, 1, 4$. The new second equation is $x_2 + x_3 = 4$. Matrix multiplication works by rows and at the same time by columns:

ROWS Each row of E acts on $[\, A \quad \boldsymbol{b}\,]$ to give a row of $[\, EA \quad E\boldsymbol{b}\,]$.

COLUMNS E acts on each column of $[\, A \quad \boldsymbol{b}\,]$ to give a column of $[\, EA \quad E\boldsymbol{b}\,]$.

Notice again that word "acts." This is essential. Matrices do something! The matrix A acts on $\boldsymbol{x}$ to produce $\boldsymbol{b}$. The matrix E operates on A to give EA. The whole process of elimination is a sequence of row operations, alias matrix multiplications. A goes to $E_{21}A$ which goes to $E_{31}E_{21}A$. Finally $E_{32}E_{31}E_{21}A$ is a triangular matrix.

The right side is included in the augmented matrix. The end result is a triangular system of equations. We stop for exercises on multiplication by E, before writing down the rules for all matrix multiplications (including block multiplication).

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = x_1$ times column $1 + \cdots + x_n$ times column n. And $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j$.
2. Identity matrix $= I$, elimination matrix $= E_{ij}$ using ℓ_{ij}, exchange matrix $= P_{ij}$.
3. Multiplying $Ax = b$ by E_{21} subtracts a multiple ℓ_{21} of equation 1 from equation 2. The number $-\ell_{21}$ is the $(2, 1)$ entry of the elimination matrix E_{21}.
4. For the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$, that elimination step gives $\begin{bmatrix} E_{21}A & E_{21}b \end{bmatrix}$.
5. When A multiplies any matrix B, it multiplies each column of B separately.

■ WORKED EXAMPLES ■

2.3 A What 3 by 3 matrix E_{21} subtracts 4 times row 1 from row 2? What matrix P_{32} exchanges row 2 and row 3? If you multiply A on the *right* instead of the left, describe the results AE_{21} and AP_{32}.

Solution By doing those operations on the identity matrix I, we find

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying by E_{21} on the right side will subtract 4 times **column 2** from **column 1**. Multiplying by P_{32} on the right will exchange **columns** 2 and 3.

2.3 B Write down the augmented matrix $[A \;\; b]$ with an extra column:

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

Apply E_{21} and then P_{32} to reach a triangular system. Solve by back substitution. What combined matrix $P_{32}\,E_{21}$ will do both steps at once?

Solution E_{21} removes the 4 in column 1. But zero appears in column 2:

$$[A \quad b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ \mathbf{4} & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad E_{21}[A \quad b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ \mathbf{0} & \mathbf{0} & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

Now P_{32} exchanges rows 2 and 3. Back substitution produces z then y and x.

$$P_{32}\,E_{21}[A \quad b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

For the matrix $P_{32}\,E_{21}$ that does both steps at once, *apply* P_{32} *to* E_{21}.

One matrix
Both steps

$$P_{32}\,E_{21} = \text{exchange the rows of } E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}.$$

2.3 C Multiply these matrices in two ways. First, rows of A times columns of B. Second, ***columns of A times rows of B***. That unusual way produces two matrices that add to AB. How many separate ordinary multiplications are needed?

Both ways

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{16} \\ \mathbf{7} & \mathbf{9} \\ \mathbf{4} & \mathbf{8} \end{bmatrix}$$

Solution Rows of A times columns of B are dot products of vectors:

$$(\text{row } 1) \cdot (\text{column } 1) = [3 \quad 4] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{10} \qquad \text{is the } (1,1) \text{ entry of } AB$$

$$(\text{row } 2) \cdot (\text{column } 1) = [1 \quad 5] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{7} \qquad \text{is the } (2,1) \text{ entry of } AB$$

We need 6 dot products, 2 multiplications each, 12 in all $(3 \cdot 2 \cdot 2)$. The same AB comes from *columns of A times rows of B*. A column times a row is a matrix.

$$AB = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} [2 \quad 4] + \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} [1 \quad 1] = \begin{bmatrix} \mathbf{6} & \mathbf{12} \\ \mathbf{2} & \mathbf{4} \\ \mathbf{4} & \mathbf{8} \end{bmatrix} + \begin{bmatrix} \mathbf{4} & \mathbf{4} \\ \mathbf{5} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Problem Set 2.3

Problems 1–15 are about elimination matrices.

1 Write down the 3 by 3 matrices that produce these elimination steps:

(a) E_{21} subtracts 5 times row 1 from row 2.

(b) E_{32} subtracts -7 times row 2 from row 3.

(c) P exchanges rows 1 and 2, then rows 2 and 3.

2 In Problem 1, applying E_{21} and then E_{32} to $\boldsymbol{b} = (1, 0, 0)$ gives $E_{32}E_{21}\boldsymbol{b} =$ ____. Applying E_{32} before E_{21} gives $E_{21}E_{32}\boldsymbol{b} =$ ____. When E_{32} comes first, row ____ feels no effect from row ____.

3 Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

4 Include $\boldsymbol{b} = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[\,A \;\; \boldsymbol{b}\,]$. Carry out the elimination steps on this augmented matrix to solve $A\boldsymbol{x} = \boldsymbol{b}$.

5 Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ____. If you change a_{33} to ____, there is no third pivot.

6 If every column of A is a multiple of $(1, 1, 1)$, then $A\boldsymbol{x}$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

7 Suppose E subtracts 7 times row 1 from row 3.

(a) To *invert* that step you should ____ 7 times row ____ to row ____.

(b) What "inverse matrix" E^{-1} takes that reverse step (so $E^{-1}E = I$)?

(c) If the reverse step is applied first (and then E) show that $EE^{-1} = I$.

8 The ***determinant*** of $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is det $M = ad - bc$. Subtract ℓ times row 1 from row 2 to produce a new M^*. Show that $\det M^* = \det M$ for every ℓ. When $\ell = c/a$, *the product of pivots equals the determinant*: $(a)(d - \ell b)$ equals $ad - bc$.

9 (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?

(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

10 (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

(b) What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1?

(c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

11 Create a matrix that has $a_{11} = a_{22} = a_{33} = 1$ but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

12 Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

13 Explain these facts. If the third column of B is all zero, the third column of EB is all zero (for any E). If the third *row* of B is all zero, the third row of EB might *not* be zero.

14 This 4 by 4 matrix will need elimination matrices E_{21} and E_{32} and E_{43}. What are those matrices?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

15 Write down the 3 by 3 matrix that has $a_{ij} = 2i - 3j$. This matrix has $a_{32} = 0$, but elimination still needs E_{32} to produce a zero in the 3, 2 position. Which previous step destroys the original zero and what is E_{32}?

Problems 16–23 are about creating and multiplying matrices.

16 Write these ancient problems in a 2 by 2 matrix form $A\boldsymbol{x} = \boldsymbol{b}$ and solve them:

(a) X is twice as old as Y and their ages add to 33.

(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

17 The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

18 Multiply these matrices in the orders EF and FE:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Also compute $E^2 = EE$ and $F^3 = FFF$. You can guess F^{100}.

19 Multiply these row exchange matrices in the orders PQ and QP and P^2:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find another non-diagonal matrix whose square is $M^2 = I$.

20 (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times ____.

(b) Suppose all rows of B are $[\,1 \;\; 2 \;\; 4\,]$. Show by example that all rows of EB are *not* $[\,1 \;\; 2 \;\; 4\,]$. It is true that those rows are ____.

21 If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE?

22 The entries of A and $\boldsymbol{x}$ are a_{ij} and x_j. So the first component of $A\boldsymbol{x}$ is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for

(a) the third component of $A\boldsymbol{x}$

(b) the $(2, 1)$ entry of $E_{21}A$

(c) the $(2, 1)$ entry of $E_{21}(E_{21}A)$

(d) the first component of $E_{21}A\boldsymbol{x}$.

23 The elimination matrix $E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ subtracts 2 times row 1 of A from row 2 of A. The result is EA. What is the effect of $E(EA)$? In the opposite order AE, we are subtracting 2 times ____ of A from ____. (Do examples.)

Problems 24–27 include the column $\boldsymbol{b}$ in the augmented matrix $[\,A \;\; \boldsymbol{b}\,]$.

24 Apply elimination to the 2 by 3 augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. What is the triangular system $U\boldsymbol{x} = \boldsymbol{c}$? What is the solution $\boldsymbol{x}$?

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

25 Apply elimination to the 3 by 4 augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. How do you know this system has no solution? Change the last number 6 so there *is* a solution.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

26 The equations $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{x}^* = \boldsymbol{b}^*$ have the same matrix A. What double augmented matrix should you use in elimination to solve both equations at once?

Solve both of these equations by working on a 2 by 4 matrix:

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

27 Choose the numbers a, b, c, d in this augmented matrix so that there is (a) no solution (b) infinitely many solutions.

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

Which of the numbers a, b, c, or d have no effect on the solvability?

28 If $AB = I$ and $BC = I$ use the associative law to prove $A = C$.

Challenge Problems

29 Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$\textbf{Eliminate column 1} \qquad E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I? Pascal's triangular matrix is exceptional, all of its multipliers are $\ell_{ij} = 1$.

30 Write $M = \left[\begin{smallmatrix} 3 & 4 \\ 5 & 7 \end{smallmatrix}\right]$ as a product of many factors $A = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

(a) What matrix E subtracts row 1 from row 2 to make row 2 of EM smaller?

(b) What matrix F subtracts row 2 of EM from row 1 to reduce row 1 of FEM?

(c) Continue E's and F's until (many E's and F's) times (M) is (A or B).

(d) E and F are the inverses of A and B! Moving all E's and F's to the right side will give you the desired result M ***= product of A's and B's.***
This is possible for integer matrices $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] > 0$ that have $ad - bc = 1$.

31 Find elimination matrices E_{21} then E_{32} then E_{43} to change K into U:

$$E_{43}\, E_{32}\, E_{21} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

Apply those three steps to the identity matrix I, to multiply $E_{43}E_{32}E_{21}$.

2.4 Rules for Matrix Operations

I will start with basic facts. A matrix is a rectangular array of numbers or "entries". When A has m rows and n columns, it is an "m by n" matrix. Matrices can be added if their shapes are the same. They can be multiplied by any constant c. Here are examples of $A + B$ and $2A$, for 3 by 2 matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix} \quad \text{and} \quad 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}.$$

Matrices are added exactly as vectors are—one entry at a time. We could even regard a column vector as a matrix with only one column (so $n = 1$). The matrix $-A$ comes from multiplication by $c = -1$ (reversing all the signs). Adding A to $-A$ leaves the *zero matrix*, with all entries zero. All this is only common sense.

The entry in row i and column j is called a_{ij} or $A(i, j)$. The n entries along the first row are $a_{11}, a_{12}, \ldots, a_{1n}$. The lower left entry in the matrix is a_{m1} and the lower right is a_{mn}. The row number i goes from 1 to m. The column number j goes from 1 to n.

Matrix addition is easy. The serious question is ***matrix multiplication.*** When can we multiply A times B, and what is the product AB? We cannot multiply when A and B are 3 by 2. They don't pass the following test:

To multiply AB: If A has n columns, B must have n rows.

When A is 3 by 2, the matrix B can be 2 by 1 (a vector) or 2 by 2 (square) or 2 by 20. ***Every column of B is multiplied by A.*** I will begin matrix multiplication the *dot product way*, and then return to this *column way*: A times columns of B. The most important rule is that ***AB times C equals A times BC.*** A Challenge Problem will prove this.

Suppose A is m by n and B is n by p. We can multiply. The product AB is m by p.

$$(m \times n)(n \times p) = (m \times p) \qquad \begin{bmatrix} \boldsymbol{m} \textbf{ rows} \\ n \textit{ columns} \end{bmatrix} \begin{bmatrix} n \textit{ rows} \\ \boldsymbol{p} \textbf{ columns} \end{bmatrix} = \begin{bmatrix} \boldsymbol{m} \textbf{ rows} \\ \boldsymbol{p} \textbf{ columns} \end{bmatrix}.$$

A row times a column is an extreme case. Then 1 by n multiplies n by 1. The result is 1 by 1. That single number is the "dot product".

In every case AB is filled with dot products. For the top corner, the $(1, 1)$ entry of AB is (row 1 of A) $\cdot$ (column 1 of B). To multiply matrices, take the dot product of ***each row of A with each column of B.***

The entry in row i and column j of AB is (row i of A) $\cdot$ (column j of B).

Figure 2.8 picks out the second row ($i = 2$) of a 4 by 5 matrix A. It picks out the third column ($j = 3$) of a 5 by 6 matrix B. Their dot product goes into row 2 and column 3 of AB. The matrix AB has *as many rows as A* (4 rows), and *as many columns as B*.

$$\begin{bmatrix} * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ * & & & \\ * & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is 4 by 5 B is 5 by 6 AB is 4 by 6

Figure 2.8: Here $i = 2$ and $j = 3$. Then $(AB)_{23}$ is (row 2) $\cdot$ (column 3) $= \Sigma a_{2k}b_{k3}$.

Example 1 Square matrices can be multiplied if and only if they have the same size:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}.$$

The first dot product is $1 \cdot 2 + 1 \cdot 3 = 5$. Three more dot products give 6, 1, and 0. Each dot product requires two multiplications—thus eight in all.

If A and B are n by n, so is AB. It contains n^2 dot products, row of A times column of B. Each dot product needs n multiplications, so ***the computation of AB uses n^3 separate multiplications***. For $n = 100$ we multiply a million times. For $n = 2$ we have $n^3 = 8$.

Mathematicians thought until recently that AB absolutely needed $2^3 = 8$ multiplications. Then somebody found a way to do it with 7 (and extra additions). By breaking n by n matrices into 2 by 2 blocks, this idea also reduced the count for large matrices. Instead of n^3 it went below $n^{2.8}$, and the exponent keeps falling.[1] The best at this moment is $n^{2.376}$. But the algorithm is so awkward that scientific computing is done the regular way: n^2 dot products in AB, and n multiplications for each one.

Example 2 Suppose A is a row vector (1 by 3) and B is a column vector (3 by 1). Then AB is 1 by 1 (only one entry, the dot product). On the other hand B times A (***a column times a row***) is a full 3 by 3 matrix. This multiplication is allowed!

Column times row
$(\boldsymbol{n} \times \mathbf{1})(\mathbf{1} \times \boldsymbol{n}) = (\boldsymbol{n} \times \boldsymbol{n})$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

A row times a column is an *"inner"* product—that is another name for dot product. A column times a row is an *"outer"* product. These are extreme cases of matrix multiplication.

Rows and Columns of AB

In the big picture, A multiplies each column of B. The result is a column of AB. In that column, we are combining the columns of A. ***Each column of AB is a combination of***

[1]Maybe 2.376 will drop to 2. No other number looks special, but no change for 10 years.

the columns of A. That is the column picture of matrix multiplication:

$$\textbf{Matrix } A \textbf{ times column of } B \qquad A\begin{bmatrix} b_1 \cdots b_p \end{bmatrix} = \begin{bmatrix} Ab_1 \cdots Ab_p \end{bmatrix}.$$

The row picture is reversed. Each row of A multiplies the whole matrix B. The result is a row of AB. It is a combination of the rows of B:

$$\textbf{Row times matrix} \qquad \begin{bmatrix} \text{row } i \text{ of } A \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \text{row } i \text{ of } AB \end{bmatrix}.$$

We see row operations in elimination (E times A). We see columns in A times $\boldsymbol{x}$. The "row-column picture" has the dot products of rows with columns. Believe it or not, ***there is also a column-row picture.*** Not everybody knows that columns $1, \ldots, n$ of A multiply rows $1, \ldots, n$ of B and add up to the same answer AB. Worked Example **2.3 C** had numbers for $n = 2$. ***Example 3 will show how to multiply*** AB ***using columns times rows.***

The Laws for Matrix Operations

May I put on record six laws that matrices do obey, while emphasizing an equation they don't obey? The matrices can be square or rectangular, and the laws involving $A + B$ are all simple and all obeyed. Here are three addition laws:

$$\begin{aligned} A + B &= B + A && \text{(commutative law)} \\ c(A + B) &= cA + cB && \text{(distributive law)} \\ A + (B + C) &= (A + B) + C && \text{(associative law).} \end{aligned}$$

Three more laws hold for multiplication, but $AB = BA$ is not one of them:

$$\begin{aligned} \boxed{AB \neq BA} & \qquad \text{(the commutative "law" is } \textit{usually broken}) \\ C(A + B) = CA + CB & \qquad \text{(distributive law from the left)} \\ (A + B)C = AC + BC & \qquad \text{(distributive law from the right)} \\ \boxed{A(BC) = (AB)C} & \qquad \text{(associative law for } ABC) \ (\textbf{\textit{parentheses not needed}}). \end{aligned}$$

When A and B are not square, AB is a different size from BA. These matrices can't be equal—even if both multiplications are allowed. For square matrices, almost any example shows that AB is different from BA:

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is true that $AI = IA$. All square matrices commute with I and also with cI. Only these matrices cI commute with all other matrices.

The law $A(B + C) = AB + AC$ is proved a column at a time. Start with $A(\boldsymbol{b} + \boldsymbol{c}) = A\boldsymbol{b} + A\boldsymbol{c}$ for the first column. That is the key to everything—***linearity***. Say no more.

The law $A(BC) = (AB)C$ ***means that you can multiply*** BC ***first or else*** AB ***first.*** The direct proof is sort of awkward (Problem 37) but this law is extremely useful. We highlighted it above; it is the key to the way we multiply matrices.

Look at the special case when $A = B = C =$ square matrix. Then (A *times* A^2) is equal to (A^2 *times* A). The product in either order is A^3. The matrix powers A^p follow the same rules as numbers:

$$A^p = AAA \cdots A \text{ (p factors)} \qquad (A^p)(A^q) = A^{p+q} \qquad (A^p)^q = A^{pq}.$$

Those are the ordinary laws for exponents. A^3 times A^4 is A^7 (seven factors). A^3 to the fourth power is A^{12} (twelve A's). When p and q are zero or negative these rules still hold, provided A has a "-1 power"—which is the *inverse matrix* A^{-1}. Then $A^0 = I$ is the identity matrix (no factors).

For a number, a^{-1} is $1/a$. For a matrix, the inverse is written A^{-1}. (It is *never* I/A, except this is allowed in MATLAB.) Every number has an inverse except $a = 0$. To decide when A has an inverse is a central problem in linear algebra. Section 2.5 will start on the answer. This section is a Bill of Rights for matrices, to say when A and B can be multiplied and how.

Block Matrices and Block Multiplication

We have to say one more thing about matrices. They can be cut into ***blocks*** (which are smaller matrices). This often happens naturally. Here is a 4 by 6 matrix broken into blocks of size 2 by 2—in this example each block is just I:

4 by 6 matrix
2 by 2 blocks

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

If B is also 4 by 6 and the block sizes match, you can add $A + B$ *a block at a time.*

We have seen block matrices before. The right side vector $\boldsymbol{b}$ was placed next to A in the "augmented matrix". Then $[\,A \quad \boldsymbol{b}\,]$ has two blocks of different sizes. Multiplying by an elimination matrix gave $[\,EA \quad E\boldsymbol{b}\,]$. No problem to multiply blocks times blocks, when their shapes permit.

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \tag{1}$$

This equation is the same as if the blocks were numbers (which are 1 by 1 blocks). We are careful to keep A's in front of B's, because BA can be different.

Main point When matrices split into blocks, it is often simpler to see how they act. The block matrix of I's above is much clearer than the original 4 by 6 matrix A.

Example 3 **(Important special case)** Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up ***columns times rows***:

Columns times rows
$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1 & - \\ & \vdots & \\ - & b_n & - \end{bmatrix} = \begin{bmatrix} a_1 b_1 + \cdots + a_n b_n \end{bmatrix}. \qquad (2)$$

This is another way to multiply matrices. Compare it with the usual rows times columns. Row 1 of A times column 1 of B gave the $(1,1)$ entry in AB. Now *column* 1 of A times *row* 1 of B gives a full matrix—not just a single number. Look at this example:

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{matrix} \text{Column 1 times row 1} \\ + \text{Column 2 times row 2} \end{matrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}. \qquad (3)$$

We stop there so you can see columns multiplying rows. If a 2 by 1 matrix (a column) multiplies a 1 by 2 matrix (a row), the result is 2 by 2. That is what we found. Dot products are *inner* products and these are *outer* products. In the top left corner the answer is $3 + 4 = 7$. This agrees with the row-column dot product of $(1,4)$ with $(3,1)$.

Summary The usual way, rows times columns, gives four dot products (8 multiplications). The new way, columns times rows, gives two full matrices (the same 8 multiplications). The 8 multiplications, and the 4 additions, are just executed in a different order.

Example 4 **(Elimination by blocks)** Suppose the first column of A contains $1, 3, 4$. To change 3 and 4 to 0 and 0, multiply the pivot row by 3 and 4 and subtract. Those row operations are really multiplications by elimination matrices E_{21} and E_{31}:

One at a time
$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

The "block idea" is to do both eliminations with one matrix E. That matrix clears out the whole first column of A below the pivot $a = 1$:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad \text{multiplies} \quad \begin{bmatrix} \mathbf{1} & x & x \\ \mathbf{3} & x & x \\ \mathbf{4} & x & x \end{bmatrix} \quad \text{to give} \quad EA = \begin{bmatrix} \mathbf{1} & x & x \\ \mathbf{0} & x & x \\ \mathbf{0} & x & x \end{bmatrix}.$$

Using inverses from 2.5, a block matrix E can do elimination on a whole (block) column of A. Suppose A has four blocks A, B, C, D. Watch how E multiplies A by blocks:

Block elimination
$$\left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right]. \qquad (4)$$

Elimination multiplies the first row $[\,A \;\; B\,]$ by CA^{-1} (previously c/a). It subtracts from C to get a zero block in the first column. It subtracts from D to get $S = D - CA^{-1}B$.

This is ordinary elimination, a column at a time—written in blocks. That final block S is $D - CA^{-1}B$, just like $d - cb/a$. This is called the ***Schur complement.***

■ REVIEW OF THE KEY IDEAS ■

1. The (i, j) entry of AB is (row i of A) $\cdot$ (column j of B).
2. An m by n matrix times an n by p matrix uses mnp separate multiplications.
3. A times BC equals AB times C (surprisingly important).
4. AB is also the sum of these matrices: (column j of A) times (row j of B).
5. Block multiplication is allowed when the block shapes match correctly.
6. Block elimination produces the *Schur complement* $D - CA^{-1}B$.

■ WORKED EXAMPLES ■

2.4 A Put yourself in the position of the author! I want to show you matrix multiplications that are *special*, but mostly I am stuck with small matrices. There is one terrific family of **Pascal matrices**, and they come in all sizes, and above all they have real meaning. I think 4 by 4 is a good size to show some of their amazing patterns.

Here is the lower triangular Pascal matrix L. Its entries come from "*Pascal's triangle*". I will multiply L times the **ones** vector, and the **powers** vector:

$$\textbf{Pascal matrix}\quad \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{4} \\ \mathbf{8} \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{x} \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1+x} \\ \mathbf{(1+x)^2} \\ \mathbf{(1+x)^3} \end{bmatrix}.$$

Each row of L leads to the next row: *Add an entry to the one on its left to get the entry below.* In symbols $\ell_{i\,j} + \ell_{i\,j-1} = \ell_{i+1\,j}$. The numbers after $1, 3, 3, 1$ would be $1, 4, 6, 4, 1$. Pascal lived in the 1600's, long before matrices, but his triangle fits perfectly into L.

Multiplying by **ones** is the same as adding up each row, to get powers of 2. By writing out L times **powers of** $\boldsymbol{x}$, you see the entries of L as the "binomial coefficients" that are so essential to gamblers:

$$\mathbf{1} + \mathbf{2}x + \mathbf{1}x^2 = (1+x)^2 \qquad \mathbf{1} + \mathbf{3}x + \mathbf{3}x^2 + \mathbf{1}x^3 = (1+x)^3$$

The number "3" counts the ways to get Heads once and Tails twice in three coin flips: HTT and THT and TTH. The other "3" counts the ways to get Heads twice: HHT and

HTH and THH. Those are examples of "i choose j" = the number of ways to get j heads in i coin flips. That number is exactly ℓ_{ij}, if we start counting rows and columns of L at $i = 0$ and $j = 0$ (and remember $0! = 1$):

$$\ell_{ij} = \binom{i}{j} = i \textit{ choose } j = \frac{i!}{j!\,(i-j)!} \qquad \binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{(2)(2)} = 6$$

There are six ways to choose two aces out of four aces. We will see Pascal's triangle and these matrices again. Here are the questions I want to ask now:

1. What is $H = L^2$? This is the "hypercube matrix".
2. Multiply H times **ones** and **powers.**
3. The last row of H is $8, 12, 6, 1$. A cube has 8 corners, 12 edges, 6 faces, 1 box. *What would the next row of H tell about a hypercube in* **4D**?

Solution Multiply L times L to get the hypercube matrix $H = L^2$:

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & & & \\ \mathbf{2} & \mathbf{1} & & \\ \mathbf{4} & \mathbf{4} & \mathbf{1} & \\ \mathbf{8} & \mathbf{12} & \mathbf{6} & \mathbf{1} \end{bmatrix} = H.$$

Now multiply H times the vectors of **ones** and **powers:**

$$\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{9} \\ \mathbf{27} \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2+x} \\ \mathbf{(2+x)^2} \\ \mathbf{(2+x)^3} \end{bmatrix}$$

If $x = 1$ we get the powers of 3. If $x = 0$ we get powers of 2. When L produces powers of $1 + x$, applying L again produces powers of $2 + x$.

How do the rows of H count corners and edges and faces of a cube? A square in **2D** has 4 corners, 4 edges, 1 face. Add one dimension at a time:

Connect two squares to get a **3D** *cube. Connect two cubes to get a* **4D** *hypercube.*

The cube has 8 corners and 12 edges: 4 edges in each square and 4 between the squares. The cube has 6 faces: 1 in each square and 4 faces between the squares. This row $8, 12, 6, 1$ will lead to the next row **16, 32, 24, 8, 1**. The rule is $2h_{i\,j} + h_{i\,j-1} = h_{i+1\,j}$.

Can you see this in four dimensions? The hypercube has 16 corners, no problem. It has 12 edges from one cube, 12 from the other cube, 8 that connect corners of those cubes: total 32 edges. It has 6 faces from each separate cube and 12 more from connecting pairs of edges: total $2 \times 6 + 12 = 24$ faces. It has one box from each cube and 6 more from connecting pairs of faces: total 8 boxes. And finally 1 hypercube.

2.4 B For these matrices, when does $AB = BA$? When does $BC = CB$? When does A times BC equal AB times C? Give the conditions on their entries p, q, r, z:

$$A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

If $p, q, r, 1, z$ are 4 by 4 blocks instead of numbers, do the answers change?

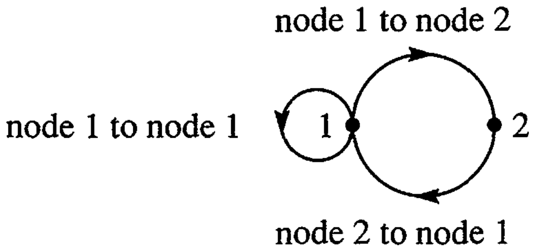
Solution First of all, A times BC *always* equals AB times C. Parentheses are not needed in $A(BC) = (AB)C = ABC$. But we must keep the matrices in this order:

$$\textbf{Usually } AB \neq BA \qquad AB = \begin{bmatrix} p & p \\ q & q+r \end{bmatrix} \qquad BA = \begin{bmatrix} p+q & r \\ q & r \end{bmatrix}.$$

$$\textbf{By chance } BC = CB \qquad BC = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \qquad CB = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}.$$

B and C happen to commute. Part of the explanation is that the diagonal of B is I, which commutes with all 2 by 2 matrices. When p, q, r, z are 4 by 4 blocks and 1 changes to I, all these products remain correct. So the answers are the same.

2.4 C A **directed graph** starts with n nodes. The n by n **adjacency matrix** has $a_{ij} = 1$ when an edge leaves node i and enters node j; if no edge then $a_{ij} = 0$.



$$A = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} = \text{adjacency matrix}$$

The i, j entry of A^2 is $\sum a_{ik}a_{kj}$. ***This is $a_{i1}a_{1j} + \cdots + a_{in}a_{nj}$.*** Why does that sum count the *two-step paths* from i to any node to j? The i, j entry of A^k counts k-step paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{matrix}\textbf{Count paths} \\ \textbf{with two edges}\end{matrix} \qquad \begin{bmatrix} \text{1 to 2 to 1, 1 to 1 to 1} & \text{1 to 1 to 2} \\ \text{2 to 1 to 1} & \text{2 to 1 to 2} \end{bmatrix}$$

List all of the 3-step paths between each pair of nodes and compare with A^3.

Solution The number $a_{ik}a_{kj}$ will be "1" if there is an edge from node i to k and an edge from k to j. This is a 2-step path. The number $a_{ik}a_{kj}$ will be "0" if either of those edges (i to k, k to j) is missing. So the sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j. Matrix multiplication is just right for this count.

The 3-step paths are counted by A^3; we look at paths to node 2:

$$A^3 = \begin{bmatrix} 3 & \mathbf{2} \\ 2 & \mathbf{1} \end{bmatrix} \quad \text{counts the paths with three steps} \quad \begin{bmatrix} \cdots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2,\ 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \cdots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

These A^k contain the Fibonacci numbers $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ coming in Section 6.2. Multiplying A by A^k involves Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ (as in $13 = 8 + 5$):

$$(A)(A^k) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} = A^{k+1}.$$

There are 13 six-step paths from node 1 to node 1, but I can't find them all.

A^k *also counts words.* A path like 1 to 1 to 2 to 1 corresponds to the word **aaba.** The letter **b** can't repeat because there is no edge from 2 to 2. The i, j entry of A^k counts the words of length $k + 1$ that start with the ith letter and end with the jth.

Problem Set 2.4

Problems 1–16 are about the laws of matrix multiplication.

1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B + C).$$

2 What rows or columns or matrices do you multiply to find

(a) the third column of AB?

(b) the first row of AB?

(c) the entry in row 3, column 4 of AB?

(d) the entry in row 1, column 1 of CDE?

3 Add AB to AC and compare with $A(B + C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

4 In Problem 3, multiply A times BC. Then multiply AB times C.

5 Compute A^2 and A^3. Make a prediction for A^5 and A^n:

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

6 Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A+B)(A+B) = A^2 + \_\_\_\_ + B^2$.

7 True or false. Give a specific example when false:

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

(d) $(AB)^2 = A^2B^2$.

8 How is each row of DA and EA related to the rows of A, when

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}?$$

How is each column of AD and AE related to the columns of A?

9 Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$:

$$EA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$\text{and} \quad (EA)F = (EA) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+c & a+c+b+d \end{bmatrix}.$$

(a) Do those steps in the opposite order. First add column 1 of A to column 2 by AF, then add row 1 of AF to row 2 by $E(AF)$.

(b) Compare with $(EA)F$. What law is obeyed by matrix multiplication?

10 Row 1 of A is again added to row 2 to produce EA. Then F adds row 2 of EA to row 1. The result is $F(EA)$:

$$F(EA) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix}.$$

(a) Do those steps in the opposite order: first add row 2 to row 1 by FA, then add row 1 of FA to row 2.

(b) What law is or is not obeyed by matrix multiplication?

11 (3 by 3 matrices) Choose the only B so that for every matrix A

(a) $BA = 4A$

(b) $BA = 4B$

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A.

12 Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are A = multiple of I.

13 Which of the following matrices are guaranteed to equal $(A - B)^2$: $A^2 - B^2$, $(B - A)^2$, $A^2 - 2AB + B^2$, $A(A - B) - B(A - B)$, $A^2 - AB - BA + B^2$?

14 True or false:

(a) If A^2 is defined then A is necessarily square.

(b) If AB and BA are defined then A and B are square.

(c) If AB and BA are defined then AB and BA are square.

(d) If $AB = B$ then $A = I$.

15 If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector $\boldsymbol{x}$ with n components?

(b) A multiplies an n by p matrix B?

(c) A multiplies itself to produce A^2? Here $m = n$.

16 For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers *and nothing more*:

(a) column 2 of AB

(b) row 2 of AB

(c) row 2 of $AA = A^2$

(d) row 2 of $AAA = A^3$.

Problems 17–19 use a_{ij} for the entry in row i, column j of A.

17 Write down the 3 by 3 matrix A whose entries are

(a) a_{ij} = minimum of i and j

(b) $a_{ij} = (-1)^{i+j}$

(c) $a_{ij} = i/j$.

18 What words would you use to describe each of these classes of matrices? Give a 3 by 3 example in each class. Which matrix belongs to all four classes?

(a) $a_{ij} = 0$ if $i \neq j$

(b) $a_{ij} = 0$ if $i < j$

(c) $a_{ij} = a_{ji}$

(d) $a_{ij} = a_{1j}$.

19 The entries of A are a_{ij}. Assuming that zeros don't appear, what is

(a) the first pivot?

(b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3?

(c) the new entry that replaces a_{32} after that subtraction?

(d) the second pivot?

Problems 20–24 involve powers of A.

20 Compute A^2, A^3, A^4 and also Av, A^2v, A^3v, A^4v for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

21 Find all the powers $A^2, A^3, \ldots$ and $AB, (AB)^2, \ldots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

22 By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \qquad BC = 0 \qquad DE = -ED \text{ (not allowing } DE = 0).$$

23 (a) Find a nonzero matrix A for which $A^2 = 0$.

(b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

24 By experiment with $n = 2$ and $n = 3$ predict A^n for these matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Problems 25–31 use column-row multiplication and block multiplication.

25 Multiply A times I using columns of A (3 by 3) times rows of I.

26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\quad\quad} = \underline{\quad\quad}.$$

27 Show that the product of upper triangular matrices is always upper triangular:

$$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}.$$

Proof using dot products (Row times column) (Row 2 of A)$\cdot$(column 1 of B)$= 0$. Which other dot products give zeros?

Proof using full matrices (Column times row) Draw x's and 0's in (column 2 of A) times (row 2 of B). Also show (column 3 of A) times (row 3 of B).

28 Draw the cuts in A (2 by 3) and B (3 by 4) and AB to show how each of the four multiplication rules is really a block multiplication:

(1) Matrix A times columns of B. **Columns of AB**

(2) Rows of A times the matrix B. **Rows of AB**

(3) Rows of A times columns of B. **Inner products** (numbers in AB)

(4) Columns of A times rows of B. **Outer products** (matrices add to AB)

29 Which matrices E_{21} and E_{31} produce zeros in the $(2,1)$ and $(3,1)$ positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

30 Block multiplication says that column 1 is eliminated by

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & D - \boldsymbol{cb}/a \end{bmatrix}.$$

In Problem 29, what are $\boldsymbol{c}$ and D and what is $D - \boldsymbol{cb}/a$?

31 With $i^2 = -1$, the product of $(A + iB)$ and $(\boldsymbol{x} + i\boldsymbol{y})$ is $A\boldsymbol{x} + iB\boldsymbol{x} + iA\boldsymbol{y} - B\boldsymbol{y}$. Use blocks to separate the real part without i from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x} - B\boldsymbol{y} \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

32 (*Very important*) Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

33 If the three solutions in Question 32 are $x_1 = (1,1,1)$ and $x_2 = (0,1,1)$ and $x_3 = (0,0,1)$, solve $Ax = b$ when $b = (3,5,8)$. Challenge problem: What is A?

34 Find all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}A$.

35 Suppose a "circle graph" has 4 nodes connected (in both directions) by edges around a circle. What is its adjacency matrix from Worked Example **2.4 C**? What is A^2? Find all the 2-step paths (or 3-letter words) predicted by A^2.

Challenge Problems

36 **Practical question** Suppose A is m by n, B is n by p, and C is p by q. Then the multiplication count for $(AB)C$ is $mnp + mpq$. The same answer comes from A times BC with $mnq + npq$ separate multiplications. Notice npq for BC.

(a) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, do you prefer $(AB)C$ or $A(BC)$?

(b) With N-component vectors, would you choose $(u^{\mathrm{T}}v)w^{\mathrm{T}}$ or $u^{\mathrm{T}}(vw^{\mathrm{T}})$?

(c) Divide by $mnpq$ to show that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

37 To prove that $(AB)C = A(BC)$, use the column vectors $b_1, \ldots, b_n$ of B. First suppose that C has only one column c with entries $c_1, \ldots, c_n$:

AB has columns $Ab_1, \ldots, Ab_n$ and then $(AB)c$ equals $c_1Ab_1 + \cdots + c_nAb_n$.

Bc has one column $c_1b_1 + \cdots + c_nb_n$ and then $A(Bc)$ equals $A(c_1b_1 + \cdots + c_nb_n)$.

Linearity gives equality of those two sums. This proves $(AB)c = A(Bc)$. The same is true for all other _____ of C. Therefore $(AB)C = A(BC)$. Apply to inverses:

If $BA = I$ and $AC = I$, prove that the left-inverse B equals the right-inverse C.

2.5 Inverse Matrices

Suppose A is a square matrix. We look for an "***inverse matrix***" A^{-1} of the same size, such that A^{-1} *times* A *equals* I. Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing to a vector, so $A^{-1}A\boldsymbol{x} = \boldsymbol{x}$. *But* A^{-1} *might not exist.*

What a matrix mostly does is to multiply a vector $\boldsymbol{x}$. Multiplying $A\boldsymbol{x} = \boldsymbol{b}$ by A^{-1} gives $A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$. ***This is*** $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. A number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix A^{-1} is called "A inverse."

DEFINITION The matrix A is ***invertible*** if there exists a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? We don't mean that we immediately calculate A^{-1}. In most problems we never compute it! Here are six "notes" about A^{-1}.

Note 1 ***The inverse exists if and only if elimination produces n pivots*** (row exchanges are allowed). Elimination solves $A\boldsymbol{x} = \boldsymbol{b}$ without explicitly using the matrix A^{-1}.

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \tag{2}$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix.*

Note 3 If A is invertible, the one and only solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$:

$$\textbf{\textit{Multiply}} \quad A\boldsymbol{x} = \boldsymbol{b} \quad \textbf{\textit{by}} \quad A^{-1}. \quad \textbf{\textit{Then}} \quad \boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

Note 4 (Important) ***Suppose there is a nonzero vector x such that*** $A\boldsymbol{x} = \boldsymbol{0}$. *Then* A *cannot have an inverse.* No matrix can bring $\boldsymbol{0}$ back to $\boldsymbol{x}$.

If A is invertible, then $A\boldsymbol{x} = \boldsymbol{0}$ can only have the zero solution $\boldsymbol{x} = A^{-1}\boldsymbol{0} = \boldsymbol{0}$.

Note 5 A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\textbf{2 by 2 Inverse:} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the *determinant* of A. A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If} \quad A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \text{then} \quad A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1 The 2 by 2 matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right]$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $Ax = \mathbf{0}$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b, the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B, the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product* AB has an inverse, if and only if the two factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in ***reverse order***:

If A and B are invertible then so is AB. The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{4}$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. Inside that is $BB^{-1} = I$:

$$\textbf{Inverse of } AB \quad (AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I. This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the ______. The same reverse order applies to three or more matrices:

$$\textbf{Reverse order} \qquad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \tag{5}$$

Example 2 ***Inverse of an elimination matrix***. If E subtracts 5 times row 1 from row 2, then E^{-1} *adds* 5 times row 1 to row 2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I. Also multiply $E^{-1}E$ to get I. We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is EE^{-1}) or subtract and then add (this is $E^{-1}E$), we are back at the start.

For square matrices, an inverse on one side is automatically an inverse on the other side. If $AB = I$ then automatically $BA = I$. In that case B is A^{-1}. This is very useful to know but we are not ready to prove it.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE. Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \tag{6}$$

The result is beautiful and correct. The product FE contains "20" but its inverse doesn't. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the *new* row 2 (changed by row 1) from row 3. ***In this order FE, row 3 feels an effect from row 1.***

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn't change again. ***In this order $E^{-1}F^{-1}$, row 3 feels no effect from row 1.***

In elimination order F follows E. In reverse order E^{-1} follows F^{-1}.
$E^{-1}F^{-1}$ is quick. The multipliers 5, 4 fall into place below the diagonal of 1's.

This special multiplication $E^{-1}F^{-1}$ and $E^{-1}F^{-1}G^{-1}$ will be useful in the next section. We will explain it again, more completely. In this section our job is A^{-1}, and we expect some serious work to compute it. Here is a way to organize that computation.

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $Ax = b$ is solved by $x = A^{-1}b$. But it is not necessary or efficient to compute A^{-1} and multiply it times b. *Elimination goes directly to x.* Elimination is also the way to calculate A^{-1}, as we now show. The Gauss-Jordan idea is to solve $AA^{-1} = I$, *finding each column of A^{-1}*.

A multiplies the first column of A^{-1} (call that x_1) to give the first column of I (call that e_1). This is our equation $Ax_1 = e_1 = (1, 0, 0)$. There will be two more equations. Each of the columns x_1, x_2, x_3 of A^{-1} is multiplied by A to produce a column of I:

3 columns of A^{-1}
$$AA^{-1} = A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = I. \tag{7}$$

To invert a 3 by 3 matrix A, we have to solve three systems of equations: $Ax_1 = e_1$ and $Ax_2 = e_2 = (0, 1, 0)$ and $Ax_3 = e_3 = (0, 0, 1)$. Gauss-Jordan finds A^{-1} this way.

The ***Gauss-Jordan method*** computes A^{-1} by solving ***all n equations together.*** Usually the "augmented matrix" $[A \quad b]$ has one extra column $\boldsymbol{b}$. Now we have three right sides $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ (when A is 3 by 3). They are the columns of I, so the augmented matrix is really the block matrix $[A \quad I]$. I take this chance to invert my favorite matrix K, with 2's on the main diagonal and -1's next to the 2's:

$$\begin{bmatrix} K & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{Start Gauss-Jordan on } K$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & \mathbf{-1} & \frac{1}{2} & \mathbf{1} & \mathbf{0} \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \quad (\tfrac{1}{2}\ \textit{row}\ \mathbf{1} + \textit{row}\ \mathbf{2})$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & \mathbf{1} \end{bmatrix} \quad (\tfrac{2}{3}\ \textit{row}\ \mathbf{2} + \textit{row}\ \mathbf{3})$$

We are halfway to K^{-1}. The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination*! He goes all the way to the ***"reduced echelon form"***. Rows are added to rows above them, to produce ***zeros above the pivots***:

$$\begin{pmatrix} \text{Zero above} \\ \text{third pivot} \end{pmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & \mathbf{0} & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{3}{4}\ \textit{row}\ \mathbf{3} + \textit{row}\ \mathbf{2})$$

$$\begin{pmatrix} \text{Zero above} \\ \text{second pivot} \end{pmatrix} \rightarrow \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \frac{3}{2} & \mathbf{1} & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{2}{3}\ \textit{row}\ \mathbf{2} + \textit{row}\ \mathbf{1})$$

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached I in the first half of the matrix, because K is invertible. ***The three columns of K^{-1} are in the second half of $[I \quad K^{-1}]$:***

$$\begin{matrix} \text{(divide by 2)} \\ \text{(divide by } \frac{3}{2}) \\ \text{(divide by } \frac{4}{3}) \end{matrix} \quad \begin{bmatrix} \mathbf{1} & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \mathbf{1} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} I & x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} I & K^{-1} \end{bmatrix}.$$

Starting from the 3 by 6 matrix $[K \quad I]$, we ended with $[I \quad K^{-1}]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix A:

Gauss-Jordan ***Multiply*** $[A \quad I]$ ***by*** A^{-1} ***to get*** $[I \quad A^{-1}]$.

The elimination steps create the inverse matrix while changing A to I. For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular K^{-1} because it is an important example. We introduce the words *symmetric*, *tridiagonal*, and *determinant*:

1. K is ***symmetric*** across its main diagonal. So is K^{-1}.

2. K is ***tridiagonal*** (only three nonzero diagonals). But K^{-1} is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.

3. The *product of pivots* is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the ***determinant*** of K.

K^{-1} involves division by the determinant
$$K^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{8}$$

This is why an invertible matrix cannot have a zero determinant.

Example 4 Find A^{-1} by Gauss-Jordan elimination starting from $A = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 7 \end{smallmatrix}\right]$. There are two row operations and then a division to put 1's in the pivots:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & 1 & 0 \\ \mathbf{4} & \mathbf{7} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} U & L^{-1} \end{bmatrix}\right)$$
$$\to \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{7} & \mathbf{-3} \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \mathbf{\frac{7}{2}} & \mathbf{-\frac{3}{2}} \\ 0 & 1 & \mathbf{-2} & \mathbf{1} \end{bmatrix} \quad \left(\text{this is } \begin{bmatrix} I & A^{-1} \end{bmatrix}\right).$$

That A^{-1} involves division by the determinant $ad - bc = 2 \cdot 7 - 3 \cdot 4 = 2$. The code for $X =$ inverse(A) can use **rref**, the "row reduced echelon form" from Chapter 3:

```
I = eye (n);               % Define the n by n identity matrix
R = rref ([A I]);          % Eliminate on the augmented matrix [A I]
X = R(:, n + 1 : n + n)    % Pick A^-1 from the last n columns of R
```

A must be invertible, or elimination cannot reduce it to I (in the left half of R).

Gauss-Jordan shows why A^{-1} is expensive. We must solve n equations for its n columns.

To solve $Ax = b$ without A^{-1}, we deal with *one* column b to find one column x.

In defense of A^{-1}, we want to say that its cost is not n times the cost of one system $Ax = b$. Surprisingly, the cost for n columns is only multiplied by 3. This saving is because the n equations $Ax_i = e_i$ all involve the same matrix A. Working with the right sides is relatively cheap, because elimination only has to be done once on A.

The complete A^{-1} needs n^3 elimination steps, where a single x needs $n^3/3$. The next section calculates these costs.

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: A^{-1} ***exists exactly when A has a full set of n pivots.*** (Row exchanges are allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $Ax_i = e_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ and A^{-1} is at least a ***right-inverse***.

2. Elimination is really a sequence of multiplications by E's and P's and D^{-1}:

Left-inverse $$(D^{-1} \cdots E \cdots P \cdots E)A = I. \tag{9}$$

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. P will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a ***left-inverse***. With n pivots we have reached $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 at the start of in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that ***A must have n pivots if*** $AC = I$. (Then we deduce that C is also a left-inverse and $CA = I$.) Here is one route to those conclusions:

1. If A doesn't have n pivots, elimination will lead to a *zero row*.
2. Those elimination steps are taken by an invertible M. *So a row of MA is zero.*
3. If $AC = I$ had been possible, then $MAC = M$. The zero row of MA, times C, gives a zero row of M itself.
4. An invertible matrix M can't have a zero row! *A must* have n pivots if $AC = I$.

That argument took four steps, but the outcome is short and important.

Elimination gives a complete test for invertibility of a square matrix. A^{-1} ***exists (and Gauss-Jordan finds it) exactly when A has n pivots.*** The argument above shows more:

$$\text{If } AC = I \text{ then } CA = I \text{ and } C = A^{-1}$$

Example 5 If L is lower triangular with 1's on the diagonal, so is L^{-1}.

A triangular matrix is invertible if and only if no diagonal entries are zero.

Here L has 1's so L^{-1} also has 1's. Use the Gauss-Jordan method to construct L^{-1}. Start by subtracting multiples of pivot rows from rows *below*. Normally this gets us halfway to the inverse, but for L it gets us all the way. L^{-1} appears on the right when I appears on the left. Notice how L^{-1} contains 11, from 3 times 5 minus 4.

Gauss-Jordan on triangular L

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & 0 & 1 & 0 \\ \mathbf{4} & \mathbf{5} & \mathbf{1} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} L & I \end{bmatrix}$$

$$\begin{matrix} \\ \rightarrow \\ \rightarrow \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \text{(3 times row 1 from row 2)} \\ \text{(4 times row 1 from row 3)} \\ \text{(then 5 times row 2 from row 3)} \end{matrix}$$

$$\begin{matrix} \\ \\ \rightarrow \end{matrix}\begin{bmatrix} 1 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{11} & \mathbf{-5} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} I & \boldsymbol{L^{-1}} \end{bmatrix}.$$

L goes to I by a product of elimination matrices $E_{32}E_{31}E_{21}$. So that product is L^{-1}. All pivots are 1's (a full set). L^{-1} is lower triangular, with the strange entry "11".

That 11 does not appear to spoil 3, 4, 5 in the good order $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

■ REVIEW OF THE KEY IDEAS ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.
2. A is invertible if and only if it has n pivots (row exchanges allowed).
3. If $A\boldsymbol{x} = \mathbf{0}$ for a nonzero vector $\boldsymbol{x}$, then A has no inverse.
4. The inverse of AB is the reverse product $B^{-1}A^{-1}$. And $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1}. The augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ is row-reduced to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

■ WORKED EXAMPLES ■

2.5 A The inverse of a triangular **difference matrix** A is a triangular **sum matrix** S:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] = \begin{bmatrix} I & A^{-1} \end{bmatrix} = \begin{bmatrix} I & \textit{sum matrix} \end{bmatrix}.$$

If I change a_{13} to -1, then all rows of A add to zero. The equation $A\boldsymbol{x} = \mathbf{0}$ will now have the nonzero solution $\boldsymbol{x} = (1, 1, 1)$. A clear signal: ***This new A can't be inverted.***

2.5 B Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $Ax = \mathbf{0}$) for the other three. The matrices are in the order A, B, C, D, S, E:

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36}\begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. E is not invertible because a combination of the columns (the second column minus the first column) is zero—in other words $Ex = \mathbf{0}$ has the solution $x = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of A, D, E.

2.5 C Apply the Gauss-Jordan method to invert this triangular "Pascal matrix" L. You see **Pascal's triangle**—adding each entry to the entry on its left gives the entry below. The entries of L are "binomial coefficients". The next row would be $1, 4, 6, 4, 1$.

$$\textbf{Triangular Pascal matrix} \quad L = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \end{bmatrix} = \texttt{abs(pascal (4,1))}$$

Solution Gauss-Jordan starts with $[\,L \;\; I\,]$ and produces zeros by subtracting row 1:

$$[\,L \;\; I\,] = \left[\begin{array}{cccc|cccc} \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \mathbf{0} & 1 & 0 & 0 & \mathbf{-1} & 1 & 0 & 0 \\ \mathbf{0} & 2 & 1 & 0 & \mathbf{-1} & 0 & 1 & 0 \\ \mathbf{0} & 3 & 3 & 1 & \mathbf{-1} & 0 & 0 & 1 \end{array}\right].$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \mathbf{0} & 1 & 0 & \mathbf{1} & \mathbf{-2} & 1 & 0 \\ 0 & \mathbf{0} & 3 & 1 & \mathbf{2} & \mathbf{-3} & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathbf{-1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 1 & 0 & \mathbf{1} & \mathbf{-2} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 & \mathbf{-1} & \mathbf{3} & \mathbf{-3} & \mathbf{1} \end{array}\right] = [\,I \;\; L^{-1}\,].$$

All the pivots were 1! So we didn't need to divide rows by pivots to get I. The inverse matrix L^{-1} looks like L itself, except odd-numbered diagonals have minus signs.

The same pattern continues to n by n Pascal matrices, L^{-1} has "alternating diagonals".

Problem Set 2.5

1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

2 For these "permutation matrices" find P^{-1} by trial and error (with 1's and 0's):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3 Solve for the first column (x, y) and second column (t, z) of A^{-1}:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4 Show that $\left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ is not invertible by trying to solve $AA^{-1} = I$ for column 1 of A^{-1}:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} \textit{For a different } A, \text{ could column 1 of } A^{-1} \\ \text{be possible to find but not column 2?} \end{pmatrix}$$

5 Find an upper triangular U (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$.

6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

(b) If $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, find two different matrices such that $AB = AC$.

7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\boldsymbol{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\boldsymbol{x} = \boldsymbol{b}$?

(c) What happens to row 3 in elimination?

8 If A has column 1 + column 2 = column 3, show that A is not invertible:

(a) Find a nonzero solution $\boldsymbol{x}$ to $A\boldsymbol{x} = \boldsymbol{0}$. The matrix is 3 by 3.

(b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

9 Suppose A is invertible and you exchange its first two rows to reach B. Is the new matrix B invertible and how would you find B^{-1} from A^{-1}?

10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

(b) Find singular matrices A and B such that $A + B$ is invertible.

12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

14 If you add row 1 of A to row 2 to get B, how do you find B^{-1} from A^{-1}?

$$\text{Notice the order.}\quad \text{The inverse of}\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} & A & \end{bmatrix} \quad \text{is} \quad \underline{\qquad}.$$

15 Prove that a matrix with a column of zeros cannot have an inverse.

16 Multiply $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ times $\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. What is the inverse of each matrix if $ad \neq bc$?

17 (a) What 3 by 3 matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.

(b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.

18 If B is the inverse of A^2, show that AB is the inverse of A.

19 Find the numbers a and b that give the inverse of 5 * eye(4) − ones(4,4):

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of 6 * eye(5) − ones(5,5)?

20 Show that A = 4 * eye(4) − ones(4,4) is *not* invertible: Multiply A * ones(4,1).

21 There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1}.

22 Change I into A^{-1} as you reduce A to I (by row operations):

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

23 Follow the 3 by 3 text example but with plus signs in A. Eliminate above and below the pivots to reduce $[\,A \;\; I\,]$ to $[\,I \;\; A^{-1}\,]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

24 Use Gauss-Jordan elimination on $[\,U \quad I\,]$ to find the upper triangular U^{-1}:

$$UU^{-1} = I \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

25 Find A^{-1} and B^{-1} (*if they exist*) by elimination on $[\,A \quad I\,]$ and $[\,B \quad I\,]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 6 \end{smallmatrix}\right]$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1}.

27 Invert these matrices A by the Gauss-Jordan method starting with $[\,A \quad I\,]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

28 Exchange rows and continue with Gauss-Jordan to find A^{-1}:

$$[\,A \quad I\,] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If A is invertible then A^{-1} and A^2 are invertible.

30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \;\; I]$. Extend to a 5 by 5 "alternating matrix" and guess its inverse; then multiply to confirm.

$$\text{Invert } A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and solve } A\boldsymbol{x} = (1,1,1,1).$$

33 Suppose the matrices P and Q have the same rows as I but in any order. They are "permutation matrices". Show that $P - Q$ is singular by solving $(P - Q)\boldsymbol{x} = \mathbf{0}$.

34 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

35 Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

36 In the Worked Example **2.5 C**, the triangular Pascal matrix L has an inverse with "alternating diagonals". Check that this L^{-1} is DLD, where the diagonal matrix D has alternating entries $1, -1, 1, -1$. Then $LDLD = I$, so what is the inverse of $LD =$ pascal (4,1)?

37 The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse invhilb(6). Then ask it to compute inv(hilb(6)). How can these be different, when the computer never makes mistakes?

38 (a) Use inv(P) to invert MATLAB's 4 by 4 symmetric matrix $P =$ pascal(4).

(b) Create Pascal's lower triangular $L =$ abs(pascal(4,1)) and test $P = LL^{\mathrm{T}}$.

39 If $A =$ ones(4) and $\boldsymbol{b} =$ rand(4,1), how does MATLAB tell you that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution? For the special $\boldsymbol{b} =$ ones(4,1), which solution to $A\boldsymbol{x} = \boldsymbol{b}$ is found by $A\backslash\boldsymbol{b}$?

Challenge Problems

40 (Recommended) A is a 4 by 4 matrix with 1's on the diagonal and $-a, -b, -c$ on the diagonal above. Find A^{-1} for this bidiagonal matrix.

41 Suppose E_1, E_2, E_3 are 4 by 4 identity matrices, except E_1 has a, b, c in column 1 and E_2 has d, e in column 2 and E_3 has f in column 3 (below the 1's). Multiply $L = E_1E_2E_3$ to show that all these nonzeros are copied into L.

$E_1E_2E_3$ is in the *opposite* order from elimination (because E_3 is acting first). But $E_1E_2E_3 = L$ is in the *correct* order to invert elimination and recover A.

42 Direct multiplications **1–4** give $MM^{-1} = I$, and I would recommend doing #3. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A:

$$\begin{array}{llll}
\mathbf{1} & M = I - uv^{\mathrm{T}} & \text{and} & M^{-1} = I + uv^{\mathrm{T}}/(1 - v^{\mathrm{T}}u) \quad \text{(rank 1 change in } I\text{)} \\
\mathbf{2} & M = A - uv^{\mathrm{T}} & \text{and} & M^{-1} = A^{-1} + A^{-1}uv^{\mathrm{T}}A^{-1}/(1 - v^{\mathrm{T}}A^{-1}u) \\
\mathbf{3} & M = I - UV & \text{and} & M^{-1} = I_n + U(I_m - VU)^{-1}V \\
\mathbf{4} & M = A - UW^{-1}V & \text{and} & M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}
\end{array}$$

The Woodbury-Morrison formula **4** is the "matrix inversion lemma" in engineering. The ***Kalman filter*** for solving block tridiagonal systems uses formula **4** at each step. The four matrices M^{-1} are in diagonal blocks when inverting these block matrices (v^{T} is 1 by n, u is n by 1, V is m by n, U is n by m).

$$\begin{bmatrix} I & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} A & u \\ v^{\mathrm{T}} & 1 \end{bmatrix} \qquad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \qquad \begin{bmatrix} A & U \\ V & W \end{bmatrix}$$

43 Second difference matrices have beautiful inverses if they start with $T_{11} = 1$ (instead of $K_{11} = 2$). Here is the 3 by 3 tridiagonal matrix T and its inverse:

$$\boldsymbol{T_{11} = 1} \qquad T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

One approach is Gauss-Jordan elimination on $[\,T \quad I\,]$. That seems too mechanical. I would rather write T as the product of first differences L times U. The inverses of L and U in Worked Example **2.5 A** are **sum matrices**, so here are T and T^{-1}:

$$LU = \underset{\textbf{difference}}{\begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix}} \underset{\textbf{difference}}{\begin{bmatrix} 1 & -1 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix}} \qquad U^{-1}L^{-1} = \underset{\textbf{sum}}{\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}} \underset{\textbf{sum}}{\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}}$$

Question. (**4 by 4**) What are the pivots of T? What is its 4 by 4 inverse? The reverse order UL gives what matrix T^*? What is the inverse of T^*?

44 Here are two more difference matrices, both important. ***But are they invertible***?

$$\textbf{Cyclic } C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \textbf{Free ends } F = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

One test is elimination—the fourth pivot fails. Another test is the determinant, we don't want that. The best way is much faster, and independent of matrix size:

Produce $x \neq 0$ so that $Cx = 0$. Do the same for $Fx = 0$. Not invertible.

Show how both equations $Cx = b$ and $Fx = b$ lead to $0 = b_1 + b_2 + \cdots + b_n$. There is no solution for other b.

45 *Elimination for a 2 by 2 block matrix*: When you multiply the first block row by CA^{-1} and subtract from the second row, the "*Schur complement*" S appears:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \qquad \begin{matrix} A \text{ and } D \text{ are square} \\ S = D - CA^{-1}B. \end{matrix}$$

Multiply on the right to subtract $A^{-1}B$ times block column 1 from block column 2.

$$\begin{bmatrix} A & B \\ 0 & S \end{bmatrix}\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = ? \quad \text{Find } S \text{ for } \begin{bmatrix} A & B \\ C & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

The block pivots are A and S. If they are invertible, so is $[\,A \quad B\,;\ \ C \quad D\,]$.

46 How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.

2.6 Elimination = Factorization: $A = LU$

Students often say that mathematics courses are too theoretical. Well, not this section. It is almost purely practical. The goal is to describe Gaussian elimination in the most useful way. Many key ideas of linear algebra, when you look at them closely, are really *factorizations* of a matrix. The original matrix A becomes the product of two or three special matrices. The first factorization—also the most important in practice—comes now from elimination. ***The factors L and U are triangular matrices. The factorization that comes from elimination is $A = LU$.***

We already know U, the upper triangular matrix with the pivots on its diagonal. The elimination steps take A to U. We will show how reversing those steps (taking U back to A) is achieved by a lower triangular L. ***The entries of L are exactly the multipliers*** ℓ_{ij}—which multiplied the pivot row j when it was subtracted from row i.

Start with a 2 by 2 example. The matrix A contains $2, 1, 6, 8$. The number to eliminate is 6. ***Subtract 3 times row 1 from row 2.*** That step is E_{21} in the forward direction with multiplier $\ell_{21} = 3$. The return step from U to A is $L = E_{21}^{-1}$ (an addition using $+3$):

$$\textit{Forward from } A \textit{ to } U: \quad E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$\textit{Back from } U \textit{ to } A: \quad E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A.$$

The second line is our factorization $LU = A$. Instead of E_{21}^{-1} we write L. Move now to larger matrices with many E's. ***Then L will include all their inverses.***

Each step from A to U multiplies by a matrix E_{ij} to produce zero in the (i, j) position. To keep this clear, we stay with the most frequent case—***when no row exchanges are involved.*** If A is 3 by 3, we multiply by E_{21} and E_{31} and E_{32}. The multipliers ℓ_{ij} produce zeros in the $(2,1)$ and $(3,1)$ and $(3,2)$ positions—all below the diagonal. Elimination ends with the upper triangular U.

Now move those E's onto the other side, *where their inverses multiply U*:

$$(E_{32}E_{31}E_{21})A = U \quad \textbf{becomes} \quad A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U \quad \textbf{which is} \quad A = LU. \quad (1)$$

The inverses go in opposite order, as they must. That product of three inverses is L. ***We have reached $A = LU$.*** Now we stop to understand it.

Explanation and Examples

First point: Every inverse matrix E^{-1} is *lower triangular.* Its off-diagonal entry is ℓ_{ij}, to undo the subtraction produced by $-\ell_{ij}$. The main diagonals of E and E^{-1} contain 1's. Our example above had $\ell_{21} = 3$ and $E = \left[\begin{smallmatrix} 1 & 0 \\ -3 & 1 \end{smallmatrix}\right]$ and $L = E^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$.

Second point: Equation (1) shows a lower triangular matrix (the product of the E_{ij}) multiplying A. It also shows all the E_{ij}^{-1} multiplying U to bring back A. ***This lower triangular product of inverses is L.***

One reason for working with the inverses is that we want to factor A, not U. The "inverse form" gives $A = LU$. Another reason is that we get something extra, almost more than we deserve. This is the third point, showing that L is exactly right.

Third point: Each multiplier ℓ_{ij} goes directly into its i, j position—*unchanged*—in the product of inverses which is L. Usually matrix multiplication will mix up all the numbers. Here that doesn't happen. The order is right for the inverse matrices, to keep the ℓ's unchanged. The reason is given below in equation (3).

Since each E^{-1} has 1's down its diagonal, the final good point is that L does too.

($A = LU$) ***This is elimination without row exchanges.*** The upper triangular U has the pivots on its diagonal. The lower triangular L has all 1's on its diagonal. ***The multipliers ℓ_{ij} are below the diagonal of L.***

Example 1 Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. The last step subtracts $\frac{2}{3}$ times row 2 from row 3. The lower triangular L has $\ell_{21} = \frac{1}{2}$ and $\ell_{32} = \frac{2}{3}$. Multiplying LU produces A:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU.$$

The $(3, 1)$ multiplier is zero because the $(3, 1)$ entry in A is zero. No operation needed.

Example 2 Change the top left entry from 2 to 1. The pivots all become 1. The multipliers are all 1. That pattern continues when A is 4 by 4:

$$\textbf{Special pattern} \quad A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}.$$

These LU examples are showing something extra, which is very important in practice. Assume no row exchanges. When can we predict *zeros* in L and U?

When a row of A starts with zeros, so does that row of L.

When a column of A starts with zeros, so does that column of U.

If a row starts with zero, we don't need an elimination step. L has a zero, which saves computer time. Similarly, zeros at the *start* of a column survive into U. But please realize: Zeros in the *middle* of a matrix are likely to be filled in, while elimination sweeps forward. We now explain why L has the multipliers ℓ_{ij} in position, with no mix-up.

The key reason why A equals LU: Ask yourself about the pivot rows that are subtracted from lower rows. Are they the original rows of A? *No*, elimination probably changed them. Are they rows of U? *Yes*, the pivot rows never change again. When computing the third row of U, we subtract multiples of earlier rows of U (*not rows of* A!):

$$\text{Row 3 of } U = (\text{Row 3 of } A) - \ell_{31}(\text{Row 1 of } U) - \ell_{32}(\text{Row 2 of } U). \tag{2}$$

Rewrite this equation to see that the row $[\,\ell_{31} \quad \ell_{32} \quad 1\,]$ is multiplying U:

$$(\text{Row 3 of } A) = \ell_{31}(\text{Row 1 of } U) + \ell_{32}(\text{Row 2 of } U) + 1(\text{Row 3 of } U). \tag{3}$$

This is exactly row **3** ***of*** $A = LU$. That row of L holds $\ell_{31}, \ell_{32}, 1$. All rows look like this, whatever the size of A. With no row exchanges, we have $A = LU$.

Better balance The LU factorization is "unsymmetric" because U has the pivots on its diagonal where L has 1's. This is easy to change. ***Divide U by a diagonal matrix D that contains the pivots.*** That leaves a new matrix with 1's on the diagonal:

$$\text{Split } U \text{ into } \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

It is convenient (but a little confusing) to keep the same letter U for this new upper triangular matrix. It has 1's on the diagonal (like L). Instead of the normal LU, the new form has D in the middle: ***Lower triangular L times diagonal D times upper triangular U.***

The triangular factorization can be written $\quad A = LU \quad$ ***or*** $\quad A = LDU.$

Whenever you see LDU, it is understood that U has 1's on the diagonal. *Each row is divided by its first nonzero entry—the pivot.* Then L and U are treated evenly in LDU:

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} \quad \text{splits further into} \quad \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & \\ & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

The pivots 2 and 5 went into D. Dividing the rows by 2 and 5 left the rows $[\,1 \quad 4\,]$ and $[\,0 \quad 1\,]$ in the new U with diagonal ones. The multiplier 3 is still in L.

My own lectures sometimes stop at this point. The next paragraphs show how elimination codes are organized, and how long they take. If MATLAB (or any software) is available, you can measure the computing time by just counting the seconds.

One Square System = Two Triangular Systems

The matrix L contains our memory of Gaussian elimination. It holds the numbers that multiplied the pivot rows, before subtracting them from lower rows. When do we need this record and how do we use it in solving $A\boldsymbol{x} = \boldsymbol{b}$?

We need L as soon as there is a *right side* $\boldsymbol{b}$. The factors L and U were completely decided by the left side (the matrix A). On the right side of $A\boldsymbol{x} = \boldsymbol{b}$, we use L^{-1} and then U^{-1}. That ***Solve*** step deals with two triangular matrices.

1 ***Factor*** (into L and U, by elimination on the left side matrix A)

2 ***Solve*** (forward elimination on $\boldsymbol{b}$ using L, then back substitution for $\boldsymbol{x}$ using U).

Earlier, we worked on A and $\boldsymbol{b}$ at the same time. No problem with that—just augment to $[A \quad \boldsymbol{b}]$. But most computer codes keep the two sides separate. The memory of elimination is held in L and U, to process $\boldsymbol{b}$ whenever we want to. The User's Guide to LAPACK remarks that "This situation is so common and the savings are so important that no provision has been made for solving a single system with just one subroutine."

How does ***Solve*** work on $\boldsymbol{b}$? First, apply forward elimination to the right side (the multipliers are stored in L, use them now). This changes $\boldsymbol{b}$ to a new right side $\boldsymbol{c}$. *We are really solving* $L\boldsymbol{c} = \boldsymbol{b}$. Then back substitution solves $U\boldsymbol{x} = \boldsymbol{c}$ as always. The original system $A\boldsymbol{x} = \boldsymbol{b}$ is factored into ***two triangular systems***:

$$\textbf{Forward and backward} \quad \textit{Solve} \quad L\boldsymbol{c} = \boldsymbol{b} \quad \textit{and then solve} \quad U\boldsymbol{x} = \boldsymbol{c}\,. \tag{5}$$

To see that $\boldsymbol{x}$ is correct, multiply $U\boldsymbol{x} = \boldsymbol{c}$ by L. Then $LU\boldsymbol{x} = L\boldsymbol{c}$ is just $A\boldsymbol{x} = \boldsymbol{b}$.

To emphasize: There is *nothing new* about those steps. This is exactly what we have done all along. We were really solving the triangular system $L\boldsymbol{c} = \boldsymbol{b}$ as elimination went forward. Then back substitution produced $\boldsymbol{x}$. An example shows what we actually did.

Example 3 Forward elimination (downward) on $A\boldsymbol{x} = \boldsymbol{b}$ ends at $U\boldsymbol{x} = \boldsymbol{c}$:

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} u + 2v &= 5 \\ v &= 1 \end{aligned} \qquad U\boldsymbol{x} = \boldsymbol{c}$$

The multiplier was 4, which is saved in L. The right side used it to change 21 to 1:

$$L\boldsymbol{c} = \boldsymbol{b} \quad \text{The lower triangular system} \quad \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \quad \text{gave} \quad \boldsymbol{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

$$U\boldsymbol{x} = \boldsymbol{c} \quad \text{The upper triangular system} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \text{gives} \quad \boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

L and U can go into the n^2 storage locations that originally held A (now forgettable).

The Cost of Elimination

A very practical question is cost—or computing time. We can solve 1000 equations on a PC. What if $n = 100{,}000$? (*Not if A is dense.*) Large systems come up all the time in scientific computing, where a three-dimensional problem can easily lead to a million unknowns. We can let the calculation run overnight, but we can't leave it for 100 years.

The first stage of elimination, on column 1, produces zeros below the first pivot. To find each new entry below the pivot row requires one multiplication and one subtraction. *We will count this first stage as n^2 multiplications and n^2 subtractions.* It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the second column below the second pivot. The working matrix is now of size $n-1$. Estimate this stage by $(n-1)^2$ multiplications and subtractions. The matrices are getting smaller as elimination goes forward. The rough count to reach U is the sum of squares $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$.

There is an exact formula $\frac{1}{3}n(n+\frac{1}{2})(n+1)$ for this sum of squares. When n is large, the $\frac{1}{2}$ and the 1 are not important. *The number that matters is* $\frac{1}{3}n^3$. The sum of squares is like the integral of x^2! The integral from 0 to n is $\frac{1}{3}n^3$:

Elimination on A requires about $\frac{1}{3}n^3$ ***multiplications and*** $\frac{1}{3}n^3$ ***subtractions.***

What about the right side $\boldsymbol{b}$? Going forward, we subtract multiples of b_1 from the lower components $b_2, \ldots, b_n$. This is $n-1$ steps. The second stage takes only $n-2$ steps, because b_1 is not involved. The last stage of forward elimination takes one step.

Now start back substitution. Computing x_n uses one step (divide by the last pivot). The next unknown uses two steps. When we reach x_1 it will require n steps ($n-1$ substitutions of the other unknowns, then division by the first pivot). The total count on the right side, from $\boldsymbol{b}$ to $\boldsymbol{c}$ to $\boldsymbol{x}$—*forward to the bottom and back to the top*—is exactly n^2:

$$[(n-1)+(n-2)+\cdots+1] \quad + \quad [1+2+\cdots+(n-1)+n] = n^2. \tag{6}$$

To see that sum, pair off $(n-1)$ with 1 and $(n-2)$ with 2. The pairings leave n terms, each equal to n. That makes n^2. The right side costs a lot less than the left side!

Solve ***Each right side needs*** n^2 ***multiplications and*** n^2 ***subtractions.***

A **band matrix** B has only w nonzero diagonals below and also above its main diagonal. The zero entries outside the band stay zero in elimination (zeros in L and U). Clearing out the first column needs w^2 multiplications and subtractions (w zeros to be produced below the pivot, each one using a pivot row of length w). Then clearing out all n columns, to reach U, needs no more than nw^2. This saves a lot of time:

Band matrices **Factor** change $\dfrac{1}{3}n^3$ to $\boldsymbol{nw^2}$ **Solve** change n^2 to $\boldsymbol{2nw}$

Here are codes to factor A into LU and to solve $A\boldsymbol{x} = \boldsymbol{b}$. The Teaching code **slu** stops right away if a number smaller than the tolerance "*tol*" appears in a pivot position. The Teaching Codes are on **web.mit.edu/18.06/www**. Professional codes will look down each column for the largest available pivot, to exchange rows and continue solving.

MATLAB's backslash command $\boldsymbol{x} = A\backslash\boldsymbol{b}$ combines **Factor** and **Solve** to reach $\boldsymbol{x}$.

```
function [L, U] = slu(A)
%    Square LU factorization with no row exchanges!
[n, n] = size(A);   tol = 1.e − 6;
for k = 1 : n
  if abs(A(k, k)) < tol
  end      % Cannot proceed without a row exchange: stop
  L(k, k) = 1;
  for i = k + 1 : n
    L(i, k) = A(i, k)/A(k, k);   % Multipliers for column k are put into L
    for j = k + 1 : n   % Elimination beyond row k and column k
      A(i, j) = A(i, j) − L(i, k) * A(k, j);   % Matrix still called A
    end
  end
  for j = k : n
     U(k, j) = A(k, j);      % row k is settled, now name it U
  end
end
```

```
function x = slv(A, b)
%    Solve Ax = b using L and U from slu(A).
[L, U] = slu(A); s = 0;   % No row exchanges!
for k = 1 : n   % Forward elimination to solve Lc = b
  for j = 1 : k − 1
    s = s + L(k, j) * c(j);   % Add L times earlier c(j) before c(k)
  end
  c(k) = b(k) − s; s = 0;   % Find c(k) and reset s for next k
end
for k = n : −1 : 1   % Going backwards from x(n) to x(1)
  for j = k + 1 : n   % Back substitution
    t = t + U(k, j) * x(j);   % U times later x(j)
  end
  x(k) = (c(k) − t)/U(k, k);   % Divide by pivot
end
x = x';   % Transpose to column vector
```

How long does it take to solve $Ax = b$? For a random matrix of order $n = 1000$, a typical time is 1 second. See **web.mit.edu/18.06** and **math.mit.edu/linearalgebra** for the times in MATLAB, Maple, Mathematica, SciLab, Python, and R. The time is multiplied by about 8 when n is multiplied by 2. For professional codes go to **netlib.org**.

According to this n^3 rule, matrices that are 10 times as large (order 10,000) will take a thousand seconds. Matrices of order 100,000 will take a million seconds. This is too expensive without a supercomputer, but remember that these matrices are full. Most matrices in practice are sparse (many zero entries). In that case $A = LU$ is much faster.

For tridiagonal matrices of order 10,000, storing only the nonzeros, solving $Ax = b$ is a breeze. Provided the code recognizes that A is tridiagonal.

■ REVIEW OF THE KEY IDEAS ■

1. Gaussian elimination (with no row exchanges) factors A into L times U.
2. The lower triangular L contains the numbers ℓ_{ij} that multiply pivot rows, going from A to U. The product LU adds those rows back to recover A.
3. On the right side we solve $Lc = b$ (forward) and $Ux = c$ (backward).
4. ***Factor*** : There are $\frac{1}{3}(n^3 - n)$ multiplications and subtractions on the left side.
5. ***Solve*** : There are n^2 multiplications and subtractions on the right side.
6. For a band matrix, change $\frac{1}{3}n^3$ to nw^2 and change n^2 to $2wn$.

■ WORKED EXAMPLES ■

2.6 A The lower triangular Pascal matrix L contains the famous "*Pascal triangle*". Gauss-Jordan found its inverse in the worked example **2.5 C**. This problem connects L to the *symmetric* Pascal matrix P and the upper triangular U. The symmetric P has Pascal's triangle tilted, so each entry is the sum of the entry above and the entry to the left. The n by n symmetric P is pascal(n) in MATLAB.

Problem: *Establish the amazing lower-upper factorization* $P = LU$.

$$\text{pascal(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 0 & 0 & \mathbf{1} & \mathbf{3} \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix} = LU.$$

Then predict and check the next row and column for 5 by 5 Pascal matrices.

Solution You could multiply LU to get P. Better to start with the symmetric P and reach the upper triangular U by elimination:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U.$$

The multipliers ℓ_{ij} that entered these steps go perfectly into L. Then $P = LU$ is a particularly neat example. *Notice that every pivot is* 1 on the diagonal of U.

The next section will show how symmetry produces a special relationship between the triangular L and U. For Pascal, U is the "**transpose**" of L.

You might expect the MATLAB command lu(pascal(4)) to produce these L and U. That doesn't happen because the **lu** subroutine chooses the largest available pivot in each column. The second pivot will change from 1 to 3. But a "Cholesky factorization" does no row exchanges: $U =$ chol(pascal(4))

The full proof of $P = LU$ for all Pascal sizes is quite fascinating. The paper "*Pascal Matrices*" is on the course web page **web.mit.edu/18.06** which is also available through MIT's *OpenCourseWare* at **ocw.mit.edu**. These Pascal matrices have so many remarkable properties—we will see them again.

2.6 B The problem is: *Solve* $Px = b = (1,0,0,0)$. This right side $=$ column of I means that x will be the first column of P^{-1}. That is Gauss-Jordan, matching the columns of $PP^{-1} = I$. We already know the Pascal matrices L and U as factors of P:

Two triangular systems $\quad Lc = b$ (forward) $\quad Ux = c$ (back).

Solution The lower triangular system $Lc = b$ is solved *top to bottom*:

$$\begin{aligned} c_1 &= 1 \\ c_1 + c_2 &= 0 \\ c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 3c_2 + 3c_3 + c_4 &= 0 \end{aligned} \quad \text{gives} \quad \begin{aligned} c_1 &= +1 \\ c_2 &= -1 \\ c_3 &= +1 \\ c_4 &= -1 \end{aligned}$$

Forward elimination is multiplication by L^{-1}. It produces the upper triangular system $Ux = c$. The solution x comes as always by back substitution, *bottom to top*:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_2 + 2x_3 + 3x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ x_4 &= -1 \end{aligned} \quad \text{gives} \quad \begin{aligned} x_1 &= +4 \\ x_2 &= -6 \\ x_3 &= +4 \\ x_4 &= -1 \end{aligned}$$

I see a pattern in that x, but I don't know where it comes from. Try **inv(pascal**(4)).

Problem Set 2.6

Problems 1–14 compute the factorization $A = LU$ (and also $A = LDU$).

1 (Important) Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}x = b$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x = c$:

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + y &= 5 \\ y &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

That step subtracted $\ell_{21} =$ ____ times row 1 from row 2. The reverse step *adds* ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L =$ ____. Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get ____ $=$ ____. In letters, L multiplies $Ux = c$ to give ____.

2 Write down the 2 by 2 triangular systems $Lc = b$ and $Ux = c$ from Problem 1. Check that $c = (5,2)$ solves the first one. Find x that solves the second one.

3 (Move to 3 by 3) Forward elimination changes $A\boldsymbol{x} = \boldsymbol{b}$ to a triangular $U\boldsymbol{x} = \boldsymbol{c}$:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned} \qquad \begin{aligned} x + y + z &= 5 \\ y + 2z &= 2 \\ 2y + 5z &= 6 \end{aligned} \qquad \begin{aligned} x + y + z &= 5 \\ y + 2z &= 2 \\ z &= 2 \end{aligned}$$

The equation $z = 2$ in $U\boldsymbol{x} = \boldsymbol{c}$ comes from the original $x + 3y + 6z = 11$ in $A\boldsymbol{x} = \boldsymbol{b}$ by subtracting $\ell_{31} =$ ____ times equation 1 and $\ell_{32} =$ ____ times the *final* equation 2. Reverse that to recover $[\,1 \;\; 3 \;\; 6 \;\; 11\,]$ in the last row of A and $\boldsymbol{b}$ from the final $[\,1 \;\; 1 \;\; 1 \;\; 5\,]$ and $[\,0 \;\; 1 \;\; 2 \;\; 2\,]$ and $[\,0 \;\; 0 \;\; 1 \;\; 2\,]$ in U and $\boldsymbol{c}$:

$$\text{Row 3 of } \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = (\ell_{31} \text{ Row 1} + \ell_{32} \text{ Row 2} + 1 \text{ Row 3}) \text{ of } \begin{bmatrix} U & \boldsymbol{c} \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $\boldsymbol{b} = L\boldsymbol{c}$.

4 What are the 3 by 3 triangular systems $L\boldsymbol{c} = \boldsymbol{b}$ and $U\boldsymbol{x} = \boldsymbol{c}$ from Problem 3? Check that $\boldsymbol{c} = (5, 2, 2)$ solves the first one. Which $\boldsymbol{x}$ solves the second one?

5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

6 What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

7 What three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into L times U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}.$$

8 Suppose A is already lower triangular with 1's on the diagonal. Then $U = I$!

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

(a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.

(b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L (nicer than E).

9 *When zero appears in a pivot position, $A = LU$ is not possible!* (We are requiring nonzero pivots in U.) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \ell & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ & f & h \\ & & i \end{bmatrix}.$$

This difficulty is fixed by a row exchange. That needs a "permutation" P.

10 Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

11 What are L and D (the diagonal ***pivot matrix***) for this matrix A? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$\textbf{Already triangular} \qquad A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

12 A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$\textbf{Symmetric} \qquad A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

13 (*Recommended*) Compute L and U for the symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

14 This nonsymmetric matrix will have the same L as in Problem **13**:

$$\textbf{Find } L \textbf{ and } U \textbf{ for} \qquad A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Problems 15-16 use L and U (without needing A) to solve $Ax = b$.

15 Solve the triangular system $Lc = b$ to find c. Then solve $Ux = c$ to find x:

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety multiply LU and solve $Ax = b$ as usual. Circle c when you see it.

16 Solve $Lc = b$ to find c. Then solve $Ux = c$ to find x. **What was** A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

17 (a) When you apply the usual elimination steps to L, what matrix do you reach?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I, what matrix do you get?

(c) When you apply the same steps to LU, what matrix do you get?

18 If $A = LDU$ and also $A = L_1D_1U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. *"The three factors are unique."*

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal 1's). Then $D = D_1$.

19 *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDL^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

20 When T is tridiagonal, its L and U factors have only two nonzero diagonals. How would you take advantage of knowing the zeros in T, in a code for Gaussian elimination? Find L and U.

Tridiagonal $$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

21 If A and B have nonzeros in the positions marked by x, which zeros (marked by 0) *stay zero* in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

22 Suppose you eliminate upwards (almost unheard of). Use the last row to produce zeros in the last column (the pivot is 1). Then use the second row to produce zero above the second pivot. Find the factors in the unusual order $A = UL$.

$$\textbf{Upper times lower} \qquad A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

23 *Easy but important.* If A has pivots $5, 9, 3$ with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix A_2 (without row 3 and column 3)?

Challenge Problems

24 Which invertible matrices allow $A = LU$ (elimination without row exchanges)? *Good question*! Look at each of the square upper left submatrices of A.

All upper left k ***by*** k ***submatrices*** A_k ***must be invertible*** (sizes $k = 1, \ldots, n$).

Explain that answer: A_k factors into _____ because $LU = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}$.

25 For the 6 by 6 second difference constant-diagonal matrix K, put the pivots and multipliers into $K = LU$. (L and U will have only two nonzero diagonals, because K has three.) Find a formula for the i, j entry of L^{-1}, by software like MATLAB using inv(L) or by looking for a nice pattern.

$$\mathbf{-1, 2, -1 \text{ matrix}} \quad K = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -1 \\ & & & & -1 & 2 \end{bmatrix} = \text{toeplitz}([2 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0])$$

26 If you print K^{-1}, it doesn't look so good. But if you print $7K^{-1}$ (when K is 6 by 6), that matrix looks wonderful. Write down $7K^{-1}$ by hand, following this pattern:

1 Row 1 and column 1 are $(6, 5, 4, 3, 2, 1)$.

2 On and above the main diagonal, row i is i times row 1.

3 On and below the main diagonal, column j is j times column 1.

Multiply K times that $7K^{-1}$ to produce $7I$. Here is that pattern for $n = 3$:

3 by 3 case
The determinant
of this K is 4

$$(K)(4K^{-1}) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix}.$$

2.7 Transposes and Permutations

We need one more matrix, and fortunately it is much simpler than the inverse. It is the ***"transpose"*** of A, which is denoted by A^{T}. *The columns of A^{T} are the rows of A.*

When A is an m by n matrix, the transpose is n by m:

Transpose If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ then $A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$.

You can write the rows of A into the columns of A^{T}. Or you can write the columns of A into the rows of A^{T}. The matrix "flips over" its main diagonal. The entry in row i, column j of A^{T} comes from row j, column i of the original A:

Exchange rows and columns $(A^{\mathrm{T}})_{ij} = A_{ji}$.

The transpose of a lower triangular matrix is upper triangular. (But the inverse is still lower triangular.) The transpose of A^{T} is A.

Note MATLAB's symbol for the transpose of A is A'. Typing $[1\ \ 2\ \ 3]$ gives a row vector and the column vector is $\boldsymbol{v} = [1\ \ 2\ \ 3]'$. To enter a matrix M with second column $\boldsymbol{w} = [\ 4\ \ 5\ \ 6\ ]'$ you could define $M = [\ \boldsymbol{v}\ \ \boldsymbol{w}\ ]$. Quicker to enter by rows and then transpose the whole matrix: $M = [1\ \ 2\ \ 3;\ \ 4\ \ 5\ \ 6]'$.

The rules for transposes are very direct. We can transpose $A+B$ to get $(A+B)^{\mathrm{T}}$. Or we can transpose A and B separately, and then add $A^{\mathrm{T}} + B^{\mathrm{T}}$—with the same result. The serious questions are about the transpose of a product AB and an inverse A^{-1}:

Sum The transpose of $A+B$ is $A^{\mathrm{T}} + B^{\mathrm{T}}$. (1)

Product The transpose of AB is $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. (2)

Inverse The transpose of A^{-1} is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$. (3)

Notice especially how $B^{\mathrm{T}}A^{\mathrm{T}}$ comes in reverse order. For inverses, this reverse order was quick to check: $B^{-1}A^{-1}$ times AB produces I. To understand $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, start with $(A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$:

$A\boldsymbol{x}$ ***combines the columns of*** A ***while*** $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$ ***combines the rows of*** A^{T}.

It is the same combination of the same vectors! In A they are columns, in A^{T} they are rows. So the transpose of the column $A\boldsymbol{x}$ is the row $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$. That fits our formula $(A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$. Now we can prove the formula $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, when B has several columns.

If $B = [\boldsymbol{x}_1\ \ \boldsymbol{x}_2]$ has two columns, apply the same idea to each column. The columns of AB are $A\boldsymbol{x}_1$ and $A\boldsymbol{x}_2$. Their transposes are the rows of $B^{\mathrm{T}}A^{\mathrm{T}}$:

$$\text{Transposing } AB = \begin{bmatrix} \\ A\boldsymbol{x}_1 & A\boldsymbol{x}_2 & \cdots \\ \\ \end{bmatrix} \text{ gives } \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}}A^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}}A^{\mathrm{T}} \\ \vdots \end{bmatrix} \text{ which is } B^{\mathrm{T}}A^{\mathrm{T}}. \tag{4}$$

The right answer $B^{\mathrm{T}}A^{\mathrm{T}}$ comes out a row at a time. Here are numbers in $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$:

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{9} & \mathbf{1} \end{bmatrix} \quad \text{and} \quad B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{9} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

The reverse order rule extends to three or more factors: $(ABC)^{\mathrm{T}}$ equals $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}$.

If $A = LDU$ ***then*** $A^{\mathrm{T}} = U^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}}$. ***The pivot matrix has*** $D = D^{\mathrm{T}}$.

Now apply this product rule to both sides of $A^{-1}A = I$. On one side, I^{T} is I. We confirm the rule that $(A^{-1})^{\mathrm{T}}$ *is the inverse of* A^{T}, because their product is I:

Transpose of inverse $\quad A^{-1}A = I \quad$ is transposed to $\quad A^{\mathrm{T}}(A^{-1})^{\mathrm{T}} = I. \qquad (5)$

Similarly $AA^{-1} = I$ leads to $(A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I$. We can invert the transpose or we can transpose the inverse. Notice especially: A^{T} ***is invertible exactly when*** A ***is invertible.***

Example 1 The inverse of $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$. The transpose is $A^{\mathrm{T}} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$.

$$(A^{-1})^{\mathrm{T}} \quad \textit{and} \quad (A^{\mathrm{T}})^{-1} \quad \textit{are both equal to} \quad \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}.$$

The Meaning of Inner Products

We know the dot product (inner product) of $\boldsymbol{x}$ and $\boldsymbol{y}$. It is the sum of numbers $x_i y_i$. Now we have a better way to write $\boldsymbol{x} \cdot \boldsymbol{y}$, without using that unprofessional dot. Use matrix notation instead:

$^{\mathrm{T}}$ **is inside**	***The dot product or inner product is*** $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$	$(1 \times n)(n \times 1)$
$^{\mathrm{T}}$ **is outside**	***The rank one product or outer product is*** $\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$	$(n \times 1)(1 \times n)$

$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ is a number, $\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ is a matrix. Quantum mechanics would write those as $< \boldsymbol{x}|\boldsymbol{y} >$ (inner) and $|\boldsymbol{x} >< \boldsymbol{y}|$ (outer). I think the world is governed by linear algebra, but physics disguises it well. Here are examples where the inner product has meaning:

From mechanics	Work = (Movements) (Forces) = $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{f}$
From circuits	Heat loss = (Voltage drops) (Currents) = $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{y}$
From economics	Income = (Quantities) (Prices) = $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}$

We are really close to the heart of applied mathematics, and there is one more point to explain. It is the deeper connection between inner products and the transpose of A.

We defined A^{T} by flipping the matrix across its main diagonal. That's not mathematics. There is a better way to approach the transpose. A^{T} *is the matrix that makes these two inner products equal for every* $\boldsymbol{x}$ *and* $\boldsymbol{y}$:

$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) \qquad$ **Inner product of** $A\boldsymbol{x}$ **with** $\boldsymbol{y}$ **= Inner product of** $\boldsymbol{x}$ **with** $A^{\mathrm{T}}\boldsymbol{y}$

Example 2 Start with $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

On one side we have $\boldsymbol{Ax}$ multiplying $\boldsymbol{y}$: $(x_2 - x_1)y_1 + (x_3 - x_2)y_2$
That is the same as $x_1(-y_1) + x_2(y_1 - y_2) + x_3(y_2)$. Now $\boldsymbol{x}$ is multiplying $A^{\mathrm{T}}\boldsymbol{y}$.

$$A^{\mathrm{T}}y \text{ must be } \begin{bmatrix} -y_1 \\ y_1 - y_2 \\ y_2 \end{bmatrix} \text{ which produces } A^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ as expected.}$$

Example 3 Will you allow me a little calculus? It is extremely important or I wouldn't leave linear algebra. (This is really linear algebra for functions $x(t)$.) **The difference matrix changes to a derivative** $A = d/dt$. Its transpose will now come from $(dx/dt, y) = (x, -dy/dt)$.

The inner product changes from a finite sum of $x_k y_k$ to an integral of $x(t)y(t)$.

$$\textbf{Inner product of functions} \qquad x^{\mathrm{T}}y = (x,y) = \int_{-\infty}^{\infty} x(t)\,y(t)\,dt \quad \text{by definition}$$

$$\textbf{Transpose rule} \quad (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y) \qquad \int_{-\infty}^{\infty} \frac{dx}{dt}y(t)\,dt = \int_{-\infty}^{\infty} x(t)\left(-\frac{dy}{dt}\right)dt \quad \text{shows } A^{\mathrm{T}} \tag{6}$$

I hope you recognize ***"integration by parts"***. The derivative moves from the first function $x(t)$ to the second function $y(t)$. During that move, a minus sign appears. This tells us that ***the "transpose" of the derivative is minus the derivative.***

The derivative is *anti-symmetric*: $\boldsymbol{A = d/dt}$ ***and*** $\boldsymbol{A^{\mathrm{T}} = -d/dt}$. Symmetric matrices have $A^{\mathrm{T}} = A$, anti-symmetric matrices have $A^{\mathrm{T}} = -A$. In some way, the 2 by 3 difference matrix above followed this pattern. The 3 by 2 matrix A^{T} was *minus* a difference matrix. It produced $y_1 - y_2$ in the middle component of $A^{\mathrm{T}}\boldsymbol{y}$ instead of the difference $y_2 - y_1$.

Symmetric Matrices

For a *symmetric matrix*, transposing A to A^{T} produces no change. Then $A^{\mathrm{T}} = A$. Its (j,i) entry across the main diagonal equals its (i,j) entry. In my opinion, these are the most important matrices of all.

DEFINITION A *symmetric matrix* has $A^{\mathrm{T}} = A$. This means that $a_{ji} = a_{ij}$.

$$\textbf{Symmetric matrices} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^{\mathrm{T}} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^{\mathrm{T}}.$$

The inverse of a symmetric matrix is also symmetric. The transpose of A^{-1} is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1}$. That says A^{-1} is symmetric (when A is invertible):

$$\textbf{Symmetric inverses} \qquad A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Now we produce symmetric matrices by ***multiplying any matrix R by $\boldsymbol{R^{\mathrm{T}}}$***.

Symmetric Products $R^{\mathrm{T}}R$ and RR^{T} and LDL^{T}

Choose any matrix R, probably rectangular. Multiply R^{T} times R. Then the product $R^{\mathrm{T}}R$ is automatically a square symmetric matrix:

$$\textit{The transpose of}\quad R^{\mathrm{T}}R \quad \textit{is}\quad R^{\mathrm{T}}(R^{\mathrm{T}})^{\mathrm{T}} \quad \textit{which is}\quad R^{\mathrm{T}}R. \tag{7}$$

That is a quick proof of symmetry for $R^{\mathrm{T}}R$. We could also look at the (i, j) entry of $R^{\mathrm{T}}R$. It is the dot product of row i of R^{T} (column i of R) with column j of R. The (j, i) entry is the same dot product, column j with column i. So $R^{\mathrm{T}}R$ is symmetric.

The matrix RR^{T} is also symmetric. (The shapes of R and R^{T} allow multiplication.) But RR^{T} is a different matrix from $R^{\mathrm{T}}R$. In our experience, most scientific problems that start with a rectangular matrix R end up with $R^{\mathrm{T}}R$ or RR^{T} or both. As in least squares.

Example 4 Multiply $R = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $R^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ in both orders.

$$RR^{\mathrm{T}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } R^{\mathrm{T}}R = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ are both symmetric matrices.}$$

The product $R^{\mathrm{T}}R$ is n by n. In the opposite order, RR^{T} is m by m. Both are symmetric, with positive diagonal (*why*?). But even if $m = n$, it is not very likely that $R^{\mathrm{T}}R = RR^{\mathrm{T}}$. Equality can happen, but it is abnormal.

Symmetric matrices in elimination $A^{\mathrm{T}} = A$ makes elimination faster, because we can work with half the matrix (plus the diagonal). It is true that the upper triangular U is probably not symmetric. ***The symmetry is in the triple product*** $A = LDU$. Remember how the diagonal matrix D of pivots can be divided out, to leave 1's on the diagonal of both L and U:

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad LU \text{ misses the symmetry of } A$$

$$= \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 0 & \mathbf{1} \end{bmatrix} \qquad \begin{array}{l} LDU \text{ captures the symmetry} \\ \textit{Now } U \textit{ is the transpose of } L. \end{array}$$

When A is symmetric, the usual form $A = LDU$ becomes $A = \boldsymbol{LDL^{\mathrm{T}}}$. The final U (with 1's on the diagonal) is the transpose of L (also with 1's on the diagonal). The diagonal matrix D containing the pivots is symmetric by itself.

If $A = A^{\mathrm{T}}$ is factored into LDU with no row exchanges, then U is exactly L^{T}.

The symmetric factorization of a symmetric matrix is $A = LDL^{\mathrm{T}}$.

Notice that the transpose of LDL^{T} is automatically $(L^{\mathrm{T}})^{\mathrm{T}}D^{\mathrm{T}}L^{\mathrm{T}}$ which is LDL^{T} again. The work of elimination is cut in half, from $n^3/3$ multiplications to $n^3/6$. The storage is also cut essentially in half. We only keep L and D, not U which is just L^{T}.

Permutation Matrices

The transpose plays a special role for a *permutation matrix*. This matrix P has a single "1" in every row and every column. Then P^{T} is also a permutation matrix—maybe the same or maybe different. Any product $P_1 P_2$ is again a permutation matrix. We now create every P from the identity matrix, by reordering the rows of I.

The simplest permutation matrix is $P = I$ (*no exchanges*). The next simplest are the row exchanges P_{ij}. Those are constructed by exchanging two rows i and j of I. Other permutations reorder more rows. By doing all possible row exchanges to I, we get all possible permutation matrices:

DEFINITION ***A permutation matrix P has the rows of the identity I in any order.***

Example 5 There are six 3 by 3 permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There are $n!$ permutation matrices of order n. The symbol $n!$ means "n factorial," the product of the numbers $(1)(2)\cdots(n)$. Thus $3! = (1)(2)(3)$ which is 6. There will be 24 permutation matrices of order $n = 4$. And 120 permutations of order 5.

There are only two permutation matrices of order 2, namely $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Important: P^{-1} is also a permutation matrix. Among the six 3 by 3 P's displayed above, the four matrices on the left are their own inverses. The two matrices on the right are inverses of each other. In all cases, a single row exchange is its own inverse. If we repeat the exchange we are back to I. But for $P_{32}P_{21}$, the inverses go in opposite order as always. The inverse is $P_{21}P_{32}$.

More important: $\boldsymbol{P^{-1}}$ ***is always the same as*** $\boldsymbol{P^{\mathrm{T}}}$**.** The two matrices on the right are transposes—and inverses—of each other. When we multiply PP^{T}, the "1" in the first row of P hits the "1" in the first column of P^{T} (since the first row of P is the first column of P^{T}). It misses the ones in all the other columns. So $PP^{\mathrm{T}} = I$.

Another proof of $P^{\mathrm{T}} = P^{-1}$ looks at P as a product of row exchanges. Every row exchange is its own transpose and its own inverse. P^{T} and P^{-1} both come from the product of row exchanges *in reverse order*. So P^{T} and P^{-1} are the same.

Symmetric matrices led to $A = LDL^{\mathrm{T}}$. Now permutations lead to $PA = LU$.

The $PA = LU$ Factorization with Row Exchanges

We sure hope you remember $A = LU$. It started with $A = (E_{21}^{-1} \cdots E_{ij}^{-1} \cdots)U$. Every elimination step was carried out by an E_{ij} and it was inverted by E_{ij}^{-1}. Those inverses were compressed into one matrix L, bringing U back to A. The lower triangular L has 1's on the diagonal, and the result is $A = LU$.

This is a great factorization, but it doesn't always work. Sometimes row exchanges are needed to produce pivots. Then $A = (E^{-1} \cdots P^{-1} \cdots E^{-1} \cdots P^{-1} \cdots)U$. Every row exchange is carried out by a P_{ij} and inverted by that P_{ij}. We now compress those row exchanges into a *single permutation matrix* P. This gives a factorization for every invertible matrix A—which we naturally want.

The main question is where to collect the P_{ij}'s. There are two good possibilities—do all the exchanges before elimination, or do them after the E_{ij}'s. The first way gives $PA = LU$. The second way has a permutation matrix P_1 in the middle.

1. The row exchanges can be done *in advance*. Their product P puts the rows of A in the right order, so that no exchanges are needed for PA. ***Then*** $\boldsymbol{PA = LU}$.

2. If we hold row exchanges until *after elimination*, the pivot rows are in a strange order. P_1 puts them in the correct triangular order in U_1. ***Then*** $\boldsymbol{A = L_1 P_1 U_1}$.

$PA = LU$ is constantly used in all computing (and in MATLAB). ***We will concentrate on this form.*** Most numerical analysts have never seen the other form.

The factorization $A = L_1 P_1 U_1$ might be more elegant. If we mention both, it is because the difference is not well known. Probably you will not spend a long time on either one. Please don't. The most important case has $P = I$, when A equals LU with no exchanges.

For this matrix A, exchange rows 1 and 2 to put the first pivot in its usual place. Then go through elimination on PA:

$$\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{\ell_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}} \rightarrow \underset{\ell_{32} = 3}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}.$$

The matrix PA has its rows in good order, and it factors as usual into LU:

$$\boldsymbol{P} = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \quad PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU. \tag{8}$$

We started with A and ended with U. *The only requirement is invertibility of* A.

> If A is invertible, a permutation P will put its rows in the right order to factor $\boldsymbol{PA = LU}$. There must be a full set of pivots after row exchanges for A to be invertible.

In MATLAB, $A([r\ k],:) = A([k\ r],:)$ exchanges row k with row r below it (where the kth pivot has been found). Then the **lu** code updates L and P and the sign of P:

This is part of $[L, U, P] = \mathsf{lu}(A)$

$$\begin{aligned} &A([r\ k],:) = A([k\ r],:);\\ &L([r\ k], 1:k-1) = L([k\ r], 1:k-1);\\ &P([r\ k],:) = P([k\ r],:);\\ &\text{sign} = -\text{sign} \end{aligned}$$

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$). An odd number of row exchanges will produce sign $= -1$. At the start, P is I and sign $= +1$. When there is a row exchange, the sign is reversed. The final value of sign is the **determinant of** P and it does not depend on the order of the row exchanges.

For PA we get back to the familiar LU. This is the usual factorization. In reality, $\mathsf{lu}(A)$ often does not use the first available pivot. Mathematically we accept a small pivot—anything but zero. It is better if the computer looks down the column for the largest pivot. (Section 9.1 explains why this "***partial pivoting***" reduces the roundoff error.) Then P may contain row exchanges that are not algebraically necessary. Still $PA = LU$.

Our advice is to understand permutations but let the computer do the work. Calculations of $A = LU$ are enough to do by hand, without P. The Teaching Code splu(A) factors $PA = LU$ and splv(A, b) solves $Ax = b$ for any invertible A. The program splu stops if no pivot can be found in column k. Then A is not invertible.

■ REVIEW OF THE KEY IDEAS ■

1. The transpose puts the rows of A into the columns of A^{T}. Then $(A^{\mathrm{T}})_{ij} = A_{ji}$.

2. The transpose of AB is $B^{\mathrm{T}}A^{\mathrm{T}}$. The transpose of A^{-1} is the inverse of A^{T}.

3. The dot product is $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$. Then $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}$ equals the dot product $\boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$.

4. When A is symmetric ($A^{\mathrm{T}} = A$), its LDU factorization is symmetric: $A = LDL^{\mathrm{T}}$.

5. A permutation matrix P has a 1 in each row and column, and $\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}^{-1}$.

6. There are $n!$ permutation matrices of size n. *Half even, half odd.*

7. If A is invertible then a permutation P will reorder its rows for $PA = LU$.

■ WORKED EXAMPLES ■

2.7 A Applying the permutation P to the rows of A destroys its symmetry:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & \mathbf{2} & 6 \\ 5 & 6 & \mathbf{3} \end{bmatrix} \qquad PA = \begin{bmatrix} 4 & \mathbf{2} & 6 \\ 5 & 6 & \mathbf{3} \\ 1 & 4 & 5 \end{bmatrix}$$

What permutation Q applied to the *columns* of PA will recover symmetry in PAQ? The numbers $1, 2, 3$ must come back to the main diagonal (not necessarily in order). Show that Q is P^{T}, so that symmetry is saved by $PAQ = PAP^{\mathrm{T}}$.

Solution To recover symmetry and put "2" back on the diagonal, column 2 of PA must move to column 1. Column 3 of PA (containing "3") must move to column 2. Then the "1" moves to the $3, 3$ position. The matrix that permutes columns is Q:

$$PA = \begin{bmatrix} 4 & \mathbf{2} & 6 \\ 5 & 6 & \mathbf{3} \\ 1 & 4 & 5 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad PAQ = \begin{bmatrix} \mathbf{2} & 6 & 4 \\ 6 & \mathbf{3} & 5 \\ 4 & 5 & 1 \end{bmatrix} \text{ is symmetric.}$$

The matrix Q is P^{T}. This choice always recovers symmetry, because PAP^{T} is guaranteed to be symmetric. (Its transpose is again PAP^{T}.) *The matrix Q is also P^{-1}, because the inverse of every permutation matrix is its transpose.*

If D is a diagonal matrix, we are finding that PDP^{T} is also diagonal. When P moves row 1 down to row 3, P^{T} on the right will move column 1 to column 3. The $(1, 1)$ entry moves down to $(3, 1)$ and over to $(3, 3)$.

2.7 B Find the symmetric factorization $A = LDL^{\mathrm{T}}$ for the matrix A above. Is this A invertible? Find also the $PQ = LU$ factorization for Q, which needs row exchanges.

Solution To factor A into LDL^{T} we eliminate below the pivots:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = U.$$

The multipliers were $\ell_{21} = 4$ and $\ell_{31} = 5$ and $\ell_{32} = 1$. The pivots $1, -14, -8$ go into D. When we divide the rows of U by those pivots, L^{T} should appear:

Symmetric factorization when $A = A^{\mathrm{T}}$

$$A = LDL^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -14 & \\ & & -8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix A is invertible because *it has three pivots*. Its inverse is $(L^{\mathrm{T}})^{-1}D^{-1}L^{-1}$ and A^{-1} is also symmetric. The numbers 14 and 8 will turn up in the denominators of A^{-1}. The "determinant" of A is the product of the pivots $(1)(-14)(-8) = 112$.

Any permutation matrix Q is invertible. Here elimination needs two row exchanges:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[1 \leftrightarrow 2]{\text{rows}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[2 \leftrightarrow 3]{\text{rows}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

With $A = Q$, the $PQ = (L)(U)$ factorization is the same as $Q^{-1}Q = (I)(I)$.

2.7 C For a rectangular A, this ***saddle-point matrix*** S is symmetric and important:

Block matrix from least squares $\quad S = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{A}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix} = S^{\mathrm{T}}$ has size $m + n$.

Apply block elimination to find a **block factorization** $S = LDL^{\mathrm{T}}$. Then test invertibility:

$$\boldsymbol{S \text{ is invertible}} \iff \boldsymbol{A^{\mathrm{T}}A \text{ is invertible}} \iff \boldsymbol{Ax \neq 0 \text{ whenever } x \neq 0}$$

Solution The first block pivot is I. The matrix to multiply row 1 is certainly A^{T}:

Block elimination $\quad S = \begin{bmatrix} I & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$ goes to $\begin{bmatrix} I & A \\ 0 & -A^{\mathrm{T}}A \end{bmatrix}$. This is U.

The block pivot matrix D contains I and $-A^{\mathrm{T}}A$. Then L and L^{T} contain A^{T} and A:

Block factorization $\quad S = LDL^{\mathrm{T}} = \begin{bmatrix} I & 0 \\ A^{\mathrm{T}} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -A^{\mathrm{T}}A \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}.$

L is certainly invertible, with diagonal 1's from I. The inverse of the middle matrix involves $(A^{\mathrm{T}}A)^{-1}$. Section 4.2 answers a key question about the matrix $A^{\mathrm{T}}A$:

When is $A^{\mathrm{T}}A$ invertible? *Answer: A must have independent columns.*
Then $Ax = 0$ only if $x = 0$. Otherwise $Ax = 0$ will lead to $A^{\mathrm{T}}Ax = 0$.

Problem Set 2.7

Questions 1–7 are about the rules for transpose matrices.

1 Find A^{T} and A^{-1} and $(A^{-1})^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

2 Verify that $(AB)^{\mathrm{T}}$ equals $B^{\mathrm{T}}A^{\mathrm{T}}$ but those are different from $A^{\mathrm{T}}B^{\mathrm{T}}$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!) how do you prove that $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$?

3 (a) The matrix $((AB)^{-1})^{\mathrm{T}}$ comes from $(A^{-1})^{\mathrm{T}}$ and $(B^{-1})^{\mathrm{T}}$. *In what order*?

(b) If U is upper triangular then $(U^{-1})^{\mathrm{T}}$ is ____ triangular.

4 Show that $A^2 = 0$ is possible but $A^{\mathrm{T}}A = 0$ is not possible (unless $A =$ zero matrix).

5 (a) The row vector $\boldsymbol{x}^{\mathrm{T}}$ times A times the column $\boldsymbol{y}$ produces what number?

$$\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \_\_\_\_ .$$

(b) This is the row $\boldsymbol{x}^{\mathrm{T}}A =$ ____ times the column $\boldsymbol{y} = (0, 1, 0)$.

(c) This is the row $\boldsymbol{x}^{\mathrm{T}} = [0 \quad 1]$ times the column $A\boldsymbol{y} =$ ____.

6 The transpose of a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is $M^{\mathrm{T}} =$ ____. Test an example. Under what conditions on A, B, C, D is the block matrix symmetric?

7 True or false:

(a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric.

(b) If A and B are symmetric then their product AB is symmetric.

(c) If A is not symmetric then A^{-1} is not symmetric.

(d) When A, B, C are symmetric, the transpose of ABC is CBA.

Questions 8–15 are about permutation matrices.

8 Why are there $n!$ permutation matrices of order n?

9 If P_1 and P_2 are permutation matrices, so is P_1P_2. This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.

10 There are 12 "*even*" permutations of $(1, 2, 3, 4)$, with an *even number of exchanges*. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, just order the numbers.

11 Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? ***Multiplying A on the right by P_2 exchanges the*** ____ ***of A.***

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

12 Explain why the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$ equals the dot product of $P\boldsymbol{x}$ and $P\boldsymbol{y}$. Then from $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ deduce that $P^{\mathrm{T}}P = I$ for any permutation. With $\boldsymbol{x} = (1, 2, 3)$ and $\boldsymbol{y} = (1, 4, 2)$ choose P to show that $P\boldsymbol{x} \cdot \boldsymbol{y}$ is not always $\boldsymbol{x} \cdot P\boldsymbol{y}$.

13 (a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$).

(b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$.

14 If P has 1's on the antidiagonal from $(1,n)$ to $(n,1)$, describe PAP. Note $P = P^{\mathrm{T}}$.

15 All row exchange matrices are symmetric: $P^{\mathrm{T}} = P$. Then $P^{\mathrm{T}}P = I$ becomes $P^2 = I$. Other permutation matrices may or may not be symmetric.

(a) If P sends row 1 to row 4, then P^{T} sends row _____ to row _____. When $P^{\mathrm{T}} = P$ the row exchanges come in pairs with no overlap.

(b) Find a 4 by 4 example with $P^{\mathrm{T}} = P$ that moves all four rows.

Questions 16–21 are about symmetric matrices and their factorizations.

16 If $A = A^{\mathrm{T}}$ and $B = B^{\mathrm{T}}$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$ (b) $(A+B)(A-B)$ (c) ABA (d) $ABAB$.

17 Find 2 by 2 symmetric matrices $A = A^{\mathrm{T}}$ with these properties:

(a) A is not invertible.

(b) A is invertible but cannot be factored into LU (row exchanges needed).

(c) A can be factored into LDL^{T} but not into LL^{T} (because of negative D).

18 (a) How many entries of A can be chosen independently, if $A = A^{\mathrm{T}}$ is 5 by 5?

(b) How do L and D (still 5 by 5) give the same number of choices in LDL^{T}?

(c) How many entries can be chosen if A is *skew-symmetric*? $(A^{\mathrm{T}} = -A)$.

19 Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^{\mathrm{T}}AR$ to show its symmetry. What shape is this matrix?

(b) Show why $R^{\mathrm{T}}R$ has no negative numbers on its diagonal.

20 Factor these symmetric matrices into $A = LDL^{\mathrm{T}}$. The pivot matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

21 After elimination clears out column 1 below the first pivot, find the symmetric 2 by 2 matrix that appears in the lower right corner:

$$\text{Start from } A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Questions 22–24 are about the factorizations $PA = LU$ and $A = L_1P_1U_1$.

22 Find the $PA = LU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

23 Find a 4 by 4 permutation matrix (call it A) that needs 3 row exchanges to reach the end of elimination. For this matrix, what are its factors P, L, and U?

24 Factor the following matrix into $PA = LU$. Factor it also into $A = L_1P_1U_1$ (hold the exchange of row 3 until 3 times row 1 is subtracted from row 2):

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}.$$

25 Extend the **slu** code in Section 2.6 to a code **splu** that factors PA into LU.

26 Prove that the identity matrix cannot be the product of three row exchanges (or five). It can be the product of two exchanges (or four).

27 (a) Choose E_{21} to remove the 3 below the first pivot. Then multiply $E_{21}AE_{21}^{\mathrm{T}}$ to remove both 3's:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 11 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{is going toward} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Choose E_{32} to remove the 4 below the second pivot. Then A is reduced to D by $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$. Invert the E's to find L in $A = LDL^{\mathrm{T}}$.

28 If every row of a 4 by 4 matrix contains the numbers $0, 1, 2, 3$ in some order, can the matrix be symmetric?

29 Prove that no reordering of rows and reordering of columns can transpose a typical matrix. (Watch the diagonal entries.)

The next three questions are about applications of the identity $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$.

30 Wires go between Boston, Chicago, and Seattle. Those cities are at voltages x_B, x_C, x_S. With unit resistances between cities, the currents between cities are in y:

$$y = Ax \quad \text{is} \quad \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}.$$

(a) Find the total currents $A^{\mathrm{T}}y$ out of the three cities.

(b) Verify that $(Ax)^{\mathrm{T}}y$ agrees with $x^{\mathrm{T}}(A^{\mathrm{T}}y)$—six terms in both.

31 Producing x_1 trucks and x_2 planes needs $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs y_1, y_2, y_3 are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane? Those are the components of $A^{\mathrm{T}}\boldsymbol{y}$.

32 $A\boldsymbol{x}$ gives the amounts of steel, rubber, and labor to produce $\boldsymbol{x}$ in Problem 31. Find A. Then $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the _____ of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the value of _____.

33 The matrix P that multiplies (x, y, z) to give (z, x, y) is also a rotation matrix. Find P and P^3. The rotation axis $\boldsymbol{a} = (1, 1, 1)$ doesn't move, it equals $P\boldsymbol{a}$. What is the angle of rotation from $\boldsymbol{v} = (2, 3, -5)$ to $P\boldsymbol{v} = (-5, 2, 3)$?

34 Write $A = \left[\begin{smallmatrix} 1 & 2 \\ 4 & 9 \end{smallmatrix}\right]$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

35 Here is a new factorization of A into *triangular* (with 1's) *times symmetric*:

$$\text{Start from } A = LDU. \text{ Then } A = L(U^{\mathrm{T}})^{-1} \text{ times } U^{\mathrm{T}}DU.$$

Why is $L(U^{\mathrm{T}})^{-1}$ triangular? Its diagonal is all 1's. Why is $U^{\mathrm{T}}DU$ symmetric?

36 A *group* of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?
Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^{\mathrm{T}} = Q^{-1}$. ***Invent two more matrix groups.***

Challenge Problems

37 A square ***northwest matrix*** B is zero in the southeast corner, below the antidiagonal that connects $(1, n)$ to $(n, 1)$. Will B^{T} and B^2 be northwest matrices? Will B^{-1} be northwest or southeast? What is the shape of BC = ***northwest times southeast***?

38 If you take powers of a permutation matrix, why is some P^k eventually equal to I?
Find a 5 by 5 permutation P so that the smallest power to equal I is P^6.

39 (a) Write down any 3 by 3 matrix A. Split A into $B + C$ where $B = B^{\mathrm{T}}$ is symmetric and $C = -C^{\mathrm{T}}$ is anti-symmetric.

(b) Find formulas for B and C involving A and A^{T}. We want $A = B + C$ with $B = B^{\mathrm{T}}$ and $C = -C^{\mathrm{T}}$.

40 Suppose Q^{T} *equals* Q^{-1} (transpose equals inverse, so $Q^{\mathrm{T}}Q = I$).

(a) Show that the columns $q_1, \ldots, q_n$ are unit vectors: $\|\boldsymbol{q}_i\|^2 = 1$.

(b) Show that every two columns of Q are perpendicular: $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$.

(c) Find a 2 by 2 example with first entry $q_{11} = \cos\theta$.

Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of Ax and AB are linear combinations of n vectors—the columns of A. This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at "spaces" of vectors. Without seeing ***vector spaces*** and especially their ***subspaces***, you haven't understood everything about $Ax = \boldsymbol{b}$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We are looking inside the calculations, to find the mathematics. The author's job is to make it clear. The chapter ends with the "*Fundamental Theorem of Linear Algebra*".

We begin with the most important vector spaces. They are denoted by $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, ... Each space $\mathbf{R}^n$ consists of a whole collection of vectors. $\mathbf{R}^5$ contains all column vectors with five components. This is called "5-dimensional space".

DEFINITION *The space* $\mathbf{R}^n$ *consists of all column vectors* $\boldsymbol{v}$ *with* n *components.*

The components of $\boldsymbol{v}$ are real numbers, which is the reason for the letter $\mathbf{R}$. A vector whose n components are complex numbers lies in the space $\mathbf{C}^n$.

The vector space $\mathbf{R}^2$ is represented by the usual xy plane. Each vector $\boldsymbol{v}$ in $\mathbf{R}^2$ has two components. The word "*space*" asks us to think of all those vectors—the whole plane. Each vector gives the x and y coordinates of a point in the plane: $\boldsymbol{v} = (x, y)$.

Similarly the vectors in $\mathbf{R}^3$ correspond to points (x, y, z) in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line (like the x axis). As before, we print vectors as a column between brackets, or along a line using commas and parentheses:

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \text{ is in } \mathbf{R}^2, \quad (1,1,0,1,1) \text{ is in } \mathbf{R}^5, \quad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ is in } \mathbf{C}^2.$$

The great thing about linear algebra is that it deals easily with five-dimensional space. We don't draw the vectors, we just need the five numbers (or n numbers).

To multiply v by 7, multiply every component by 7. Here 7 is a "scalar". To add vectors in $\mathbf{R}^5$, add them a component at a time. The two essential vector operations go on *inside the vector space*, and they produce ***linear combinations***:

We can add any vectors in $\mathbf{R}^n$***, and we can multiply any vector*** v ***by any scalar*** c.

"Inside the vector space" means that ***the result stays in the space.*** If v is the vector in $\mathbf{R}^4$ with components $1, 0, 0, 1$, then $2v$ is the vector in $\mathbf{R}^4$ with components $2, 0, 0, 2$. (In this case 2 is the scalar.) A whole series of properties can be verified in $\mathbf{R}^n$. The commutative law is $v + w = w + v$; the distributive law is $c(v + w) = cv + cw$. There is a unique "zero vector" satisfying $\mathbf{0} + v = v$. Those are three of the eight conditions listed at the start of the problem set.

These eight conditions are required of every vector space. There are vectors other than column vectors, and vector spaces other than $\mathbf{R}^n$, and all vector spaces have to obey the eight reasonable rules.

A real vector space is a set of "vectors" *together with rules for vector addition and for multiplication by real numbers.* The addition and the multiplication must produce vectors that are in the space. And the eight conditions must be satisfied (which is usually no problem). Here are three vector spaces other than $\mathbf{R}^n$:

M	The vector space of ***all real 2 by 2 matrices.***
F	The vector space of ***all real functions*** $f(x)$.
Z	The vector space that consists only of a ***zero vector.***

In **M** the "vectors" are really matrices. In **F** the vectors are functions. In **Z** the only addition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$. In each case we can add: matrices to matrices, functions to functions, zero vector to zero vector. We can multiply a matrix by 4 or a function by 4 or the zero vector by 4. The result is still in **M** or **F** or **Z**. The eight conditions are all easily checked.

The function space **F** is infinite-dimensional. A smaller function space is **P**, or $\mathbf{P}_n$, containing all polynomials $a_0 + a_1x + \cdots + a_nx^n$ of degree n.

The space **Z** is zero-dimensional (by any reasonable definition of dimension). It is the smallest possible vector space. We hesitate to call it $\mathbf{R}^0$, which means no components—you might think there was no vector. The vector space **Z** contains exactly *one vector* (zero). No space can do without that zero vector. Each space has its own zero vector—the zero matrix, the zero function, the vector $(0, 0, 0)$ in $\mathbf{R}^3$.

Subspaces

At different times, we will ask you to think of matrices and functions as vectors. But at all times, the vectors that we need most are ordinary column vectors. They are vectors with n components—but *maybe not all* of the vectors with n components. There are important vector spaces ***inside*** $\mathbf{R}^n$. Those are ***subspaces*** of $\mathbf{R}^n$.

Start with the usual three-dimensional space $\mathbf{R}^3$. Choose a plane through the origin $(0, 0, 0)$. ***That plane is a vector space in its own right.*** If we add two vectors in the plane, their sum is in the plane. If we multiply an in-plane vector by 2 or -5, it is still in the plane.

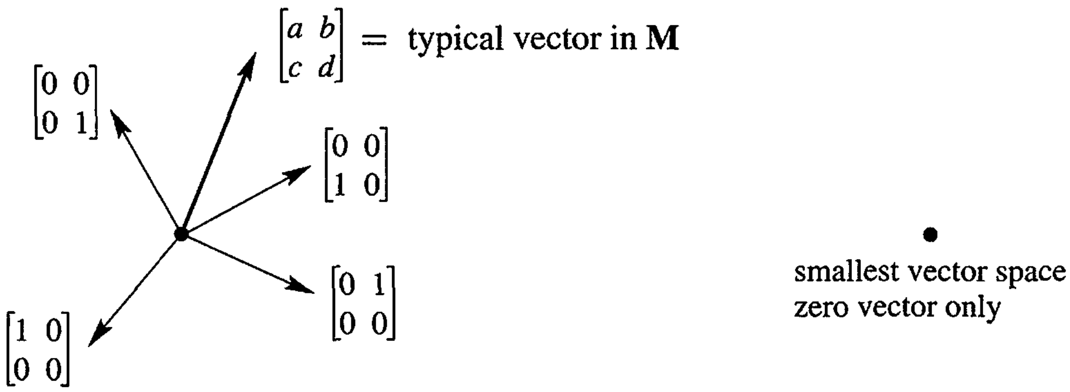


Figure 3.1: "Four-dimensional" matrix space $\mathbf{M}$. The "zero-dimensional" space $\mathbf{Z}$.

A plane in three-dimensional space is not $\mathbf{R}^2$ (even if it looks like $\mathbf{R}^2$). The vectors have three components and they belong to $\mathbf{R}^3$. The plane is a vector space *inside* $\mathbf{R}^3$.

This illustrates one of the most fundamental ideas in linear algebra. The plane going through $(0, 0, 0)$ is a ***subspace*** of the full vector space $\mathbf{R}^3$.

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If v and w are vectors in the subspace and c is any scalar, then***
(i) $v + w$ is in the subspace
(ii) cv is in the subspace.

In other words, the set of vectors is "closed" under addition $v + w$ and multiplication cv (and cw). Those operations leave us in the subspace. We can also subtract, because $-w$ is in the subspace and its sum with v is $v - w$. In short, ***all linear combinations stay in the subspace.***

All these operations follow the rules of the host space, so the eight required conditions are automatic. We just have to check the requirements for a subspace, so that we can take linear combinations.

First fact: ***Every subspace contains the zero vector.*** The plane in $\mathbf{R}^3$ has to go through $(0, 0, 0)$. We mention this separately, for extra emphasis, but it follows directly from rule (**ii**). Choose $c = 0$, and the rule requires $0v$ to be in the subspace.

Planes that don't contain the origin fail those tests. When v is on such a plane, $-v$ and $0v$ are *not* on the plane. A plane that misses the origin is not a subspace.

Lines through the origin are also subspaces. When we multiply by 5, or add two vectors on the line, we stay on the line. But the line must go through $(0, 0, 0)$.

Another subspace is all of $\mathbf{R}^3$. The whole space is a subspace (*of itself*). Here is a list of all the possible subspaces of $\mathbf{R}^3$:

(**L**) Any line through $(0,0,0)$	($\mathbf{R}^3$) The whole space
(**P**) Any plane through $(0,0,0)$	(**Z**) The single vector $(0,0,0)$

If we try to keep only *part* of a plane or line, the requirements for a subspace don't hold. Look at these examples in $\mathbf{R}^2$.

Example 1 Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane). The vector $(2,3)$ is included but $(-2,-3)$ is not. So rule (**ii**) is violated when we try to multiply by $c = -1$. ***The quarter-plane is not a subspace.***

Example 2 Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement (**ii**) is satisfied; we can multiply by any c. But rule (**i**) now fails. The sum of $\boldsymbol{v} = (2,3)$ and $\boldsymbol{w} = (-3,-2)$ is $(-1,1)$, which is outside the quarter-planes. ***Two quarter-planes don't make a subspace.***

Rules (i) and (ii) involve vector addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication by scalars like c and d. The rules can be combined into a single requirement—*the rule for subspaces*:

A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

Example 3 Inside the vector space **M** of all 2 by 2 matrices, here are two subspaces:

$$(\mathbf{U}) \text{ All upper triangular matrices } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad (\mathbf{D}) \text{ All diagonal matrices } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Add any two matrices in **U**, and the sum is in **U**. Add diagonal matrices, and the sum is diagonal. In this case **D** is also a subspace of **U**! Of course the zero matrix is in these subspaces, when a, b, and d all equal zero.

To find a smaller subspace of diagonal matrices, we could require $a = d$. The matrices are multiples of the identity matrix I. The sum $2I + 3I$ is in this subspace, and so is 3 times $4I$. The matrices cI form a "line of matrices" inside **M** and **U** and **D**.

Is the matrix I a subspace by itself? Certainly not. Only the zero matrix is. Your mind will invent more subspaces of 2 by 2 matrices—write them down for Problem 5.

The Column Space of A

The most important subspaces are tied directly to a matrix A. We are trying to solve $A\boldsymbol{x} = \boldsymbol{b}$. If A is not invertible, the system is solvable for some $\boldsymbol{b}$ and not solvable for other $\boldsymbol{b}$. We want to describe the good right sides $\boldsymbol{b}$—the vectors that *can* be written as A times some vector $\boldsymbol{x}$. Those $\boldsymbol{b}'s$ form the "*column space*" of A.

Remember that $A\boldsymbol{x}$ is a combination of the columns of A. To get every possible $\boldsymbol{b}$, we use every possible $\boldsymbol{x}$. So start with the columns of A, and ***take all their linear combinations. This produces the column space of A.*** **It is a vector space made up of column vectors.**

$\boldsymbol{C}(A)$ contains not just the n columns of A, but all their combinations $A\boldsymbol{x}$.

DEFINITION The ***column space*** consists of ***all linear combinations of the columns.*** The combinations are all possible vectors Ax. They fill the column space $C(A)$.

This column space is crucial to the whole book, and here is why. ***To solve $Ax = b$ is to express b as a combination of the columns.*** The right side b has to be *in the column space* produced by A on the left side, or no solution!

The system $Ax = b$ is solvable if and only if b is in the column space of A.

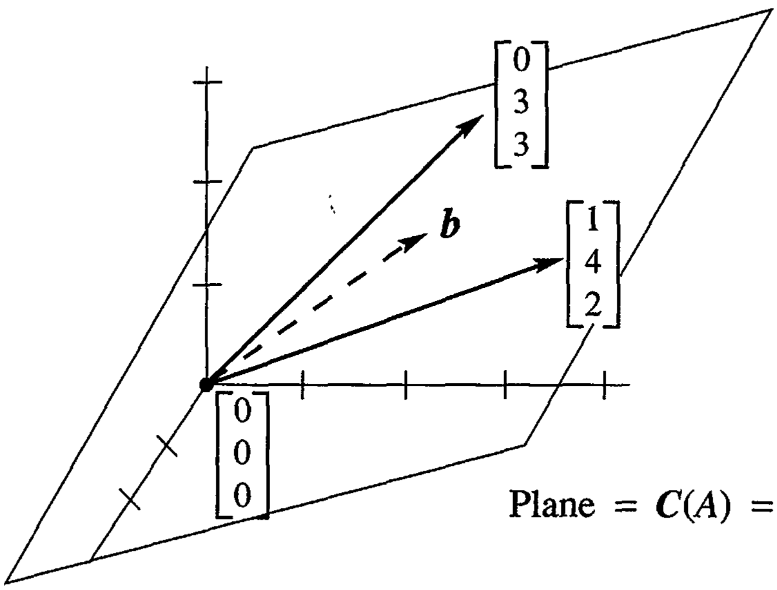
When b is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution x to the system $Ax = b$.

Suppose A is an m by n matrix. Its columns have m components (not n). So the columns belong to $\mathbf{R}^m$. ***The column space of A is a subspace of $\mathbf{R}^m$ (not $\mathbf{R}^n$).*** The set of all column combinations Ax satisfies rules (i) and (ii) for a subspace: When we add linear combinations or multiply by scalars, we still produce combinations of the columns. The word "subspace" is justified *by taking all linear combinations.*

Here is a 3 by 2 matrix A, whose column space is a subspace of $\mathbf{R}^3$. The column space of A is a plane in Figure 3.2.

Example 4

$$Ax \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{which is} \quad x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Plane $= C(A) =$ all vectors Ax

Figure 3.2: The column space $C(A)$ is a plane containing the two columns. $Ax = b$ is solvable when b is on that plane. Then b is a combination of the columns.

The column space of all combinations of the two columns *fills up a plane in* $\mathbf{R}^3$. We drew one particular $\boldsymbol{b}$ (a combination of the columns). This $\boldsymbol{b} = A\boldsymbol{x}$ lies on the plane. The plane has zero thickness, so most right sides $\boldsymbol{b}$ in $\mathbf{R}^3$ are *not* in the column space. For most $\boldsymbol{b}$ there is no solution to our 3 equations in 2 unknowns.

Of course $(0, 0, 0)$ is in the column space. The plane passes through the origin. There is certainly a solution to $A\boldsymbol{x} = \mathbf{0}$. That solution, always available, is $\boldsymbol{x} =$ ____.

To repeat, the attainable right sides $\boldsymbol{b}$ are exactly the vectors in the column space. One possibility is the first column itself—take $x_1 = 1$ and $x_2 = 0$. Another combination is the second column—take $x_1 = 0$ and $x_2 = 1$. The new level of understanding is to see *all* combinations—the whole subspace is generated by those two columns.

Notation The column space of A is denoted by $\boldsymbol{C}(A)$. Start with the columns and take all their linear combinations. We might get the whole $\mathbf{R}^m$ or only a subspace.

Important Instead of columns in $\mathbf{R}^m$, we could start with any set $\mathbf{S}$ of vectors in a vector space $\mathbf{V}$. To get a subspace $\mathbf{SS}$ of $\mathbf{V}$, we take *all combinations* of the vectors in that set:

$$\begin{aligned} \mathbf{S} &= \text{set of vectors in } \mathbf{V} \text{ (probably } \textit{not} \text{ a subspace)} \\ \mathbf{SS} &= \text{all combinations of vectors in } \mathbf{S} \end{aligned}$$

$$\mathbf{SS} = \text{all } c_1\boldsymbol{v}_1 + \cdots + c_N\boldsymbol{v}_N = \textbf{the subspace of V “spanned” by S}$$

When $\mathbf{S}$ is the set of columns, $\mathbf{SS}$ is the column space. When there is only one nonzero vector $\boldsymbol{v}$ in $\mathbf{S}$, the subspace $\mathbf{SS}$ is the line through $\boldsymbol{v}$. *Always* $\mathbf{SS}$ *is the smallest subspace containing* $\mathbf{S}$. This is a fundamental way to create subspaces and we will come back to it.

The subspace SS is the “span” of S, containing all combinations of vectors in S.

Example 5 Describe the column spaces (they are subspaces of $\mathbf{R}^2$) for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The column space of I is the *whole space* $\mathbf{R}^2$. Every vector is a combination of the columns of I. In vector space language, $\boldsymbol{C}(I)$ is $\mathbf{R}^2$.

The column space of A is only a line. The second column $(2, 4)$ is a multiple of the first column $(1, 2)$. Those vectors are different, but our eye is on vector ***spaces***. The column space contains $(1, 2)$ and $(2, 4)$ and all other vectors $(c, 2c)$ along that line. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is only solvable when $\boldsymbol{b}$ is on the line.

For the third matrix (with three columns) the column space $\boldsymbol{C}(B)$ is all of $\mathbf{R}^2$. Every $\boldsymbol{b}$ is attainable. The vector $\boldsymbol{b} = (5, 4)$ is column 2 plus column 3, so $\boldsymbol{x}$ can be $(0, 1, 1)$. The same vector $(5, 4)$ is also 2(column 1) + column 3, so another possible $\boldsymbol{x}$ is $(2, 0, 1)$. This matrix has the same column space as I—any $\boldsymbol{b}$ is allowed. But now $\boldsymbol{x}$ has extra components and there are more solutions—more combinations that give $\boldsymbol{b}$.

The next section creates a vector space $\boldsymbol{N}(A)$, to describe all the solutions of $A\boldsymbol{x} = \mathbf{0}$. This section created the column space $\boldsymbol{C}(A)$, to describe all the attainable right sides $\boldsymbol{b}$.

■ REVIEW OF THE KEY IDEAS ■

1. $\mathbf{R}^n$ contains all column vectors with n real components.
2. $\mathbf{M}$ (2 by 2 matrices) and $\mathbf{F}$ (functions) and $\mathbf{Z}$ (zero vector alone) are vector spaces.
3. A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all their combinations $c\boldsymbol{v} + d\boldsymbol{w}$.
4. The combinations of the columns of A form the ***column space*** $\boldsymbol{C(A)}$. Then the column space is "spanned" by the columns.
5. $A\boldsymbol{x} = \boldsymbol{b}$ has a solution exactly when $\boldsymbol{b}$ is in the column space of A.

■ WORKED EXAMPLES ■

3.1 A We are given three different vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. Construct a matrix so that the equations $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ are solvable, but $A\boldsymbol{x} = \boldsymbol{b}_3$ is not solvable. How can you decide if this is possible? How could you construct A?

Solution We want to have $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the column space of A. Then $A\boldsymbol{x} = \boldsymbol{b}_1$ and $A\boldsymbol{x} = \boldsymbol{b}_2$ will be solvable. *The quickest way is to make $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ the two columns of A.* Then the solutions are $\boldsymbol{x} = (1, 0)$ and $\boldsymbol{x} = (0, 1)$.

Also, we don't want $A\boldsymbol{x} = \boldsymbol{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the question is:

$$\text{Is } A\boldsymbol{x} = \begin{bmatrix} & & \\ \boldsymbol{b}_1 & & \boldsymbol{b}_2 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{b}_3 \text{ solvable?} \qquad \text{Is } \boldsymbol{b}_3 \text{ a combination of } \boldsymbol{b}_1 \text{ and } \boldsymbol{b}_2\text{?}$$

If the answer is *no*, we have the desired matrix A. If the answer is *yes*, then it is *not possible* to construct A. When the column space contains $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, it will have to contain all their linear combinations. So $\boldsymbol{b}_3$ would necessarily be in that column space and $A\boldsymbol{x} = \boldsymbol{b}_3$ would necessarily be solvable.

3.1 B Describe a subspace $\mathbf{S}$ of each vector space $\mathbf{V}$, and then a subspace $\mathbf{SS}$ of $\mathbf{S}$.

$\mathbf{V}_1$ = all combinations of $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$
$\mathbf{V}_2$ = all vectors perpendicular to $\boldsymbol{u} = (1, 2, 1)$, so $\boldsymbol{u} \cdot \boldsymbol{v} = 0$
$\mathbf{V}_3$ = all symmetric 2 by 2 matrices (a subspace of $\mathbf{M}$)
$\mathbf{V}_4$ = all solutions to the equation $d^4y/dx^4 = 0$ (a subspace of $\mathbf{F}$)

Describe each $\mathbf{V}$ two ways: *All combinations of ...., all solutions of the equations ....*

Solution $\mathbf{V}_1$ starts with three vectors. A subspace $\mathbf{S}$ comes from all combinations of the first two vectors $(1,1,0,0)$ and $(1,1,1,0)$. A subspace $\mathbf{SS}$ of $\mathbf{S}$ comes from all multiples $(c,c,0,0)$ of the first vector. So many possibilities.

A subspace $\mathbf{S}$ of $\mathbf{V}_2$ is the line through $(1,-1,1)$. This line is perpendicular to $\boldsymbol{u}$. The vector $\boldsymbol{x} = (0,0,0)$ is in $\mathbf{S}$ and all its multiples $c\boldsymbol{x}$ give the smallest subspace $\mathbf{SS} = \mathbf{Z}$.

The diagonal matrices are a subspace $\mathbf{S}$ of the symmetric matrices. The multiples cI are a subspace $\mathbf{SS}$ of the diagonal matrices.

$\mathbf{V}_4$ contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $d^4y/dx^4 = 0$. The quadratic polynomials give a subspace $\mathbf{S}$. The linear polynomials are one choice of $\mathbf{SS}$. The constants could be $\mathbf{SSS}$.

In all four parts we could take $\mathbf{S} = \mathbf{V}$ itself, and $\mathbf{SS} =$ the zero subspace $\mathbf{Z}$.

Each $\mathbf{V}$ can be described as *all combinations of* .... and as *all solutions of* ....:

$\mathbf{V}_1 =$ all combinations of the 3 vectors $\quad$ $\mathbf{V}_1 =$ all solutions of $v_1 - v_2 = 0$

$\mathbf{V}_2 =$ all combinations of $(1,0,-1)$ and $(1,-1,1)$ are solutions of $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.

$\mathbf{V}_3 =$ all combinations of $\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$. $\mathbf{V}_3 =$ all solutions $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ of $b = c$

$\mathbf{V}_4 =$ all combinations of $1, x, x^2, x^3$ $\quad$ $\mathbf{V}_4 =$ all solutions to $d^4y/dx^4 = 0$.

Problem Set 3.1

The first problems 1–8 are about vector spaces in general. The vectors in those spaces are not necessarily column vectors. In the definition of a *vector space*, vector addition $\boldsymbol{x}+\boldsymbol{y}$ and scalar multiplication $c\boldsymbol{x}$ must obey the following eight rules:

(1) $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$

(2) $\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z}) = (\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}$

(3) There is a unique "zero vector" such that $\boldsymbol{x}+\mathbf{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$

(4) For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x}+(-\boldsymbol{x}) = \mathbf{0}$

(5) 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$

(6) $(c_1c_2)\boldsymbol{x} = c_1(c_2\boldsymbol{x})$

(7) $c(\boldsymbol{x}+\boldsymbol{y}) = c\boldsymbol{x}+c\boldsymbol{y}$

(8) $(c_1+c_2)\boldsymbol{x} = c_1\boldsymbol{x}+c_2\boldsymbol{x}$.

1 Suppose $(x_1,x_2)+(y_1,y_2)$ is defined to be (x_1+y_2, x_2+y_1). With the usual multiplication $c\boldsymbol{x} = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

2 Suppose the multiplication $c\boldsymbol{x}$ is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2). With the usual addition in $\mathbf{R}^2$, are the eight conditions satisfied?

3 (a) Which rules are broken if we keep only the positive numbers $x > 0$ in $\mathbf{R}^1$? Every c must be allowed. The half-line is not a subspace.

(b) The positive numbers with $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ redefined to equal the usual xy and x^c *do* satisfy the eight rules. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $\boldsymbol{x} + \boldsymbol{y} = 2$ and $c\boldsymbol{x} = 8$.) Which number acts as the "zero vector"?

4 The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

5 (a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

6 The functions $f(x) = x^2$ and $g(x) = 5x$ are "vectors" in $\mathbf{F}$. This is the vector space of all real functions. (The functions are defined for $-\infty < x < \infty$.) The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______.

7 Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$? Keep the usual addition $f(x) + g(x)$.

8 If the sum of the "vectors" $f(x)$ and $g(x)$ is defined to be the function $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$ and find two rules that are broken.

Questions 9–18 are about the "subspace requirements": $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ (and then all linear combinations $c\boldsymbol{x} + d\boldsymbol{y}$) stay in the subspace.

9 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{x} + \boldsymbol{y}$ stays in the set but $\frac{1}{2}\boldsymbol{x}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{x}$ stays in the set but $\boldsymbol{x} + \boldsymbol{y}$ may be outside.

10 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1, 4, 0)$ and $\boldsymbol{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

11 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12 Let P be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in P! Find two vectors in P and check that their sum is not in P.

13 Let $\mathbf{P}_0$ be the plane through $(0, 0, 0)$ parallel to the previous plane P. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

14 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0, 0, 0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

15 (a) The intersection of two planes through $(0, 0, 0)$ is probably a ____ but it could be a ____. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ____ but it could be a ____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both subspaces. *Check the requirements on* $\boldsymbol{x} + \boldsymbol{y}$ *and* $c\boldsymbol{x}$.

16 Suppose $\mathbf{P}$ is a plane through $(0, 0, 0)$ and $\mathbf{L}$ is a line through $(0, 0, 0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.

17 (a) Show that the set of *invertible* matrices in $\mathbf{M}$ is not a subspace.

(b) Show that the set of *singular* matrices in $\mathbf{M}$ is not a subspace.

18 True or false (check addition in each case by an example):

(a) The symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = A$) form a subspace.

(b) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(c) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

Questions 19–27 are about column spaces $C(A)$ and the equation $A\boldsymbol{x} = \boldsymbol{b}$.

19 Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

20 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

21 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of (B or C ?) is also a combination of the columns of A. Which two matrices have the same column _____ ?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

22 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{x} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$?

24 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

25 Suppose $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{y} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, *then* $\boldsymbol{b} + \boldsymbol{b}^*$ is in $\boldsymbol{C}(A)$.

26 If A is any 5 by 5 invertible matrix, then its column space is _____. Why?

27 True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

28 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

29 If the 9 by 12 system $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A) =$ _____.

Challenge Problems

30 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

(a) **Definition**: The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Show that $\mathbf{S} + \mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ or $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$. (Section 3.5 returns to this word "span".)

31 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

32 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$. Important point:

An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an _____ matrix.

3.2 The Nullspace of A: Solving $Ax = 0$

This section is about the subspace containing all solutions to $A\boldsymbol{x} = \mathbf{0}$. The m by n matrix A can be square or rectangular. *One immediate solution is* $\boldsymbol{x} = \mathbf{0}$. For invertible matrices this is the only solution. For other matrices, not invertible, there are nonzero solutions to $A\boldsymbol{x} = \mathbf{0}$. *Each solution* $\boldsymbol{x}$ *belongs to the nullspace of* A.

Elimination will find all solutions and identify this very important subspace.

> ***The nullspace of*** A ***consists of all solutions to*** $A\boldsymbol{x} = \mathbf{0}$. **These vectors** $\boldsymbol{x}$ **are in** $\mathbf{R}^n$. The nullspace containing all solutions of $A\boldsymbol{x} = \mathbf{0}$ is denoted by $\boldsymbol{N}(A)$.

Check that the solution vectors form a subspace. Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the nullspace (this means $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$). The rules of matrix multiplication give $A(\boldsymbol{x} + \boldsymbol{y}) = \mathbf{0} + \mathbf{0}$. The rules also give $A(c\boldsymbol{x}) = c\mathbf{0}$. The right sides are still zero. Therefore $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are also in the nullspace $\boldsymbol{N}(A)$. Since we can add and multiply without leaving the nullspace, it is a subspace.

To repeat: The solution vectors $\boldsymbol{x}$ have n components. They are vectors in $\mathbf{R}^n$, so *the nullspace is a subspace of* $\mathbf{R}^n$. The column space $\boldsymbol{C}(A)$ is a subspace of $\mathbf{R}^m$.

If the right side $\boldsymbol{b}$ is not zero, the solutions of $A\boldsymbol{x} = \boldsymbol{b}$ do *not* form a subspace. The vector $\boldsymbol{x} = \mathbf{0}$ is only a solution if $\boldsymbol{b} = \mathbf{0}$. When the set of solutions does not include $\boldsymbol{x} = \mathbf{0}$, it cannot be a subspace. Section 3.4 will show how the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ (if there are any solutions) are shifted away from the origin by one particular solution.

Example 1 $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [1 \quad 2 \quad 3]$. This equation $A\boldsymbol{x} = \mathbf{0}$ produces a plane through the origin $(0, 0, 0)$. The plane is a subspace of $\mathbf{R}^3$. *It is the nullspace of* A.

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular!

Solution Apply elimination to the linear equations $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{array}{r} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{array} \quad \rightarrow \quad \begin{array}{r} x_1 + 2x_2 = 0 \\ \mathbf{0 = 0} \end{array}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $\boldsymbol{N}(A)$. It contains all solutions (x_1, x_2).

To describe this line of solutions, here is an efficient way. Choose one point on the line (one "***special solution***"). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. The special solution $\boldsymbol{s}$ is $(-2, 1)$:

Special solution The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

This is the best way to describe the nullspace, by computing special solutions to $Ax = \mathbf{0}$. This example has one special solution and the nullspace is a line.

The nullspace consists of all combinations of the special solutions.

The plane $x + 2y + 3z = 0$ in Example 1 had *two* special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has the special solutions } s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Those vectors s_1 and s_2 lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. All vectors on the plane are combinations of s_1 and s_2.

Notice what is special about s_1 and s_2. They have ones and zeros in the last two components. *Those components are "free" and we choose them specially.* Then the first components -2 and -3 are determined by the equation $Ax = \mathbf{0}$.

The first column of $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ contains the *pivot*, so the first component of x is *not free*. The free components correspond to columns without pivots. This description of special solutions will be completed after one more example.

The special choice (one or zero) is only for the free variables.

Example 3 Describe the nullspaces of these three matrices A, B, C:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $Ax = \mathbf{0}$ has only the zero solution $x = \mathbf{0}$. *The nullspace is* $\mathbf{Z}$. It contains only the single point $x = \mathbf{0}$ in $\mathbf{R}^2$. This comes from elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}.$$

A is invertible. There are no special solutions. All columns of this A have pivots.

The rectangular matrix B has the same nullspace $\mathbf{Z}$. The first two equations in $Bx = \mathbf{0}$ again require $x = \mathbf{0}$. The last two equations would also force $x = \mathbf{0}$. When we add extra equations, the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors x in the nullspace.

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector x has *four* components. Elimination will produce pivots in the first two columns of C, but the last two columns are "free". They don't have pivots:

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \end{matrix}$$

pivot columns **free columns**

For the free variables x_3 and x_4, we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are determined by the equation $Ux = \mathbf{0}$. We get two special solutions in the nullspace of C (which is also the nullspace of U). The special solutions are s_1 and s_2:

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{ pivot} \\ \leftarrow \quad \text{variables} \\ \leftarrow \text{ free} \\ \leftarrow \quad \text{variables} \end{array}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U! We can continue to make this matrix simpler, in two ways:

1. ***Produce zeros above the pivots***, **by eliminating upward.**
2. ***Produce ones in the pivots***, **by dividing the whole row by its pivot.**

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easiest to see when we reach the ***reduced row echelon form*** R. It has I in the pivot columns:

Reduced form R

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$
$$\uparrow \; \uparrow$$

now the pivot columns contain I

I subtracted row 2 of U from row 1, and then multiplied row 2 by $\frac{1}{2}$. The original two equations have simplified to $x_1 + 2x_3 = 0$ and $x_2 + 2x_4 = 0$.

The first special solution is still $s_1 = (-2, 0, 1, 0)$, and s_2 is also unchanged. Special solutions are much easier to find from the reduced system $Rx = \mathbf{0}$.

Before moving to m by n matrices A and their nullspaces $N(A)$ and special solutions, allow me to repeat one comment. For many matrices, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. Their nullspaces $N(A) = \mathbf{Z}$ contain only that zero vector. The only combination of the columns that produces $b = \mathbf{0}$ is then the "zero combination" or "trivial combination". The solution is trivial (just $x = \mathbf{0}$) but the idea is not trivial.

This case of a zero nullspace $\mathbf{Z}$ is of the greatest importance. It says that the columns of A are **independent**. No combination of columns gives the zero vector (except the zero combination). All columns have pivots, and no columns are free. You will see this idea of independence again . . .

Solving $Ax = \mathbf{0}$ by Elimination

This is important. ***A is rectangular and we still use elimination***. We solve m equations in n unknowns when $b = \mathbf{0}$. After A is simplified by row operations, we read off the solution (or solutions). Remember the two stages (forward and back) in solving $Ax = \mathbf{0}$:

1. **Forward elimination** takes A to a triangular U (or its reduced form R).

2. **Back substitution** in $Ux = \mathbf{0}$ or $Rx = \mathbf{0}$ produces x.

You will notice a difference in back substitution, when A and U have fewer than n pivots. *We are allowing all matrices in this chapter*, not just the nice ones (which are square matrices with inverses).

Pivots are still nonzero. The columns below the pivots are still zero. But it might happen that a column has no pivot. That free column doesn't stop the calculation. ***Go on to the next column***. The first example is a 3 by 4 matrix with two pivots:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}.$$

Certainly $a_{11} = 1$ is the first pivot. Clear out the 2 and 3 below that pivot:

$$A \to \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \qquad \begin{matrix} \\ \text{(subtract } 2 \times \text{ row 1)} \\ \text{(subtract } 3 \times \text{ row 1)} \end{matrix}$$

The second column has a zero in the pivot position. We look below the zero for a nonzero entry, ready to do a row exchange. *The entry below that position is also zero*. Elimination can do nothing with the second column. This signals trouble, which we expect anyway for a rectangular matrix. There is no reason to quit, and we go on to the third column.

The second pivot is 4 (but it is in the third column). Subtracting row 2 from row 3 clears out that column below the pivot. **The pivot columns are 1 and 3:**

$$\textbf{Triangular } U: \quad U = \begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textit{\textbf{Only two pivots}} \\ \textit{\textbf{The last equation}} \\ \textit{\textbf{became}}\ 0 = 0 \end{matrix}$$

The fourth column also has a zero in the pivot position—but nothing can be done. There is no row below it to exchange, and forward elimination is complete. The matrix has three rows, four columns, and *only two pivots*. The original $Ax = \mathbf{0}$ seemed to involve three different equations, but the third equation is the sum of the first two. It is automatically satisfied ($0 = 0$) when the first two equations are satisfied. Elimination reveals the inner truth about a system of equations. Soon we push on from U to R.

Now comes back substitution, to find all solutions to $Ux = \mathbf{0}$. With four unknowns and only two pivots, there are many solutions. The question is how to write them all down. A good method is to separate the ***pivot variables*** from the ***free variables***.

P	The ***pivot*** variables are x_1 and x_3.	**Columns 1 and 3 contain pivots.**
F	The ***free*** variables are x_2 and x_4.	**Columns 2 and 4 have no pivots.**

The free variables x_2 and x_4 can be given any values whatsoever. Then back substitution finds the pivot variables x_1 and x_3. (In Chapter 2 no variables were free. When A is invertible, all variables are pivot variables.) The simplest choices for the free variables are ones and zeros. Those choices give the *special solutions*.

Special solutions to $x_1 + x_2 + 2x_3 + 3x_4 = 0$ and $4x_3 + 4x_4 = 0$

- Set $x_2 = 1$ and $x_4 = 0$. By back substitution $x_3 = 0$. Then $x_1 = -1$.
- Set $x_2 = 0$ and $x_4 = 1$. By back substitution $x_3 = -1$. Then $x_1 = -1$.

These special solutions solve $Ux = \mathbf{0}$ and therefore $Ax = \mathbf{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions*.

Complete solution to $Ax = \mathbf{0}$

$$\boldsymbol{x} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\textbf{special}} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\textbf{special}} = \underbrace{\begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}}_{\textbf{complete}}. \tag{1}$$

Please look again at that answer. It is the main goal of this section. The vector $s_1 = (-1, 1, 0, 0)$ is the special solution when $x_2 = 1$ and $x_4 = 0$. The second special solution has $x_2 = 0$ and $x_4 = 1$. ***All solutions are linear combinations of s_1 and s_2.*** The special solutions are in the nullspace $N(A)$, and their combinations fill out the whole nullspace.

The MATLAB code **nullbasis** computes these special solutions. They go into the columns of a ***nullspace matrix*** N. The complete solution to $Ax = \mathbf{0}$ is a combination of those columns. Once we have the special solutions, we have the whole nullspace.

There is a special solution for each free variable. If no variables are free—this means there are n pivots—then the only solution to $Ux = \mathbf{0}$ and $Ax = \mathbf{0}$ is the trivial solution $x = \mathbf{0}$. All variables are pivot variables. In that case the nullspaces of A and U contain only the zero vector. With no free variables, and pivots in every column, the output from **nullbasis** is an empty matrix. The nullspace with n pivots is **Z**.

Example 4 Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of U has no pivot. So x_2 is free. The special solution has $x_2 = 1$. Back substitution into $9x_3 = 0$ gives $x_3 = 0$. Then $x_1 + 5x_2 = 0$ or $x_1 = -5$. The solutions to $Ux = \mathbf{0}$ are multiples of one special solution:

$$x = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of U is a line in $\mathbf{R}^3$.
It contains multiples of the special solution $s = (-5, 1, 0)$.
One variable is free, and $N =$ **nullbasis** (U) has one column s.

In a minute elimination will get zeros above the pivots and ones in the pivots. By continuing elimination on U, the 7 is removed and the pivot changes from 9 to 1.

The final result will be the **reduced row echelon form R**:

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{rref}(U).$$

This makes it even clearer that the special solution (column of N) is $s = (-5, 1, 0)$.

Echelon Matrices

Forward elimination goes from A to U. It acts by row operations, including row exchanges. It goes on to the next column when no pivot is available in the current column. The m by n "staircase" U is an ***echelon matrix***.

Here is a 4 by 7 echelon matrix with the three pivots $\boldsymbol{p}$ highlighted in boldface:

$$U = \begin{bmatrix} \boldsymbol{p} & x & x & x & x & x & x \\ 0 & \boldsymbol{p} & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{p} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Three pivot variables $\mathbf{x_1, x_2, x_6}$
Four free variables $\mathbf{x_3, x_4, x_5, x_7}$
Four special solutions in $N(U)$

Question What are the column space and the nullspace for this matrix?

Answer The columns have four components so they lie in $\mathbf{R}^4$. (Not in $\mathbf{R}^3$!) The fourth component of every column is zero. Every combination of the columns—every vector in the column space—has fourth component zero. *The column space $\boldsymbol{C}(U)$ consists of all vectors of the form* $(b_1, b_2, b_3, 0)$. For those vectors we can solve $Ux = \boldsymbol{b}$ by back substitution. These vectors $\boldsymbol{b}$ are all possible combinations of the seven columns.

The nullspace $N(U)$ is a subspace of $\mathbf{R}^7$. The solutions to $Ux = \mathbf{0}$ are all the combinations of the four special solutions—*one for each free variable*:

1. Columns $3, 4, 5, 7$ have no pivots. So the free variables are x_3, x_4, x_5, x_7.

2. Set one free variable to 1 and set the other free variables to zero.

3. Solve $Ux = \mathbf{0}$ for the pivot variables x_1, x_2, x_6.

4. This gives one of the four special solutions in the nullspace matrix N.

The nonzero rows of an echelon matrix go down in a staircase pattern. The pivots are the first nonzero entries in those rows. There is a column of zeros below every pivot.

Counting the pivots leads to an extremely important theorem. Suppose A has more columns than rows. ***With $n > m$ there is at least one free variable.*** The system $Ax = \mathbf{0}$ has at least one special solution. This solution is *not zero*!

Suppose $Ax = \mathbf{0}$ has more unknowns than equations (**$\mathbf{n > m}$, more columns than rows**). Then there are **nonzero solutions**. There must be free columns, without pivots.

A short wide matrix ($n > m$) always has nonzero vectors in its nullspace. There must be at least $n - m$ free variables, since the number of pivots cannot exceed m. (The matrix only has m rows, and a row never has two pivots.) Of course a row might have *no* pivot—which means an extra free variable. But here is the point: When there is a free variable, it can be set to 1. Then the equation $Ax = \mathbf{0}$ has a nonzero solution.

To repeat: There are at most m pivots. With $n > m$, the system $Ax = \mathbf{0}$ has a nonzero solution. Actually there are infinitely many solutions, since any multiple $c\boldsymbol{x}$ is also a solution. The nullspace contains at least a line of solutions. With two free variables, there are two special solutions and the nullspace is even larger.

The nullspace is a subspace. Its "dimension" is the number of free variables. This central idea—the ***dimension*** of a subspace—is defined and explained in this chapter.

The Reduced Row Echelon Matrix R

From an echelon matrix U we go one more step. Continue with a 3 by 4 example:

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can divide the second row by 4. Then both pivots equal 1. *We can subtract* 2 *times this new row* $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ *from the row above.* ***The reduced row echelon matrix R has zeros above the pivots as well as below***:

Reduced row echelon matrix $$R = \text{rref}(A) = \begin{bmatrix} \mathbf{1} & 1 & \mathbf{0} & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ **Pivot rows contain I**

R *has* 1*'s as pivots. Zeros above pivots come from upward elimination.*

Important *If A is invertible, its reduced row echelon form is the identity matrix $R = I$.* This is the ultimate in row reduction. Of course the nullspace is then **Z**.

The zeros in R make it easy to find the special solutions (the same as before):

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = 0$.

Those numbers -1 and 0 are sitting in column 2 of R (with plus signs).

2. Set $x_2 = 0$ and $x_4 = 1$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = -1$.

Those numbers -1 and -1 are sitting in column 4 (with plus signs).

By reversing signs we can read off the special solutions directly from R. The nullspace $N(A) = N(U) = N(R)$ contains all combinations of the special solutions:

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\textbf{\textit{complete solution of }} Ax = \mathbf{0}).$$

The next section of the book moves firmly from U to the row reduced form R. The MATLAB command $[\,R, pivcol\,] = \text{rref}(A)$ produces R and also a list of the pivot columns.

■ REVIEW OF THE KEY IDEAS ■

1. The nullspace $N(A)$ is a subspace of $\mathbf{R}^n$. It contains all solutions to $Ax = \mathbf{0}$.

2. Elimination produces an echelon matrix U, and then a row reduced R, with pivot columns and free columns.

3. Every free column of U or R leads to a special solution. The free variable equals 1 and the other free variables equal 0. Back substitution solves $Ax = \mathbf{0}$.

4. The complete solution to $Ax = \mathbf{0}$ is a combination of the special solutions.

5. If $n > m$ then A has at least one column without pivots, giving a special solution. So there are nonzero vectors x in the nullspace of this rectangular A.

■ WORKED EXAMPLES ■

3.2 A Create a 3 by 4 matrix whose special solutions to $Ax = \mathbf{0}$ are s_1 and s_2:

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix} \qquad \begin{array}{l} \text{pivot columns 1 and 3} \\ \text{free variables } x_2 \text{ and } x_4 \end{array}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required nullspace $N(A) =$ all combinations of s_1 and s_2.

Solution The reduced matrix R has pivots $= 1$ in columns 1 and 3. There is no third pivot, so the third row of R is all zeros. The free columns 2 and 4 will be combinations of the pivot columns:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad Rs_1 = \mathbf{0} \quad \text{and} \quad Rs_2 = \mathbf{0}.$$

The entries $3, 2, 6$ in R are the negatives of $-3, -2, -6$ in the special solutions!

R is only one matrix (one possible A) with the required nullspace. We could do any elementary operations on R—exchange rows, multiply a row by any $c \neq 0$, subtract any multiple of one row from another. **R can be multiplied** (*on the left*) **by any invertible matrix, without changing its nullspace.**

Every 3 by 4 matrix has at least one special solution. *These matrices have two.*

3.2 B Find the special solutions and describe the *complete solution* to $Ax = \mathbf{0}$ for

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} A_2 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is R in each case?

Solution $A_1 x = \mathbf{0}$ has four special solutions. They are the columns s_1, s_2, s_3, s_4 of the 4 by 4 identity matrix. The nullspace is all of $\mathbf{R}^4$. The complete solution to $A_1 x = \mathbf{0}$ is any $x = c_1 s_1 + c_2 s_2 + c_3 s_3 + c_4 s_4$ in $\mathbf{R}^4$. There are no pivot columns; all variables are free; the reduced R is the same zero matrix as A_1.

$A_2 x = \mathbf{0}$ has only one special solution $s = (-2, 1)$. The multiples $x = cs$ give the complete solution. The first column of A_2 is its pivot column, and x_2 is the free variable. The row reduced matrices R_2 for A_2 and R_3 for $A_3 = [\, A_2 \;\; A_2 \,]$ have 1's in the pivot:

$$A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \to R_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} A_2 & A_2 \end{bmatrix} \to R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that R_3 has only one pivot column (the first column). All the variables x_2, x_3, x_4 are free. There are three special solutions to $A_3 x = \mathbf{0}$ (and also $R_3 x = \mathbf{0}$):

$s_1 = (-2, 1, 0, 0)$ $s_2 = (-1, 0, 1, 0)$ $s_3 = (-2, 0, 0, 1)$ **Complete** $x = c_1 s_1 + c_2 s_2 + c_3 s_3$.

With r pivots, A has $n - r$ free variables. $Ax = \mathbf{0}$ has $n - r$ special solutions.

Problem Set 3.2

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

1 Reduce these matrices to their ordinary echelon forms U:

$$\text{(a)}\; A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \qquad \text{(b)}\quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

2 For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)

3 By combining the special solutions in Problem 2, describe every solution to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. The nullspace contains only $x = \mathbf{0}$ when there are no ______.

4 By further row operations on each U in Problem 1, find the reduced echelon form R. *True or false*: The nullspace of R equals the nullspace of U.

5 By row operations reduce each matrix to its echelon form U. Write down a 2 by 2 lower triangular L such that $B = LU$.

(a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$.

6 For the same A and B, find the special solutions to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is _____.

7 In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors $\boldsymbol{x}$ that satisfy those equations as combinations of the special solutions.

8 Reduce the echelon forms U in Problem 5 to R. For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

Questions 9–17 are about free variables and pivot variables.

9 True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An m by n matrix has no more than n pivot variables.

(d) An m by n matrix has no more than m pivot variables.

10 Construct 3 by 3 matrices A to satisfy these requirements (if possible):

(a) A has no zero entries but $U = I$.

(b) A has no zero entries but $R = I$.

(c) A has no zero entries but $R = U$.

(d) $A = U = 2R$.

11 Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot columns are

(a) $2, 4, 5$

(b) $1, 3, 6, 7$

(c) 4 and 6.

12 Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix R so that the free columns are

(a) $2, 4, 5, 6$

(b) $1, 3, 6, 7, 8$.

13 Suppose column 4 of a 3 by 5 matrix is all zero. Then x_4 is certainly a _____ variable. The special solution for this variable is the vector $\boldsymbol{x} =$ _____.

14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.

15 Suppose an m by n matrix has r pivots. The number of special solutions is ____. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $r =$ ____. The column space is all of $\mathbf{R}^m$ when $r =$ ____.

16 The nullspace of a 5 by 5 matrix contains only $\boldsymbol{x} = \mathbf{0}$ when the matrix has ____ pivots. The column space is $\mathbf{R}^5$ when there are ____ pivots. Explain why.

17 The equation $x - 3y - z = 0$ determines a plane in $\mathbf{R}^3$. What is the matrix A in this equation? Which are the free variables? The special solutions are $(3, 1, 0)$ and ____.

18 (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

19 Prove that U and $A = LU$ have the same nullspace when L is invertible:

$$\text{If } U\boldsymbol{x} = \mathbf{0} \text{ then } LU\boldsymbol{x} = \mathbf{0}. \quad \text{If } LU\boldsymbol{x} = \mathbf{0}, \text{ how do you know } U\boldsymbol{x} = \mathbf{0}?$$

20 Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Questions 21–28 ask for matrices (if possible) with specific properties.

21 Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

22 Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

23 Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

24 Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

25 Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.

26 Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.

27 Why does no 3 by 3 matrix have a nullspace that equals its column space?

28 If $AB = 0$ then the column space of B is contained in the ____ of A. Give an example of A and B.

29 The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What R is virtually certain if the random A is 4 by 3?

30 Show by example that these three statements are generally *false*:

(a) A and A^{T} have the same nullspace.
(b) A and A^{T} have the same free variables.
(c) If R is the reduced form $\mathbf{rref}(A)$ then R^{T} is $\mathbf{rref}(A^{\mathrm{T}})$.

31 If the nullspace of A consists of all multiples of $x = (2, 1, 0, 1)$, how many pivots appear in U? What is R?

32 If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N, go backward to find the nonzero rows of the reduced matrices R:

$$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \text{ (empty 3 by 1).}$$

33 (a) What are the five 2 by 2 reduced echelon matrices R whose entries are all 0's and 1's?

(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R?

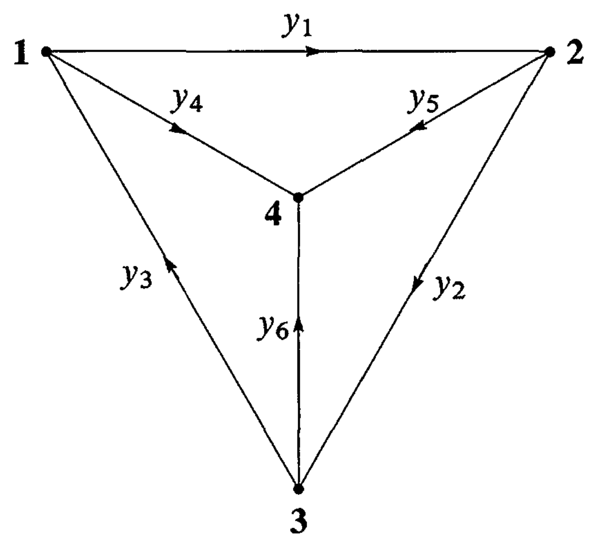
34 Explain why A and $-A$ always have the same reduced echelon form R.

Challenge Problems

35 If A is 4 by 4 and invertible, describe all vectors in the nullspace of the 4 by 8 matrix $B = [A \;\; A]$.

36 How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

37 Kirchhoff's Law says that *current in* = *current out* at every node. This network has six currents $y_1, \ldots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $Ay = \mathbf{0}$ for Kirchhoff's Law at the four nodes. Find three special solutions in the nullspace of A.



3.3 The Rank and the Row Reduced Form

The numbers m and n give the size of a matrix—but not necessarily the *true size* of a linear system. An equation like $0 = 0$ should not count. If there are two identical rows in A, the second one disappears in elimination. Also if row 3 is a combination of rows 1 and 2, then row 3 will become all zeros in the triangular U and the reduced echelon form R. We don't want to count rows of zeros. ***The true size of A is given by its rank***:

DEFINITION *The rank of A is the number of pivots. This number is r.*

That definition is computational, and I would like to say more about the rank r. The matrix will eventually be reduced to r nonzero rows. Start with a 3 by 4 example.

Four columns
How many pivots?

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}. \tag{1}$$

The first two columns are $(1, 1, 1)$ and $(1, 2, 3)$, going in different directions. Those will be pivot columns. The third column $(2, 2, 2)$ is a multiple of the first. We won't see a pivot in that third column. The fourth column $(4, 5, 6)$ is a combination of the first three (their sum). That column will also be without a pivot.

The fourth column is actually a combination $3(1, 1, 1) + (1, 2, 3)$ of the two pivot columns. ***Every "free column" is a combination of earlier pivot columns.*** It is the *special solutions* $\boldsymbol{s}$ that tell us those combinations of pivot columns:

$$\begin{array}{lll} \text{Column 3} = \mathbf{2} \text{ (column 1)} & \boldsymbol{s}_1 = (-\mathbf{2}, 0, 1, 0) & A\boldsymbol{s}_1 = \mathbf{0} \\ \text{Column 4} = \mathbf{3} \text{ (column 1)} + \mathbf{1} \text{ (column 2)} & \boldsymbol{s}_2 = (-\mathbf{3}, -\mathbf{1}, 0, 1) & A\boldsymbol{s}_2 = \mathbf{0} \end{array}$$

With nice numbers we can see the right combinations. The systematic way to find $\boldsymbol{s}$ is by elimination! This will change the columns but it won't change the combinations, because $A\boldsymbol{x} = \mathbf{0}$ is equivalent to $U\boldsymbol{x} = \mathbf{0}$ and also $R\boldsymbol{x} = \mathbf{0}$. I will go from A to U and then to R:

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 1 & 2 & 4 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

U already shows the two pivots in the pivot columns. **The rank of A (and U) is 2.** Continuing to R we see the combinations of pivot columns that produce the free columns:

$$U = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{\text{row 1} - \text{row 2}}{\overset{\text{Subtract}}{\longrightarrow}} \quad R = \begin{bmatrix} \mathbf{1} & 0 & 2 & 3 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Clearly the $(3, 1, 0)$ column equals 3 (column 1) + column 2. Moving all columns to the "left side" will reverse signs to -3 and -1, which go in the special solution $\boldsymbol{s}$:

$$-3 \text{ (column 1)} - \text{(column 2)} + \text{(column 4)} = \mathbf{0} \qquad \boldsymbol{s} = (-3, -1, 0, 1).$$

Rank One

Matrices of ***rank one*** have only *one pivot.* When elimination produces zero in the first column, it produces zero in all the columns. *Every row is a multiple of the pivot row.* At the same time, every column is a multiple of the pivot column!

$$\textbf{Rank one matrix} \qquad A = \begin{bmatrix} \mathbf{1} & 3 & 10 \\ \mathbf{2} & 6 & 20 \\ \mathbf{3} & 9 & 30 \end{bmatrix} \longrightarrow R = \begin{bmatrix} \mathbf{1} & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The column space of a rank one matrix is "one-dimensional". Here all columns are on the line through $\boldsymbol{u} = (1, 2, 3)$. The columns of A are $\boldsymbol{u}$ and $3\boldsymbol{u}$ and $10\boldsymbol{u}$. Put those numbers into the row $\boldsymbol{v}^{\mathrm{T}} = [\,1 \quad 3 \quad 10\,]$ and you have the special rank one form $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \textbf{column times row} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \qquad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} [1 \quad 3 \quad 10]. \tag{3}$$

With rank one, the solutions to $A\boldsymbol{x} = \mathbf{0}$ are easy to understand. That equation $\boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x}) = \mathbf{0}$ leads us to $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x} = 0$. All vectors $\boldsymbol{x}$ in the nullspace must be orthogonal to $\boldsymbol{v}$ in the row space. This is the geometry: *row space = line, nullspace = perpendicular plane.* Now describe the special solutions with numbers:

Pivot row $[1 \;\; 3 \;\; 10]$
Pivot variable x_1
Free variables x_2 **and** x_3

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

The nullspace contains all combinations of s_1 and s_2. This produces the plane $x + 3y + 10z = 0$, perpendicular to the row $(1, 3, 10)$. **Nullspace** (plane) **perpendicular to row space** (line).

Example 1 When all rows are multiples of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \end{bmatrix} \text{ all have rank 1.}$$

For those matrices, the reduced row echelon $R =$ **rref** (A) can be checked by eye:

$$R = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \end{bmatrix} \text{ have only one pivot.}$$

Our second definition of rank will be at a higher level. It deals with entire rows and entire columns—vectors and not just numbers. The matrices A and U and R have r *independent* rows (the pivot rows). They also have r *independent* columns (the pivot columns). Section 3.5 says what it means for rows or columns to be independent.

A third definition of rank, at the top level of linear algebra, will deal with *spaces* of vectors. ***The rank r is the "dimension" of the column space.*** It is also the dimension of the row space. The great thing is that r also reveals the dimension of the nullspace.

The Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else. The r pivot columns taken together contain an r by r identity matrix I. It sits above $m - r$ rows of zeros. The numbers of the pivot columns are in the list *pivcol*.

The pivot columns of A are probably not obvious from A itself. But their column numbers are given by the *same list pivcol*. The r columns of A that eventually have pivots (in U and R) are the pivot columns of A. This example has *pivcol* $= (1, 3)$:

Pivot Columns
$$A = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{1} & 3 & \mathbf{1} & 6 & -4 \end{bmatrix} \text{ yields } R = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}.$$

The column spaces of A and R are different! All columns of this R end with zeros. Elimination subtracts rows 1 and 2 of A from row 3, to produce that zero row in R:

$\boldsymbol{EA = R}$
$\boldsymbol{A = E^{-1}R}$
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The r pivot columns of A are also the first r columns of E^{-1}. The r by r identity matrix inside R just picks out the first r columns of E^{-1} as columns of $A = E^{-1}R$.

One more fact about pivot columns. Their definition has been purely computational, based on R. Here is a direct mathematical description of the pivot columns of A:

The pivot columns are not combinations of earlier columns. The free columns ***are*** combinations of earlier columns. These combinations are the special solutions!

A pivot column of R (with 1 in the pivot row) cannot be a combination of earlier columns (with 0's in that row). The same column of A can't be a combination of earlier columns, because $A\boldsymbol{x} = \mathbf{0}$ *exactly when* $R\boldsymbol{x} = \mathbf{0}$.

Now we look at the special solution $\boldsymbol{x}$ from each free column.

The Special Solutions

Each special solution to $A\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \mathbf{0}$ has one free variable equal to 1. The other free variables in $\boldsymbol{x}$ are all zero. The solutions come directly from the echelon form R:

Free columns
Free variables
in boldface
$$R\boldsymbol{x} = \begin{bmatrix} 1 & \mathbf{3} & 0 & \mathbf{2} & \mathbf{-1} \\ 0 & \mathbf{0} & 1 & \mathbf{4} & \mathbf{-3} \\ 0 & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_1 \\ \boldsymbol{x_2} \\ x_3 \\ \boldsymbol{x_4} \\ \boldsymbol{x_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Set the first free variable to $x_2 = 1$ with $x_4 = x_5 = 0$. The equations give the pivot variables $x_1 = -3$ and $x_3 = 0$. The special solution is $s_1 = (-3, 1, 0, 0, 0)$.

The next special solution has $x_4 = 1$. The other free variables are $x_2 = x_5 = 0$. The solution is $s_2 = (-2, 0, -4, 1, 0)$. Notice -2 and -4 in R, with plus signs.

The third special solution has $x_5 = 1$. With $x_2 = 0$ and $x_4 = 0$ we find $s_3 = (1, 0, 3, 0, 1)$. The numbers $x_1 = 1$ and $x_3 = 3$ are in column 5 of R, again with opposite signs. This is a general rule as we soon verify. The nullspace matrix N contains the three special solutions in its columns, so $AN =$ *zero matrix*:

Nullspace matrix
$n - r = 5 - 2$
3 special solutions

$$N = \begin{array}{c} \begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \\ \text{free} \end{array}$$

The linear combinations of these three columns give all vectors in the nullspace. This is the complete solution to $A\boldsymbol{x} = \mathbf{0}$ (and $R\boldsymbol{x} = \mathbf{0}$). Where R had the identity matrix (2 by 2) in its pivot columns, N has the identity matrix (3 by 3) in its free rows.

There is a special solution for every free variable. Since r columns have pivots, that leaves $n - r$ free variables. This is the key to $A\boldsymbol{x} = \mathbf{0}$ and the nullspace:

$A\boldsymbol{x} = \mathbf{0}$ ***has*** r ***pivots and*** $n - r$ ***free variables***: n columns minus r pivot columns. The ***nullspace matrix*** N contains the $n - r$ special solutions. Then $AN = 0$.

When we introduce the idea of "independent" vectors, we will show that the special solutions are independent. You can see in N that no column is a combination of the other columns. The beautiful thing is that the count is exactly right:

$A\boldsymbol{x} = \mathbf{0}$ ***has*** r ***independent equations so it has*** $\mathbf{n - r}$ ***independent solutions.***

The special solutions are easy for $R\boldsymbol{x} = \mathbf{0}$. Suppose that the first r columns are the pivot columns. Then the reduced row echelon form looks like

$$R = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{array}{l} r \text{ pivot rows} \\ m - r \text{ zero rows} \end{array} \tag{4}$$

r pivot columns $\quad n - r$ free columns

The pivot variables in the $n - r$ special solutions come by changing F to $-F$:

$$\textbf{Nullspace matrix} \quad N = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} \begin{array}{l} r \text{ pivot variables} \\ n - r \text{ free variables} \end{array} \tag{5}$$

Check $RN = 0$. The first block row of RN is (I times $-F$) + (F times I) = zero. The columns of N solve $R\boldsymbol{x} = \mathbf{0}$. When the free part of $R\boldsymbol{x} = \mathbf{0}$ moves to the right side, the left side just holds the identity matrix:

$$Rx = \mathbf{0} \quad \text{means} \quad I\begin{bmatrix}\text{pivot}\\ \text{variables}\end{bmatrix} = -F\begin{bmatrix}\text{free}\\ \text{variables}\end{bmatrix}. \tag{6}$$

In each special solution, the free variables are a column of I. Then the pivot variables are a column of $-F$. Those special solutions give the nullspace matrix N.

The idea is still true if the pivot columns are mixed in with the free columns. Then I and F are mixed together. You can still see $-F$ in the solutions. Here is an example where $I = [1]$ comes first and $F = \begin{bmatrix} 2 & 3 \end{bmatrix}$ comes last.

Example 2 The special solutions of $Rx = x_1 + 2x_2 + 3x_3 = 0$ are the columns of N:

$$R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rank is one. There are $n - r = 3 - 1$ special solutions $(-2, 1, 0)$ and $(-3, 0, 1)$.

Final Note How can I write confidently about R not knowing which steps MATLAB will take? A could be reduced to R in different ways. Very likely you and Mathematica and Maple would do the elimination differently. The key is that ***the final R is always the same.*** *The original A completely determines the I and F and zero rows in R.*

For proof I will determine the pivot columns (which locate I) and free columns (which contain F) in an "algebra way"—two rules that have nothing to do with any particular elimination steps. Here are those rules:

1. The pivot columns *are not* combinations of earlier columns of A.
2. The free columns *are* combinations of earlier columns (F tells the combinations).

A small example with rank one will show two E's that produce the correct $EA = R$:

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{rref}(A) \quad \text{and no other } R.$$

You could multiply row 1 of A by $\frac{1}{2}$, and subtract row 1 from row 2:

$$\textbf{Two steps give } E \qquad \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} = E.$$

Or you could exchange rows in A, and then subtract 2 times row 1 from row 2:

$$\textbf{Two different steps give } E_{\text{new}} \qquad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = E_{\text{new}}.$$

Multiplication gives $EA = R$ and also $E_{\text{new}}A = R$. ***Different E's but the same R.***

Codes for Row Reduction

There is no way that rref will ever come close in importance to lu. The Teaching Code elim for this book uses rref. Of course rref(R) would give R again!

MATLAB: $[R, \textit{pivcol}] =$ rref(A) Teaching Code: $[E, R] =$ elim(A)

The extra output *pivcol* gives the numbers of the pivot columns. They are the same in A and R. The extra output E in the Teaching Code is an m by m ***elimination matrix*** that puts the original A (whatever it was) into its row reduced form R:

$$EA = R.$$

The square matrix E is the product of elementary matrices E_{ij} and also P_{ij} and D^{-1}. P_{ij} exchanges rows. The diagonal D^{-1} divides rows by their pivots to produce 1's.

If we want E, we can apply row reduction to the matrix $\begin{bmatrix} A & I \end{bmatrix}$ with $n + m$ columns. All the elementary matrices that multiply A (to produce R) will also multiply I (to produce E). The whole augmented matrix is being multiplied by E:

$$E\,[A \;\; I] \quad = \quad [R \;\; E] \tag{7}$$

This is exactly what "Gauss-Jordan" did in Chapter 2 to compute A^{-1}. ***When A is square and invertible, its reduced row echelon form is I.*** Then $EA = R$ becomes $EA = I$. In this invertible case, E is A^{-1}. This chapter is going further, to every A.

■ REVIEW OF THE KEY IDEAS ■

1. The rank r of A is the number of pivots (which are 1's in $R =$ rref(A)).
2. The r pivot columns of A and R are in the same list *pivcol*.
3. Those r pivot columns are not combinations of earlier columns.
4. The $n - r$ free columns *are* combinations of earlier columns (pivot columns).
5. Those combinations (using $-F$ taken from R) give the $n - r$ special solutions to $A\boldsymbol{x} = \mathbf{0}$ and $R\boldsymbol{x} = \mathbf{0}$. They are the $n - r$ columns of the nullspace matrix N.

■ WORKED EXAMPLES ■

3.3 A Find the reduced echelon form of A. What is the rank? What is the special solution to $A\boldsymbol{x} = \mathbf{0}$?

Second differences $-1, 2, -1$
Notice $A_{11} = A_{44} = 1$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution Add row 1 to row 2. Then add row 2 to row 3. Then add row 3 to row 4:

First differences 1, −1
$$U = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now add row 3 to row 2. Then add row 2 to row 1:

Reduced form
$$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

The rank is $r = 3$. There is one free variable ($n - r = 1$). *The special solution is* $s = (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$. *Every row adds to* 0. Notice $-F = (1, 1, 1)$ in the pivot variables of s.

3.3 B Factor these rank one matrices into $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{(find } d \text{ from } a, b, c \text{ if } a \neq 0\text{)}$$

Split this rank two matrix into $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$ = (3 by 2) times (2 by 4) using R:

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E^{-1}R.$$

Solution For the 3 by 3 matrix A, all rows are multiples of $\boldsymbol{v}^{\mathrm{T}} = [1 \ \ 2 \ \ 3]$. All columns are multiples of the column $\boldsymbol{u} = (1, 2, 3)$. This symmetric matrix has $\boldsymbol{u} = \boldsymbol{v}$ and A is $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. Every rank one symmetric matrix will have this form or else $-\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$.

If the 2 by 2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has rank one, it must be singular. In Chapter 5, its determinant is $ad - bc = 0$. In this chapter, row 2 is c/a times row 1.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 \\ c/a \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}. \quad \text{So} \quad d = \frac{bc}{a}.$$

The 3 by 4 matrix of rank two is a sum of *two matrices of rank one*. All columns of A are combinations of the pivot columns 1 and 2. All rows are combinations of the nonzero rows of R. The pivot columns are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and those rows are $\boldsymbol{v}_1^{\mathrm{T}}$ and $\boldsymbol{v}_2^{\mathrm{T}}$. Then A is $\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$, multiplying r columns of E^{-1} times r rows of R:

Columns times rows
$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$$

3.3 C Find the row reduced form R and the rank r of A and B (*those depend on* c). Which are the pivot columns of A? What are the special solutions and the matrix N?

Find special solutions $$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & c \\ c & c \end{bmatrix}.$$

Solution The matrix A has rank $r = 2$ ***except if*** $c = 4$. The pivots are in columns 1 and 3. The second variable x_2 is free. Notice the form of R:

$$c \neq 4 \quad R = \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{bmatrix} \qquad c = 4 \quad R = \begin{bmatrix} \mathbf{1} & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Two pivots leave one free variable x_2. But when $c = 4$, the only pivot is in column 1 (rank one). The second and third variables are free, producing two special solutions:

$$c \neq 4 \quad \text{Special solution with } x_2 = 1 \text{ goes into} \quad N = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$$c = 4 \quad \text{Another special solution goes into} \quad N = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 2 by 2 matrix $\left[\begin{smallmatrix} c & c \\ c & c \end{smallmatrix}\right]$ has rank $r = 1$ ***except if*** $c = 0$, when the rank is zero!

$$c \neq 0 \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{Nullspace} = \text{line}$$

The matrix has *no pivot columns* if $c = 0$. Then both variables are free:

$$c = 0 \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{Nullspace} = \mathbf{R}^2.$$

Problem Set 3.3

1 Which of these rules gives a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.

(b) The number of columns minus the total number of rows.

(c) The number of columns minus the number of free columns.

(d) The number of 1's in the matrix R.

2 Find the reduced row echelon forms R and the rank of these matrices:

(a) The 3 by 4 matrix with all entries equal to 4.
(b) The 3 by 4 matrix with $a_{ij} = i + j - 1$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.

3 Find the reduced R for each of these (block) matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

4 Suppose all the pivot variables come *last* instead of first. Describe all four blocks in the reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N containing the special solutions?

5 (Silly problem) Describe all 2 by 3 matrices A_1 and A_2, with row echelon forms R_1 and R_2, such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is is true that $R_1 = A_1$ and $R_2 = A_2$ in this case? Does $R_1 - R_2$ equal $\mathbf{rref}(A_1 - A_2)$?

6 If A has r pivot columns, how do you know that A^{T} has r pivot columns? Give a 3 by 3 example with different column numbers in *pivcol* for A and A^{T}.

7 What are the special solutions to $R\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{y}^{\mathrm{T}}R = \mathbf{0}$ for these R?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problems 8–11 are about matrices of rank $r = 1$.

8 Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

9 If A is an m by n matrix with $r = 1$, its columns are multiples of one column and its rows are multiples of one row. The column space is a _____ in $\mathbf{R}^m$. The nullspace is a _____ in $\mathbf{R}^n$. The nullspace matrix N has shape _____.

10 Choose vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

$A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is the natural form for every matrix that has rank $r = 1$.

11 If A is a rank one matrix, the second row of U is _____. Do an example.

Problems 12–14 are about r by r invertible matrices inside A.

12 *If A has rank r, then it has an r by r submatrix S that is invertible.* Remove $m-r$ rows and $n-r$ columns to find an invertible submatrix S inside A, B, and C. You could keep the pivot rows and pivot columns:

$$A=\begin{bmatrix}1&2&3\\1&2&4\end{bmatrix} \qquad B=\begin{bmatrix}1&2&3\\2&4&6\end{bmatrix} \qquad C=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}.$$

13 Suppose P contains only the r pivot columns of an m by n matrix. Explain why this m by r submatrix P has rank r.

14 Transpose P in problem 13. Then find the r pivot columns of P^{T}. Transposing back, **this produces an r by r invertible submatrix S inside P and A:**

$$\text{For } A=\begin{bmatrix}1&2&3\\2&4&6\\2&4&7\end{bmatrix} \text{ find } P \text{ (3 by 2) and then the invertible } S \text{ (2 by 2).}$$

Problems 15–20 show that rank(AB) is not greater than rank(A) or rank(B).

15 Find the ranks of AB and AC (rank one matrix times rank one matrix):

$$A=\begin{bmatrix}1&2\\2&4\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}2&1&4\\3&1.5&6\end{bmatrix} \quad\text{and}\quad C=\begin{bmatrix}1&b\\c&bc\end{bmatrix}.$$

16 The rank one matrix $\boldsymbol{uv}^{\mathrm{T}}$ times the rank one matrix $\boldsymbol{wz}^{\mathrm{T}}$ is $\boldsymbol{uz}^{\mathrm{T}}$ times the number _____. This product $\boldsymbol{uv}^{\mathrm{T}}\boldsymbol{wz}^{\mathrm{T}}$ also has rank one unless _____ $=0$.

17 (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so **rank$(AB) \le$ rank(B).**

(b) Find A_1 and A_2 so that rank$(A_1B)=1$ and rank$(A_2B)=0$ for $B=\begin{bmatrix}1&1\\1&1\end{bmatrix}$.

18 Problem 17 proved that rank$(AB) \le$ rank(B). Then the same reasoning gives **rank$(B^{\mathrm{T}}A^{\mathrm{T}})$** $\le$ rank(A^{T}). How do you deduce that **rank$(AB) \le$ rank A**?

19 *(Important)* Suppose A and B are n by n matrices, and $AB=I$. Prove from rank$(AB) \le$ rank(A) that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA=I$ *(which is not so obvious!).*

20 If A is 2 by 3 and B is 3 by 2 and $AB=I$, show from its rank that $BA \ne I$. Give an example of A and B with $AB=I$. For $m<n$, a right inverse is not a left inverse.

21 Suppose A and B have the *same* reduced row echelon form R.

(a) Show that A and B have the same nullspace and the same row space.

(b) We know $E_1A = R$ and $E_2B = R$. So A equals an _____ matrix times B.

22 Express A and then B as the sum of two rank one matrices:

$$\textbf{rank} = \textbf{2} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

23 Answer the same questions as in Worked Example **3.3 C** for

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

24 What is the nullspace matrix N (containing the special solutions) for A, B, C?

$$A = \begin{bmatrix} I & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = [I \ \ I \ \ I].$$

25 *Neat fact* ***Every m by n matrix of rank r reduces to*** (m by r) ***times*** (r by n):

$$A = (\text{pivot columns of } A)\,(\text{first } r \text{ rows of } R) = \textbf{(COL)(ROW)}.$$

Write the 3 by 4 matrix A in equation (1) at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R.

Challenge Problems

26 Suppose A is an m by n matrix of rank r. Its reduced echelon form is R. Describe exactly the matrix Z (its shape and all its entries) that comes from *transposing the reduced row echelon form of* R' (prime means transpose):

$$R = \text{rref}(A) \quad \text{and} \quad Z = (\text{rref}(R'))'.$$

27 Suppose R is m by n of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}.$$

(a) What are the shapes of those four blocks?

(b) Find a *right-inverse* B with $RB = I$ if $r = m$.

(c) Find a *left-inverse* C with $CR = I$ if $r = n$.

(d) What is the reduced row echelon form of R^{T} (with shapes)?

(e) What is the reduced row echelon form of $R^{\mathrm{T}}R$ (with shapes)?

Prove that $R^{\mathrm{T}}R$ has the same nullspace as R. Later we show that $A^{\mathrm{T}}A$ always has the same nullspace as A (a valuable fact).

28 Suppose you allow elementary *column* operations on A as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank r?

3.4 The Complete Solution to $Ax = b$

The last sections totally solved $Ax = \mathbf{0}$. Elimination converted the problem to $Rx = \mathbf{0}$. The free variables were given special values (one and zero). Then the pivot variables were found by back substitution. We paid no attention to the right side $\boldsymbol{b}$ because it started and ended as zero. The solution $\boldsymbol{x}$ was in the nullspace of A.

Now $\boldsymbol{b}$ is not zero. Row operations on the left side must act also on the right side. $A\boldsymbol{x} = \boldsymbol{b}$ is reduced to a simpler system $R\boldsymbol{x} = \boldsymbol{d}$. One way to organize that is to ***add b as an extra column of the matrix.*** I will *"augment"* A with the right side $(b_1, b_2, b_3) = (1, 6, 7)$ and reduce the bigger matrix $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \begin{array}{l} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{array} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 1 & 3 & 1 & 6 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}.$$

The augmented matrix is just $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$. When we apply the usual elimination steps to A, we also apply them to $\boldsymbol{b}$. That keeps all the equations correct.

In this example we subtract row 1 from row 3 and then subtract row 2 from row 3. This produces a *complete row of zeros* in R, and it changes $\boldsymbol{b}$ to a new right side $\boldsymbol{d} = (1, 6, 0)$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{has the} \\ \text{augmented} \\ \text{matrix} \end{array} \quad \begin{bmatrix} 1 & 3 & 0 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 4 & \mathbf{6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}.$$

That very last zero is crucial. The third equation has become $0 = 0$ and the equations can be solved. In the original matrix A, the first row plus the second row equals the third row. If the equations are consistent, this must be true on the right side of the equations also! The all-important property on the right side was $1 + 6 = 7$.

Here are the same augmented matrices for a general $\boldsymbol{b} = (b_1, b_2, b_3)$:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

Now we get $0 = 0$ in the third equation provided $b_3 - b_1 - b_2 = 0$. This is $b_1 + b_2 = b_3$.

One Particular Solution

For an easy solution $\boldsymbol{x}$, *choose the free variables to be* $x_2 = x_4 = 0$. Then the two nonzero equations give the two pivot variables $x_1 = 1$ and $x_3 = 6$. Our particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ (and also $R\boldsymbol{x} = \boldsymbol{d}$) is $\boldsymbol{x}_p = (1, 0, 6, 0)$. This particular solution is my favorite: *free variables = zero, pivot variables from* $\boldsymbol{d}$. The method always works.

For a solution to exist, zero rows in R must also be zero in d. Since I is in the pivot rows and pivot columns of R, the pivot variables in $x_{\text{particular}}$ come from d:

$$Rx_p = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 \\ \mathbf{0} & 0 & \mathbf{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{6} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{6} \\ 0 \end{bmatrix} \qquad \begin{array}{l} \textbf{Pivot variables 1, 6} \\ \textbf{Free variables } 0, 0 \end{array}.$$

Notice how we *choose* the free variables (as zero) and *solve* for the pivot variables. After the row reduction to R, those steps are quick. When the free variables are zero, the pivot variables for x_p are already seen already seen in the right side vector d.

$x_{\text{particular}}$	***The particular solution solves***	$Ax_p = b$
$x_{\text{nullspace}}$	***The $n-r$ special solutions solve***	$Ax_n = \mathbf{0}$.

That particular solution is $(1,0,6,0)$. The two special (nullspace) solutions to $Rx = \mathbf{0}$ come from the two free columns of R, by reversing signs of $3, 2$, and 4. ***Please notice how I write the complete solution $x_p + x_n$ to $Ax = b$:***

Complete solution
one x_p
many x_n

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

Question Suppose A is a square invertible matrix, $m = n = r$. What are x_p and x_n?
Answer The particular solution is the one and *only* solution $A^{-1}b$. There are no special solutions or free variables. $R = I$ has no zero rows. The only vector in the nullspace is $x_n = \mathbf{0}$. The complete solution is $x = x_p + x_n = A^{-1}b + \mathbf{0}$.

This was the situation in Chapter 2. We didn't mention the nullspace in that chapter. $N(A)$ contained only the zero vector. Reduction goes from $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1}b \end{bmatrix}$. The original $Ax = b$ is reduced all the way to $x = A^{-1}b$ which is d. This is a special case here, but square invertible matrices are the ones we see most often in practice. So they got their own chapter at the start of the book.

For small examples we can reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} R & d \end{bmatrix}$. For a large matrix, MATLAB does it better. One particular solution (not necessarily ours) is $A\backslash b$ from backslash. Here is an example with *full column rank*. Both columns have pivots.

Example 1 Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This condition puts b in the column space of A. Find the complete $x = x_p + x_n$.

Solution Use the augmented matrix, with its extra column $\boldsymbol{b}$. Subtract row 1 of $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ from row 2, and add 2 times row 1 to row 3 to reach $\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix}.$$

The last equation is $0 = 0$ provided $b_3 + b_1 + b_2 = 0$. This is the condition to put $\boldsymbol{b}$ in the column space; then $A\boldsymbol{x} = \boldsymbol{b}$ will be solvable. The rows of A add to the zero row. So for consistency (these are equations!) the entries of $\boldsymbol{b}$ must also add to zero.

This example has no free variables since $n - r = 2 - 2$. Therefore no special solutions. The nullspace solution is $\boldsymbol{x}_n = \boldsymbol{0}$. The particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and $R\boldsymbol{x} = \boldsymbol{d}$ is at the top of the augmented column $\boldsymbol{d}$:

$$\textbf{Only solution} \qquad \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $b_3 + b_1 + b_2$ is not zero, there is no solution to $A\boldsymbol{x} = \boldsymbol{b}$ ($\boldsymbol{x}_p$ doesn't exist).

This example is typical of an extremely important case: A has *full column rank*. Every column has a pivot. *The rank is* $r = n$. The matrix is tall and thin ($m \geq n$). Row reduction puts I at the top, when A is reduced to R with rank n:

$$\textbf{Full column rank} \quad R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} \tag{1}$$

There are no free columns or free variables. The nullspace matrix is empty!

We will collect together the different ways of recognizing this type of matrix.

Every matrix A with **full column rank** ($r = n$) has all these properties:

1. All columns of A are pivot columns.
2. There are no free variables or special solutions.
3. The nullspace $\boldsymbol{N}(A)$ contains only the zero vector $\boldsymbol{x} = \boldsymbol{0}$.
4. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution (it might not) then it has only *one solution*.

In the essential language of the next section, **this A has *independent columns*.** $A\boldsymbol{x} = \boldsymbol{0}$ only happens when $\boldsymbol{x} = \boldsymbol{0}$. In Chapter 4 we will add one more fact to the list: *The square matrix $A^{\mathrm{T}}A$ is invertible when the rank is n.*

In this case the nullspace of A (and R) has shrunk to the zero vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is *unique* (if it exists). There will be $m - n$ (here $3 - 2$) zero rows in R. So there are $m - n$ conditions in order to have $0 = 0$ in those rows, and $\boldsymbol{b}$ in the column space. With full column rank, $A\boldsymbol{x} = \boldsymbol{b}$ has *one* solution or *no* solution ($m > n$ is overdetermined).

The Complete Solution

The other extreme case is full row rank. Now $Ax = b$ has *one or infinitely many* solutions. In this case A must be *short and wide* ($m \le n$). A matrix has ***full row rank*** if $r = m$ ("*independent rows*"). Every row has a pivot, and here is an example.

Example 2 There are $n = 3$ unknowns but only $m = 2$ equations:

$$\textbf{Full row rank} \qquad \begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \end{aligned} \qquad (\text{rank } r = m = 2)$$

These are two planes in xyz space. The planes are not parallel so they intersect in a line. This line of solutions is exactly what elimination will find. ***The particular solution will be one point on the line. Adding the nullspace vectors x_n will move us along the line.*** Then $x = x_p + x_n$ gives the whole line of solutions.

We find x_p and x_n by elimination on $\begin{bmatrix} A & b \end{bmatrix}$. Subtract row 1 from row 2 and then subtract row 2 from row 1:

$$\begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 1 & 2 & -1 & \mathbf{4} \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & \mathbf{3} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & \mathbf{2} \\ 0 & 1 & -2 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} R & d \end{bmatrix}.$$

The particular solution has free variable $x_3 = 0$. The special solution has $x_3 = 1$:

$x_{\text{particular}}$ comes directly from d on the right side: $x_p = (2, 1, 0)$
x_{special} comes from the third column (free column) of R: $s = (-3, 2, 1)$

It is wise to check that x_p and s satisfy the original equations $Ax_p = b$ and $As = \mathbf{0}$:

$$\begin{aligned} 2 + 1 &= 3 \\ 2 + 2 &= 4 \end{aligned} \qquad\qquad \begin{aligned} -3 + 2 + 1 &= 0 \\ -3 + 4 - 1 &= 0 \end{aligned}$$

The nullspace solution x_n is any multiple of s. It moves along the line of solutions, starting at $x_{\text{particular}}$. ***Please notice again how to write the answer:***

$$\textbf{Complete solution} \qquad x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

This line is drawn in Figure 3.3. Any point on the line *could* have been chosen as the particular solution; we chose the point with $x_3 = 0$.

The particular solution is *not* multiplied by an arbitrary constant! The special solution is, and you understand why.

Now we summarize this short wide case of *full row rank*. If $m < n$ the equation $Ax = b$ is **underdetermined** (many solutions).

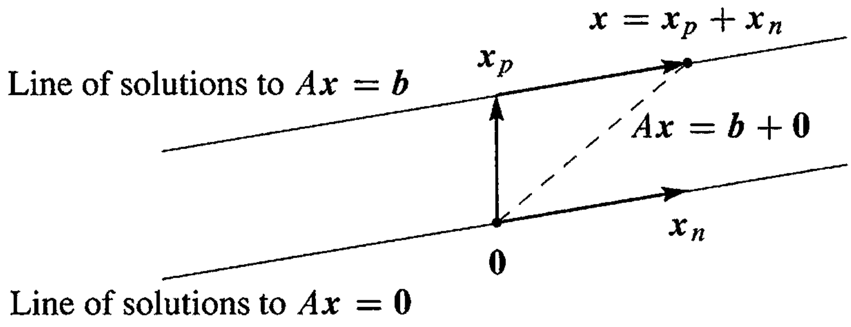


Figure 3.3: Complete solution = *one* particular solution + *all* nullspace solutions.

Every matrix A with ***full row rank*** ($r = m$) has all these properties:

1. All rows have pivots, and R has no zero rows.

2. $Ax = b$ has a solution for every right side b.

3. The column space is the whole space $\mathbf{R}^m$.

4. There are $n - r = n - m$ special solutions in the nullspace of A.

In this case with m pivots, the rows are "***linearly independent***". So the columns of A^{T} are linearly independent. We are more than ready for the definition of linear independence, as soon as we summarize the four possibilities—which depend on the rank. Notice how r, m, n are the critical numbers.

The four possibilities for linear equations depend on the rank r:

$\boldsymbol{r = m}$	and	$\boldsymbol{r = n}$	*Square and invertible*	$Ax = b$	has 1 solution
$\boldsymbol{r = m}$	and	$r < n$	*Short and wide*	$Ax = b$	has ∞ solutions
$r < m$	and	$\boldsymbol{r = n}$	*Tall and thin*	$Ax = b$	has 0 or 1 solution
$r < m$	and	$r < n$	*Not full rank*	$Ax = b$	has 0 or ∞ solutions

The reduced R will fall in the same category as the matrix A. In case the pivot columns happen to come first, we can display these four possibilities for R. For $Rx = d$ (and the original $Ax = b$) to be solvable, d must end in $m - r$ zeros.

Four types	$R = \begin{bmatrix} I \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
Their ranks	$\boldsymbol{r = m = n}$	$\boldsymbol{r = m < n}$	$\boldsymbol{r = n < m}$	$\boldsymbol{r < m, r < n}$

Cases 1 and 2 have full row rank $r = m$. Cases 1 and 3 have full column rank $r = n$. Case 4 is the most general in theory and it is the least common in practice.

Note My classes used to stop at U before reaching R. Instead of reading the complete solution directly from $Rx = d$, we found it by back substitution from $Ux = c$. This reduction to U and back substitution for x is slightly faster. Now we prefer the complete reduction: a single "1" in each pivot column. Everything is so clear in R (and the computer should do the hard work anyway) that we reduce all the way.

■ REVIEW OF THE KEY IDEAS ■

1. The rank r is the number of pivots. The matrix R has $m - r$ zero rows.
2. $Ax = b$ is solvable if and only if the last $m - r$ equations reduce to $0 = 0$.
3. One particular solution x_p has all free variables equal to zero.
4. The pivot variables are determined after the free variables are chosen.
5. Full column rank $r = n$ means no free variables: one solution or none.
6. Full row rank $r = m$ means one solution if $m = n$ or infinitely many if $m < n$.

■ WORKED EXAMPLES ■

3.4 A This question connects elimination (**pivot columns and back substitution**) to **column space-nullspace-rank-solvability** (the full picture). A has rank 2:

$$Ax = b \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

1. Reduce $[\,A \;\; b\,]$ to $[\,U \;\; c\,]$, so that $Ax = b$ becomes a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 for $Ax = b$ to have a solution.
3. Describe the column space of A. Which plane in $\mathbf{R}^3$?
4. Describe the nullspace of A. Which special solutions in $\mathbf{R}^4$?
5. Find a particular solution to $Ax = (0, 6, -6)$ and then the complete solution.
6. Reduce $[\,U \;\; c\,]$ to $[\,R \;\; d\,]$: Special solutions from R, particular solution from d.

Solution

1. The multipliers in elimination are 2 and 3 and -1. They take $[\,A \;\; b\,]$ into $[\,U \;\; c\,]$.

$$\begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 2 & 4 & 8 & 12 & \mathbf{b_2} \\ 3 & 6 & 7 & 13 & \mathbf{b_3} \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & -2 & -2 & \mathbf{b_3 - 3b_1} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & \mathbf{b_1} \\ 0 & 0 & 2 & 2 & \mathbf{b_2 - 2b_1} \\ 0 & 0 & 0 & 0 & \mathbf{b_3 + b_2 - 5b_1} \end{array}\right]$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. **First description**: The column space is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$. The pivots are in columns 1 and 3. **Second description**: The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $A\boldsymbol{x} = \boldsymbol{b}$ solvable, so $\boldsymbol{b}$ is in the column space. *All columns of A pass this test* $b_3 + b_2 - 5b_1 = 0$. *This is the equation for the plane in the first description*!
4. The special solutions have free variables $x_2 = 1, x_4 = 0$ and then $x_2 = 0, x_4 = 1$:

Special solutions to $A\boldsymbol{x} = \boldsymbol{0}$
Back substitution in $U\boldsymbol{x} = \boldsymbol{0}$

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The nullspace $N(A)$ in $\mathbf{R}^4$ contains all $\boldsymbol{x}_n = c_1 s_1 + c_2 s_2$.

5. One particular solution $\boldsymbol{x}_p$ has free variables = zero. Back substitute in $U\boldsymbol{x} = \boldsymbol{c}$:

Particular solution to $A\boldsymbol{x}_p = \boldsymbol{b} = (0, 6, -6)$
This vector $\boldsymbol{b}$ satisfies $b_3 + b_2 - 5b_1 = 0$

$$\boldsymbol{x}_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

The complete solution to $A\boldsymbol{x} = (0, 6, -6)$ is $\boldsymbol{x} = \boldsymbol{x}_p + \text{all } \boldsymbol{x}_n$.

6. In the reduced form R, the third column changes from $(3, 2, 0)$ in U to $(0, 1, 0)$. The right side $\boldsymbol{c} = (0, 6, 0)$ becomes $\boldsymbol{d} = (-9, 3, 0)$ showing -9 and 3 in $\boldsymbol{x}_p$:

$$[U \;\; \boldsymbol{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{0} \\ 0 & 0 & 2 & 2 & \mathbf{6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow [R \;\; \boldsymbol{d}] = \begin{bmatrix} 1 & 2 & 0 & 2 & \mathbf{-9} \\ 0 & 0 & 1 & 1 & \mathbf{3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

3.4 B If you have this information about the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ for a specific $\boldsymbol{b}$, what does that tell you about the shape of A (and A itself)? And possibly about $\boldsymbol{b}$.

1. There is exactly one solution.
2. All solutions to $A\boldsymbol{x} = \boldsymbol{b}$ have the form $\boldsymbol{x} = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.
3. There are no solutions.
4. All solutions to $A\boldsymbol{x} = \boldsymbol{b}$ have the form $\boldsymbol{x} = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right]$
5. There are infinitely many solutions.

Solution In case **1**, with exactly one solution, A must have full column rank $r = n$. The nullspace of A contains only the zero vector. Necessarily $m \geq n$.

In case **2**, A must have $n = 2$ columns (and m is arbitrary). With $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ in the nullspace of A, column 2 is the *negative* of column 1. Also $A \neq 0$: the rank is 1. With $\boldsymbol{x} = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ as a solution, $\boldsymbol{b} = 2(\text{column } 1) + (\text{column } 2)$. My choice for $\boldsymbol{x}_p$ would be $(1, 0)$.

In case **3** we only know that $\boldsymbol{b}$ is not in the column space of A. The rank of A must be less than m. I guess we know $\boldsymbol{b} \neq \boldsymbol{0}$, otherwise $\boldsymbol{x} = \boldsymbol{0}$ would be a solution.

In case **4**, A must have $n = 3$ columns. With $(1, 0, 1)$ in the nullspace of A, column 3 is the negative of column 1. Column 2 must *not* be a multiple of column 1, or the nullspace would contain another special solution. So the rank of A is $3 - 1 = 2$. Necessarily A has $m \geq 2$ rows. The right side $\boldsymbol{b}$ is column 1 + column 2.

In case **5** with infinitely many solutions, the nullspace must contain nonzero vectors. The rank r must be less than n (not full column rank), and $\boldsymbol{b}$ must be in the column space of A. We don't know if *every* $\boldsymbol{b}$ is in the column space, so we don't know if $r = m$.

3.4 C Find the complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ by forward elimination on $[A \;\; \boldsymbol{b}]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.$$

Find numbers y_1, y_2, y_3 so that y_1 (row 1) + y_2 (row 2) + y_3 (row 3) = ***zero row***. Check that $\boldsymbol{b} = (4, 2, 10)$ satisfies the condition $y_1 b_1 + y_2 b_2 + y_3 b_3 = 0$. Why is this the condition for the equations to be solvable and $\boldsymbol{b}$ to be in the column space?

Solution Forward elimination on $[A \;\; \boldsymbol{b}]$ produces a zero row in $[U \;\; \boldsymbol{c}]$. The third equation becomes $0 = 0$ and the equations are consistent (and solvable):

$$\begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 2 & 4 & 4 & 8 & \mathbf{2} \\ 4 & 8 & 6 & 8 & \mathbf{10} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}.$$

Columns 1 and 3 contain pivots. The variables x_2 and x_4 are free. If we set those to zero we can solve (back substitution) for the particular solution $\boldsymbol{x}_p = (7, 0, -3, 0)$. We see 7 and -3 again if elimination continues all the way to $[R \;\; \boldsymbol{d}]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 2 & 8 & \mathbf{-6} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & \mathbf{4} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -4 & \mathbf{7} \\ 0 & 0 & 1 & 4 & \mathbf{-3} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}.$$

For the nullspace part $\boldsymbol{x}_n$ with $\boldsymbol{b} = \boldsymbol{0}$, set the free variables x_2, x_4 to $1, 0$ and also $0, 1$:

Special solutions $\quad s_1 = (-2, 1, 0, 0)$ **and** $s_2 = (4, 0, -4, 1)$

Then the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$ (and $R\boldsymbol{x} = \boldsymbol{d}$) is $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + c_1 s_1 + c_2 s_2$.

The rows of A produced the zero row from 2(row 1) + (row 2) − (row 3) = $(0, 0, 0, 0)$. Thus $\boldsymbol{y} = (2, 1, -1)$. The same combination for $\boldsymbol{b} = (4, 2, 10)$ gives $2(4) + (2) - (10) = 0$.

If a combination of the rows (on the left side) gives the zero row, then the same combination must give zero on the right side. Of course! *Otherwise no solution.*

Later we will say this again in different words: If every column of A is perpendicular to $\boldsymbol{y} = (2, 1, -1)$, then any combination $\boldsymbol{b}$ of those columns must also be perpendicular to $\boldsymbol{y}$. Otherwise $\boldsymbol{b}$ is not in the column space and $A\boldsymbol{x} = \boldsymbol{b}$ is not solvable.

And again: If $\boldsymbol{y}$ is in the nullspace of A^{T} then $\boldsymbol{y}$ must be perpendicular to every $\boldsymbol{b}$ in the column space of A. Just looking ahead...

Problem Set 3.4

1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $Ax = b$ to be solvable. Together these two conditions put b into the ______ space (two planes give a line):

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

Questions 3–15 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n. Use the augmented matrix with last column b.

3 Write the complete solution as x_p plus any multiple of s in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Under what condition on b_1, b_2, b_3 is this system solvable? Include b as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find x in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination of the rows of A gives the zero row?

8 Which vectors (b_1, b_2, b_3) are in the column space of A? Which combinations of the rows of A give zero?

$$\text{(a)}\ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad \text{(b)}\ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

9 (a) The Worked Example **3.4 A** reached $[\,U \;\; \boldsymbol{c}\,]$ from $[\,A \;\; \boldsymbol{b}\,]$. Put the multipliers into L and verify that LU equals A and $L\boldsymbol{c}$ equals $\boldsymbol{b}$.

(b) Combine the pivot columns of A with the numbers -9 and 3 in the particular solution $\boldsymbol{x}_p$. What is that linear combination and why?

10 Construct a 2 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$ with particular solution $\boldsymbol{x}_p = (2, 4, 0)$ and homogeneous solution $\boldsymbol{x}_n =$ any multiple of $(1, 1, 1)$.

11 Why can't a 1 by 3 system have $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_n =$ any multiple of $(1, 1, 1)$?

12 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has two solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, find two solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

(b) Then find another solution to $A\boldsymbol{x} = \boldsymbol{0}$ and another solution to $A\boldsymbol{x} = \boldsymbol{b}$.

13 Explain why these are all false:

(a) The complete solution is any linear combination of $\boldsymbol{x}_p$ and $\boldsymbol{x}_n$.

(b) A system $A\boldsymbol{x} = \boldsymbol{b}$ has at most one particular solution.

(c) The solution $\boldsymbol{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\boldsymbol{x}\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution $\boldsymbol{x}_n$ in the nullspace.

14 Suppose column 5 of U has no pivot. Then x_5 is a _____ variable. The zero vector (is) (is not) the only solution to $A\boldsymbol{x} = \boldsymbol{0}$. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has _____ solutions.

15 Suppose row 3 of U has no pivot. Then that row is _____. The equation $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided _____. The equation $A\boldsymbol{x} = \boldsymbol{b}$ (*is*) (*is not*) (*might not be*) solvable.

Questions 16–20 are about matrices of "full rank" $r = m$ or $r = n$.

16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ (*always exists*) (*is unique*). The column space of A is _____. An example is $A =$ _____.

17 The largest possible rank of a 6 by 4 matrix is _____. Then there is a pivot in every _____ of U and R. The solution to $Ax = b$ (*always exists*) (*is unique*). The nullspace of A is _____. An example is $A =$ _____.

18 Find by elimination the rank of A and also the rank of A^{T}:

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \text{ (rank depends on } q\text{).}$$

19 Find the rank of A and also of $A^{\mathrm{T}}A$ and also of AA^{T}:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

20 Reduce A to its echelon form U. Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

21 Find the complete solution in the form $x_p + x_n$ to these full rank systems:

(a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$

22 If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right side) to have only one solution? Could $Ax = B$ have no solution?

23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

24 Give examples of matrices A for which the number of solutions to $Ax = b$ is

(a) 0 or 1, depending on b

(b) ∞, regardless of b

(c) 0 or ∞, depending on b

(d) 1, regardless of b.

25 Write down all known relations between r and m and n if $Ax = b$ has

(a) no solution for some b

(b) infinitely many solutions for every $\boldsymbol{b}$

(c) exactly one solution for some $\boldsymbol{b}$, no solution for other $\boldsymbol{b}$

(d) exactly one solution for every $\boldsymbol{b}$.

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix R.

26 Continue elimination from U to R. Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R:

$$U=\begin{bmatrix}2&4&4\\0&3&6\\0&0&0\end{bmatrix} \quad\text{and}\quad U=\begin{bmatrix}2&4&4\\0&3&6\\0&0&5\end{bmatrix}.$$

27 Suppose U is square with n pivots (an invertible matrix). *Explain why $R = I$.*

28 Apply Gauss-Jordan elimination to $U\boldsymbol{x} = \boldsymbol{0}$ and $U\boldsymbol{x} = \boldsymbol{c}$. Reach $R\boldsymbol{x} = \boldsymbol{0}$ and $R\boldsymbol{x} = \boldsymbol{d}$:

$$\begin{bmatrix}U & \boldsymbol{0}\end{bmatrix}=\begin{bmatrix}1&2&3&\mathbf{0}\\0&0&4&\mathbf{0}\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}U & \boldsymbol{c}\end{bmatrix}=\begin{bmatrix}1&2&3&\mathbf{5}\\0&0&4&\mathbf{8}\end{bmatrix}.$$

Solve $R\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}_n$ (its free variable is $x_2 = 1$). Solve $R\boldsymbol{x} = \boldsymbol{d}$ to find $\boldsymbol{x}_p$ (its free variable is $x_2 = 0$).

29 Apply Gauss-Jordan elimination to reduce to $R\boldsymbol{x} = \boldsymbol{0}$ and $R\boldsymbol{x} = \boldsymbol{d}$:

$$\begin{bmatrix}U & 0\end{bmatrix}=\begin{bmatrix}3&0&6&\mathbf{0}\\0&0&2&\mathbf{0}\\0&0&0&\mathbf{0}\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}U & \boldsymbol{c}\end{bmatrix}=\begin{bmatrix}3&0&6&\mathbf{9}\\0&0&2&\mathbf{4}\\0&0&0&\mathbf{5}\end{bmatrix}.$$

Solve $U\boldsymbol{x} = \boldsymbol{0}$ or $R\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}_n$ (free variable $= 1$). What are the solutions to $R\boldsymbol{x} = \boldsymbol{d}$?

30 Reduce to $U\boldsymbol{x} = \boldsymbol{c}$ (Gaussian elimination) and then $R\boldsymbol{x} = \boldsymbol{d}$ (Gauss-Jordan):

$$A\boldsymbol{x}=\begin{bmatrix}1&0&2&3\\1&3&2&0\\2&0&4&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2\\5\\10\end{bmatrix}=\boldsymbol{b}.$$

Find a particular solution $\boldsymbol{x}_p$ and all homogeneous solutions $\boldsymbol{x}_n$.

31 Find matrices A and B with the given property or explain why you can't:

(a) The only solution of $A\boldsymbol{x} = \begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}$.

(b) The only solution of $B\boldsymbol{x} = \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{bmatrix}$.

32 Find the LU factorization of A and the complete solution to $Ax = b$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

33 The complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A.

Challenge Problems

34 Suppose you know that the 3 by 4 matrix A has the vector $s = (2, 3, 1, 0)$ as the only special solution to $Ax = \mathbf{0}$.

(a) What is the *rank* of A and the complete solution to $Ax = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A?

(c) How do you know that $Ax = b$ can be solved for all b ?

35 Suppose K is the 9 by 9 second difference matrix (2's on the diagonal, -1's on the diagonal above and also below). Solve the equation $Kx = b = (10, \ldots, 10)$. If you graph $x_1, \ldots, x_9$ above the points $1, \ldots, 9$ on the x axis, I think the nine points fall on a parabola.

36 Suppose $Ax = b$ and $Cx = b$ have the same (complete) solutions for every b. Is it true that $A = C$?

3.5 Independence, Basis and Dimension

This important section is about the true size of a subspace. There are n columns in an m by n matrix. But the true "dimension" of the column space is not necessarily n. The dimension is measured by counting ***independent columns***—and we have to say what that means. We will see that *the true dimension of the column space is the rank* r.

The idea of independence applies to any vectors $v_1, \ldots, v_n$ in any vector space. Most of this section concentrates on the subspaces that we know and use—especially the column space and the nullspace of A. In the last part we also study "vectors" that are not column vectors. They can be matrices and functions; they can be linearly independent (or dependent). First come the key examples using column vectors.

The goal is to understand a ***basis***: **independent vectors that "span the space".**

Every vector in the space is a unique combination of the basis vectors.

We are at the heart of our subject, and we cannot go on without a basis. The four essential ideas in this section (with first hints at their meaning) are:

1. **Independent vectors** *(no extra vectors)*
2. **Spanning a space** *(enough vectors to produce the rest)*
3. **Basis for a space** *(not too many or too few)*
4. **Dimension of a space** *(the number of vectors in a basis)*

Linear Independence

Our first definition of independence is not so conventional, but you are ready for it.

DEFINITION The columns of A are *linearly independent* when the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. *No other combination Ax of the columns gives the zero vector.*

The columns are independent when the nullspace $N(A)$ contains only the zero vector. Let me illustrate linear independence (and dependence) with three vectors in $\mathbf{R}^3$:

1. If three vectors are *not* in the same plane, they are independent. No combination of v_1, v_2, v_3 in Figure 3.4 gives zero except $0v_1 + 0v_2 + 0v_3$.
2. If three vectors w_1, w_2, w_3 are *in the same plane*, they are dependent.

This idea of independence applies to 7 vectors in 12-dimensional space. If they are the columns of A, and independent, the nullspace only contains $x = \mathbf{0}$. None of the vectors is a combination of the other six vectors.

Now we choose different words to express the same idea. The following definition of independence will apply to any sequence of vectors in any vector space. When the vectors are the columns of A, the two definitions say exactly the same thing.

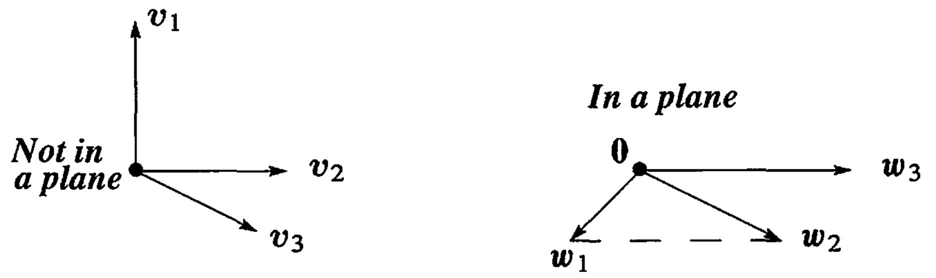


Figure 3.4: Independent vectors $v_1,\ v_2,\ v_3$. Only $0v_1 + 0v_2 + 0v_3$ gives the vector $\mathbf{0}$. Dependent vectors w_1, w_2, w_3. The combination $w_1 - w_2 + w_3$ is $(0, 0, 0)$.

DEFINITION The sequence of vectors $v_1, \ldots, v_n$ is ***linearly independent*** if the only combination that gives the zero vector is $0v_1 + 0v_2 + \cdots + 0v_n$.

Linear independence

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = \mathbf{0} \quad \text{only happens when all } x\text{'s are zero.} \tag{1}$$

If a combination gives $\mathbf{0}$, when the x's are not all zero, the vectors are *dependent*.

Correct language: "The sequence of vectors is linearly independent." *Acceptable shortcut*: "The vectors are independent." *Unacceptable*: "The matrix is independent."

A sequence of vectors is either dependent or independent. They can be combined to give the zero vector (with nonzero x's) or they can't. So the key question is: Which combinations of the vectors give zero? We begin with some small examples in $\mathbf{R}^2$:

(a) The vectors $(1, 0)$ and $(0, 1)$ are independent.

(b) The vectors $(1, 0)$ and $(1, 0.00001)$ are independent.

(c) The vectors $(1, 1)$ and $(-1, -1)$ are *dependent*.

(d) The vectors $(1, 1)$ and $(0, 0)$ are *dependent* because of the zero vector.

(e) In $\mathbf{R}^2$, any three vectors (a, b) and (c, d) and (e, f) are *dependent*.

Geometrically, $(1, 1)$ and $(-1, -1)$ are on a line through the origin. They are dependent. To use the definition, find numbers x_1 and x_2 so that $x_1(1, 1) + x_2(-1, -1) = (0, 0)$. This is the same as solving $Ax = \mathbf{0}$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } x_1 = 1 \text{ and } x_2 = 1.$$

The columns are dependent exactly when *there is a nonzero vector in the nullspace.*

If one of the v's is the zero vector, independence has no chance. Why not?

Three vectors in $\mathbf{R}^2$ cannot be independent! One way to see this: the matrix A with those three columns must have a free variable and then a special solution to $Ax = \mathbf{0}$. Another way: If the first two vectors are independent, some combination will produce the third vector. See the second highlight below.

Now move to three vectors in $\mathbf{R}^3$. If one of them is a multiple of another one, these vectors are dependent. But the complete test involves all three vectors at once. We put them in a matrix and try to solve $Ax = \mathbf{0}$.

Example 1 The columns of this A are dependent. $Ax = \mathbf{0}$ has a nonzero solution:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank is only $r = 2$. *Independent columns produce full column rank* $r = n = 3$.

In that matrix the rows are also dependent. Row 1 minus row 3 is the zero row. For a *square matrix*, we will show that dependent columns imply dependent rows.

Question How to find that solution to $Ax = \mathbf{0}$? The systematic way is elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution $x = (-3, 1, 1)$ was exactly the special solution. It shows how the free column (column 3) is a combination of the pivot columns. That kills independence!

Full column rank The columns of A are independent exactly when the rank is $r = n$. There are n pivots and no free variables. Only $x = \mathbf{0}$ is in the nullspace.

One case is of special importance because it is clear from the start. Suppose seven columns have five components each ($m = 5$ is less than $n = 7$). Then the columns *must be dependent.* Any seven vectors from $\mathbf{R}^5$ are dependent. The rank of A cannot be larger than 5. There cannot be more than five pivots in five rows. $Ax = \mathbf{0}$ has at least $7 - 5 = 2$ free variables, so it has nonzero solutions—which means that the columns are dependent.

Any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$.

This type of matrix has more columns than rows—it is short and wide. The columns are certainly dependent if $n > m$, because $Ax = \mathbf{0}$ has a nonzero solution.

The columns might be dependent or might be independent if $n \le m$. Elimination will reveal the r pivot columns. *It is those r pivot columns that are independent.*

Note Another way to describe linear dependence is this: "*One vector is a combination of the other vectors.*" That sounds clear. Why don't we say this from the start? Our definition was longer: "*Some combination gives the zero vector, other than the trivial combination with every $x = 0$.*" We must rule out the easy way to get the zero vector.

That trivial combination of zeros gives every author a headache. If one vector is a combination of the others, that vector has coefficient $x = 1$.

The point is, our definition doesn't pick out one particular vector as guilty. All columns of A are treated the same. We look at $A\boldsymbol{x} = \mathbf{0}$, and it has a nonzero solution or it hasn't. In the end that is better than asking if the last column (or the first, or a column in the middle) is a combination of the others.

Vectors that Span a Subspace

The first subspace in this book was the column space. Starting with columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, the subspace was filled out by including all combinations $x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n$. *The column space consists of all combinations $A\boldsymbol{x}$ of the columns.* We now introduce the single word "span" to describe this: The column space is ***spanned*** by the columns.

DEFINITION A set of vectors ***spans*** a space if their linear combinations fill the space.

The columns of a matrix span its column space. They might be dependent.

Example 2 $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space $\mathbf{R}^2$.

Example 3 $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbf{R}^2$.

Example 4 $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbf{R}^2$. So does $\boldsymbol{w}_1$ by itself.

Think of two vectors coming out from $(0, 0, 0)$ in 3-dimensional space. Generally they span a plane. Your mind fills in that plane by taking linear combinations. Mathematically you know other possibilities: two vectors could span a line, three vectors could span all of $\mathbf{R}^3$, or only a plane. It is even possible that three vectors span only a line, or ten vectors span only a plane. They are certainly not independent!

The columns span the column space. Here is a new subspace—*which is spanned by the rows.* ***The combinations of the rows produce the "row space".***

DEFINITION The ***row space*** of a matrix is the subspace of $\mathbf{R}^n$ spanned by the rows.
The row space of A is $C(A^{\mathrm{T}})$. It is the column space of A^{T}.

The rows of an m by n matrix have n components. They are vectors in $\mathbf{R}^n$—or they would be if they were written as column vectors. There is a quick way to fix that: *Transpose the matrix.* Instead of the rows of A, look at the columns of A^{T}. Same numbers, but now in the column space $C(A^{\mathrm{T}})$. This row space of A is a subspace of $\mathbf{R}^n$.

Example 5 Describe the column space and the row space of A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}. \text{ Here } m = 3 \text{ and } n = 2.$$

The column space of A is the plane in $\mathbf{R}^3$ spanned by the two columns of A. *The row space of A is spanned by the three rows of A* (which are columns of A^{T}). This row space is all of $\mathbf{R}^2$. Remember: The rows are in $\mathbf{R}^n$ spanning the row space. The columns are in $\mathbf{R}^m$ spanning the column space. Same numbers, different vectors, different spaces.

A Basis for a Vector Space

Two vectors can't span all of $\mathbf{R}^3$, even if they are independent. Four vectors can't be independent, even if they span $\mathbf{R}^3$. We want ***enough independent vectors to span the space*** (and not more). A "***basis***" is just right.

DEFINITION A ***basis*** for a vector space is a sequence of vectors with two properties:

The basis vectors are linearly independent and they span the space.

This combination of properties is fundamental to linear algebra. Every vector $\boldsymbol{v}$ in the space is a combination of the basis vectors, because they span the space. More than that, the combination that produces $\boldsymbol{v}$ is *unique*, because the basis vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are independent:

There is one and only one way to write $\boldsymbol{v}$ as a combination of the basis vectors.

Reason: Suppose $\boldsymbol{v} = a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n$ and also $\boldsymbol{v} = b_1\boldsymbol{v}_1 + \cdots + b_n\boldsymbol{v}_n$. By subtraction $(a_1 - b_1)\boldsymbol{v}_1 + \cdots + (a_n - b_n)\boldsymbol{v}_n$ is the zero vector. From the independence of the $\boldsymbol{v}$'s, each $a_i - b_i = 0$. Hence $a_i = b_i$, and there are not two ways to produce $\boldsymbol{v}$.

Example 6 The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbf{R}^2$.

$$\text{The basis vectors} \quad \boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{are independent. They span } \mathbf{R}^2.$$

Everybody thinks of this basis first. The vector $\boldsymbol{i}$ goes across and $\boldsymbol{j}$ goes straight up. The columns of the 3 by 3 identity matrix are the standard basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The columns of the n by n identity matrix give the "**standard basis**" for $\mathbf{R}^n$.

Now we find many other bases (infinitely many). The basis is not unique!

Example 7 (Important) The columns of *every invertible n by n matrix* give a basis for $\mathbf{R}^n$:

$$\begin{array}{ll} \textbf{Invertible matrix} & \\ \text{Independent columns} & A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ \text{Column space is } \mathbf{R}^3 & \end{array} \qquad \begin{array}{ll} \textbf{Singular matrix} & \\ \text{Dependent columns} & B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}. \\ \text{Column space} \neq \mathbf{R}^3 & \end{array}$$

The only solution to $Ax = \mathbf{0}$ is $x = A^{-1}\mathbf{0} = \mathbf{0}$. The columns are independent. They span the whole space $\mathbf{R}^n$—because every vector $\boldsymbol{b}$ is a combination of the columns. $Ax = \boldsymbol{b}$ can always be solved by $x = A^{-1}\boldsymbol{b}$. Do you see how everything comes together for invertible matrices? Here it is in one sentence:

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are a ***basis for*** $\mathbf{R}^n$ exactly when they are ***the columns of an n by n invertible matrix.*** Thus $\mathbf{R}^n$ has infinitely many different bases.

When the columns are dependent, we keep only the *pivot columns*—the first two columns of B above, with its two pivots. They are independent and they span the column space.

The pivot columns of A are a basis for its column space. The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R.

Example 8 This matrix is not invertible. Its columns are not a basis for anything!

$$\begin{array}{l} \textbf{One pivot column} \\ \textbf{One pivot row } (r = 1) \end{array} \qquad A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Column 1 of A is the pivot column. That column alone is a basis for its column space. The second column of A would be a different basis. So would any nonzero multiple of that column. There is no shortage of bases. One definite choice is the pivot columns.

Notice that the pivot column $(1, 0)$ of this R ends in zero. That column is a basis for the column space of R, but it doesn't belong to the column space of A. The column spaces of A and R are different. Their bases are different. (Their dimensions are the same.)

The row space of A is the *same* as the row space of R. It contains $(2, 4)$ and $(1, 2)$ and all other multiples of those vectors. As always, there are infinitely many bases to choose from. One natural choice is to pick the nonzero rows of R (rows with a pivot). So this matrix A with rank one has only one vector in the basis:

$$\text{Basis for the column space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \quad \text{Basis for the row space: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The next chapter will come back to these bases for the column space and row space. We are happy first with examples where the situation is clear (and the idea of a basis is still new). The next example is larger but still clear.

Example 9 Find bases for the column and row spaces of this rank two matrix:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 are the pivot columns. They are a basis for the column space (of R!). The vectors in that column space all have the form $\boldsymbol{b} = (x, y, 0)$. The column space of R is the "xy plane" inside the full 3-dimensional xyz space. That plane is not $\mathbf{R}^2$, it is a subspace of $\mathbf{R}^3$. Columns 2 and 3 are also a basis for the same column space. Which pairs of columns of R are *not* a basis for its column space?

The row space of R is a subspace of $\mathbf{R}^4$. The simplest basis for that row space is the two nonzero rows of R. The third row (the zero vector) is in the row space too. But it is not in a *basis* for the row space. The basis vectors must be independent.

Question Given five vectors in $\mathbf{R}^7$, ***how do you find a basis for the space they span?***

First answer Make them the rows of A, and eliminate to find the nonzero rows of R.
Second answer Put the five vectors into the columns of A. Eliminate to find the pivot columns (of A not R). The program **colbasis** uses the column numbers from *pivcol*.

Could another basis have more vectors, or fewer? This is a crucial question with a good answer: *No.* ***All bases for a vector space contain the same number of vectors.***

The number of vectors, in any and every basis, is the "dimension" of the space.

Dimension of a Vector Space

We have to prove what was just stated. There are many choices for the basis vectors, but the *number* of basis vectors doesn't change.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ are both bases for the same vector space, then $m = n$.

Proof Suppose that there are more $\boldsymbol{w}$'s than $\boldsymbol{v}$'s. From $n > m$ we want to reach a contradiction. The $\boldsymbol{v}$'s are a basis, so $\boldsymbol{w}_1$ must be a combination of the $\boldsymbol{v}$'s. If $\boldsymbol{w}_1$ equals $a_{11}\boldsymbol{v}_1 + \cdots + a_{m1}\boldsymbol{v}_m$, this is the first column of a matrix multiplication VA:

Each $\boldsymbol{w}$ is a combination of the $\boldsymbol{v}$'s

$$W = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \ldots & \boldsymbol{w}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_m \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = VA.$$

We don't know each a_{ij}, but we know the shape of A (it is m by n). The second vector $\boldsymbol{w}_2$ is also a combination of the $\boldsymbol{v}$'s. The coefficients in that combination fill the second column of A. The key is that A has a row for every $\boldsymbol{v}$ and a column for every $\boldsymbol{w}$. A is a short wide matrix, since we assumed $n > m$. So $A\boldsymbol{x} = \boldsymbol{0}$ ***has a nonzero solution.***

$A\boldsymbol{x} = 0$ gives $VA\boldsymbol{x} = \boldsymbol{0}$ which is $W\boldsymbol{x} = \boldsymbol{0}$. *A combination of the $\boldsymbol{w}$'s gives zero*! Then the $\boldsymbol{w}$'s could not be a basis—our assumption $n > m$ is **not possible** for two bases.

If $m > n$ we exchange the $\boldsymbol{v}$'s and $\boldsymbol{w}$'s and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$.

The number of basis vectors depends on the space—not on a particular basis. The number is the same for every basis, and it counts the "degrees of freedom" in the space.

The dimension of the space $\mathbf{R}^n$ is n. We now introduce the important word ***dimension*** for other vector spaces too.

DEFINITION The ***dimension of a space*** is the ***number of vectors*** in every basis.

This matches our intuition. The line through $v = (1, 5, 2)$ has dimension one. It is a subspace with this one vector v in its basis. Perpendicular to that line is the plane $x + 5y + 2z = 0$. This plane has dimension 2. To prove it, we find a basis $(-5, 1, 0)$ and $(-2, 0, 1)$. The dimension is 2 because the basis contains two vectors.

The plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$, which has two free variables. Our basis vectors $(-5, 1, 0)$ and $(-2, 0, 1)$ are the "special solutions" to $Ax = \mathbf{0}$. The next section shows that the $n - r$ special solutions always give *a basis for the nullspace*. $\boldsymbol{C}(A)$ has dimension r and the nullspace $\boldsymbol{N}(A)$ has dimension $n - r$.

Note about the language of linear algebra We never say "the rank of a space" or "the dimension of a basis" or "the basis of a matrix". Those terms have no meaning. It is the ***dimension of the column space*** that equals the ***rank of the matrix.***

Bases for Matrix Spaces and Function Spaces

The words "independence" and "basis" and "dimension" are not at all restricted to column vectors. We can ask whether three matrices A_1, A_2, A_3 are independent. When they are in the space of all 3 by 4 matrices, some combination might give the zero matrix. We can also ask the dimension of the full 3 by 4 matrix space. (It is 12.)

In differential equations, $d^2y/dx^2 = y$ has a space of solutions. One basis is $y = e^x$ and $y = e^{-x}$. Counting the basis functions gives the dimension 2 for the space of all solutions. (The dimension is 2 because of the second derivative.)

Matrix spaces and function spaces may look a little strange after $\mathbf{R}^n$. But in some way, you haven't got the ideas of basis and dimension straight until you can apply them to "vectors" other than column vectors.

Matrix spaces The vector space $\mathbf{M}$ contains all 2 by 2 matrices. Its dimension is 4.

$$\textbf{One basis is} \quad A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Those matrices are linearly independent. We are not looking at their columns, but at the whole matrix. Combinations of those four matrices can produce any matrix in $\mathbf{M}$, so they span the space:

$$\begin{array}{l}\textbf{Every } A \textbf{ combines} \\ \textbf{the basis matrices}\end{array} \qquad c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = A.$$

A is zero only if the c's are all zero—this proves independence of A_1, A_2, A_3, A_4.

The three matrices A_1, A_2, A_4 are a basis for a subspace—the upper triangular matrices. Its dimension is 3. A_1 and A_4 are a basis for the diagonal matrices. What is a basis for the symmetric matrices? Keep A_1 and A_4, and throw in $A_2 + A_3$.

To push this further, think about the space of all n by n matrices. One possible basis uses matrices that have only a single nonzero entry (that entry is 1). There are n^2 positions for that 1, so there are n^2 basis matrices:

The dimension of the whole n by n matrix space is n^2.

The dimension of the subspace of *upper triangular* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$.

The dimension of the subspace of *diagonal* matrices is n.

The dimension of the subspace of *symmetric* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$ (why ?).

Function spaces The equations $d^2y/dx^2 = 0$ and $d^2y/dx^2 = -y$ and $d^2y/dx^2 = y$ involve the second derivative. In calculus we solve to find the functions $y(x)$:

$$\begin{array}{ll} y'' = 0 & \text{is solved by any linear function } y = cx + d \\ y'' = -y & \text{is solved by any combination } y = c\sin x + d\cos x \\ y'' = y & \text{is solved by any combination } y = ce^x + de^{-x}. \end{array}$$

That solution space for $y'' = -y$ has two basis functions: $\sin x$ and $\cos x$. The space for $y'' = 0$ has x and 1. It is the "nullspace" of the second derivative! The dimension is 2 in each case (these are second-order equations).

The solutions of $y'' = 2$ don't form a subspace—the right side $b = 2$ is not zero. A particular solution is $y(x) = x^2$. The complete solution is $y(x) = x^2 + cx + d$. All those functions satisfy $y'' = 2$. Notice the particular solution plus any function $cx + d$ in the nullspace. A linear differential equation is like a linear matrix equation $Ax = \boldsymbol{b}$. But we solve it by calculus instead of linear algebra.

We end here with the space $\mathbf{Z}$ that contains only the zero vector. The dimension of this space is *zero*. ***The empty set*** (containing no vectors) ***is a basis for*** $\mathbf{Z}$. We can never allow the zero vector into a basis, because then linear independence is lost.

■ REVIEW OF THE KEY IDEAS ■

1. The columns of A are ***independent*** if $\boldsymbol{x} = \mathbf{0}$ is the only solution to $A\boldsymbol{x} = \mathbf{0}$.
2. The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ ***span*** a space if their combinations fill that space.
3. ***A basis consists of linearly independent vectors that span the space.*** Every vector in the space is a *unique* combination of the basis vectors.
4. All bases for a space have the same number of vectors. This number of vectors in a basis is the ***dimension*** of the space.
5. The pivot columns are one basis for the column space. The dimension is r.

■ WORKED EXAMPLES ■

3.5 A Start with the vectors $v_1 = (1,2,0)$ and $v_2 = (2,3,0)$. **(a)** Are they linearly independent? **(b)** Are they a basis for any space? **(c)** What space **V** do they span? **(d)** What is the dimension of **V**? **(e)** Which matrices A have **V** as their column space? **(f)** Which matrices have **V** as their nullspace? **(g)** Describe all vectors v_3 that complete a basis v_1, v_2, v_3 for $\mathbf{R}^3$.

Solution

(a) v_1 and v_2 are independent—the only combination to give $\mathbf{0}$ is $0v_1 + 0v_2$.

(b) Yes, they are a basis for the space they span.

(c) That space **V** contains all vectors $(x, y, 0)$. It is the xy plane in $\mathbf{R}^3$.

(d) The dimension of **V** is 2 since the basis contains two vectors.

(e) This **V** is the column space of any 3 by n matrix A of rank 2, if every column is a combination of v_1 and v_2. In particular A could just have columns v_1 and v_2.

(f) This **V** is the nullspace of any m by 3 matrix B of rank 1, if every row is a multiple of $(0,0,1)$. In particular take $B = [0\ \ 0\ \ 1]$. Then $Bv_1 = \mathbf{0}$ and $Bv_2 = \mathbf{0}$.

(g) Any third vector $v_3 = (a,b,c)$ will complete a basis for $\mathbf{R}^3$ provided $c \neq 0$.

3.5 B Start with three independent vectors w_1, w_2, w_3. Take combinations of those vectors to produce v_1, v_2, v_3. Write the combinations in matrix form as $V = WM$:

$$\begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_3 &= \phantom{w_1 +{}} w_2 + cw_3 \end{aligned} \qquad \text{which is} \qquad \begin{bmatrix} & & \\ v_1 & v_2 & v_3 \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ w_1 & w_2 & w_3 \\ & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix V to see if its columns are linearly independent? If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent. If $c = 1$ show that the v's are linearly *dependent*.

Solution The test on V for independence of its columns was in our first definition: *The nullspace of V must contain only the zero vector.* Then $x = (0,0,0)$ is the only combination of the columns that gives $Vx =$ zero vector.

If $c = 1$ in our problem, we can see *dependence* in two ways. First, $v_1 + v_3$ will be the same as v_2. (If you add $w_1 + w_2$ to $w_2 + w_3$ you get $w_1 + 2w_2 + w_3$ which is v_2.) In other words $v_1 - v_2 + v_3 = \mathbf{0}$—which says that the v's are not independent.

The other way is to look at the nullspace of M. If $c = 1$, the vector $x = (1,-1,1)$ is in that nullspace, and $Mx = \mathbf{0}$. Then certainly $WMx = \mathbf{0}$ which is the same as $Vx = 0$. So the v's are dependent. This specific $x = (1,-1,1)$ from the nullspace tells us again that $v_1 - v_2 + v_3 = \mathbf{0}$.

Now suppose $c \neq 1$. Then the matrix M is invertible. So if $\boldsymbol{x}$ is *any nonzero vector* we know that $M\boldsymbol{x}$ is nonzero. Since the $\boldsymbol{w}$'s are given as independent, we further know that $WM\boldsymbol{x}$ is nonzero. Since $V = WM$, this says that $\boldsymbol{x}$ is *not* in the nullspace of V. In other words $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent.

The general rule is "independent $\boldsymbol{v}$'s from independent $\boldsymbol{w}$'s when M is invertible". And if these vectors are in $\mathbf{R}^3$, they are not only independent—they are a basis for $\mathbf{R}^3$. "Basis of $\boldsymbol{v}$'s from basis of $\boldsymbol{w}$'s when the change of basis matrix M is invertible."

3.5 C (***Important example***) Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is a basis for $\mathbf{R}^n$ and the n by n matrix A is invertible. Show that $A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n$ is also a basis for $\mathbf{R}^n$.

Solution In *matrix language*: Put the basis vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ in the columns of an invertible(!) matrix V. Then $A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n$ are the columns of AV. Since A is invertible, so is AV and its columns give a basis.

In *vector language*: Suppose $c_1A\boldsymbol{v}_1 + \cdots + c_nA\boldsymbol{v}_n = \mathbf{0}$. This is $A\boldsymbol{v} = \mathbf{0}$ with $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$. Multiply by A^{-1} to reach $\boldsymbol{v} = \mathbf{0}$. By linear independence of the $\boldsymbol{v}$'s, all $c_i = 0$. This shows that the $A\boldsymbol{v}$'s are independent.

To show that the $A\boldsymbol{v}$'s span $\mathbf{R}^n$, solve $c_1A\boldsymbol{v}_1 + \cdots + c_nA\boldsymbol{v}_n = \boldsymbol{b}$ which is the same as $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = A^{-1}\boldsymbol{b}$. Since the $\boldsymbol{v}$'s are a basis, this must be solvable.

Problem Set 3.5

Questions 1–10 are about linear independence and linear dependence.

1 Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \mathbf{0}$ or $A\boldsymbol{x} = \mathbf{0}$. The $\boldsymbol{v}$'s go in the columns of A.

2 (Recommended) Find the largest possible number of independent vectors among

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

4 If a, d, f in Question 3 are all nonzero, show that the only solution to $Ux = \mathbf{0}$ is $x = \mathbf{0}$. Then the upper triangular U has independent columns.

5 Decide the dependence or independence of

(a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$

(b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

6 Choose three independent columns of U. Then make two other choices. Do the same for A.

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

7 If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$ and $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are *dependent*. Find a combination of the v's that gives zero. Which matrix A in $[\, v_1 \ v_2 \ v_3 \,] = [\, w_1 \ w_2 \ w_3 \,]\, A$ is singular?

8 If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$ and $v_2 = w_1 + w_3$ and $v_3 = w_1 + w_2$ are *independent*. (Write $c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}$ in terms of the w's. Find and solve equations for the c's, to show they are zero.)

9 Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^3$.

(a) These four vectors are dependent because _____ .

(b) The two vectors v_1 and v_2 will be dependent if _____ .

(c) The vectors v_1 and $(0, 0, 0)$ are dependent because _____ .

10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbf{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–15 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

11 Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$

(b) the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

12 The vector $\boldsymbol{b}$ is in the subspace spanned by the columns of A when _____ has a solution. The vector $\boldsymbol{c}$ is in the row space of A when _____ has a solution.

True or false: If the zero vector is in the row space, the rows are dependent.

13 Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of A, (b) column space of U, (c) row space of A, (d) row space of U:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

14 $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ are combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$. Write $\boldsymbol{v}$ and $\boldsymbol{w}$ as combinations of $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$. The two pairs of vectors _____ the same space. When are they a basis for the same space?

Questions 15–25 are about the requirements for a basis.

15 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, the space they span has dimension _____. These vectors are a _____ for that space. If the vectors are the columns of an m by n matrix, then m is _____ than n. If $m = n$, that matrix is _____.

16 Find a basis for each of these subspaces of $\mathbf{R}^4$:

(a) All vectors whose components are equal.

(b) All vectors whose components add to zero.

(c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.

(d) The column space and the nullspace of I (4 by 4).

17 Find three different bases for the column space of $U = \left[\begin{smallmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{smallmatrix}\right]$. Then find two different bases for the row space of U.

18 Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ are six vectors in $\mathbf{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbf{R}^4$.

(b) Those vectors (are)(are not)(might be) linearly independent.

(c) Any four of those vectors (are)(are not)(might be) a basis for $\mathbf{R}^4$.

19 The columns of A are n vectors from $\mathbf{R}^m$. If they are linearly independent, what is the rank of A? If they span $\mathbf{R}^m$, what is the rank? If they are a basis for $\mathbf{R}^m$, what then? *Looking ahead*: The rank r counts the number of _____ columns.

20 Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

21 Suppose the columns of a 5 by 5 matrix A are a basis for $\mathbf{R}^5$.

(a) The equation $A\boldsymbol{x} = \boldsymbol{0}$ has only the solution $\boldsymbol{x} = \boldsymbol{0}$ because _____.

(b) If $\boldsymbol{b}$ is in $\mathbf{R}^5$ then $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because the basis vectors _____ $\mathbf{R}^5$.

Conclusion: A is invertible. Its rank is 5. Its rows are also a basis for $\mathbf{R}^5$.

22 Suppose S is a 5-dimensional subspace of $\mathbf{R}^6$. True or false (example if false):

(a) Every basis for S can be extended to a basis for $\mathbf{R}^6$ by adding one more vector.

(b) Every basis for $\mathbf{R}^6$ can be reduced to a basis for S by removing one vector.

23 U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

24 True or false (give a good reason):

(a) If the columns of a matrix are dependent, so are the rows.

(b) The column space of a 2 by 2 matrix is the same as its row space.

(c) The column space of a 2 by 2 matrix has the same dimension as its row space.

(d) The columns of a matrix are a basis for the column space.

25 For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Questions 26–30 are about spaces where the "vectors" are matrices.

26 Find a basis (and the dimension) for each of these subspaces of 3 by 3 matrices:

(a) All diagonal matrices.

(b) All symmetric matrices ($A^{\mathrm{T}} = A$).

(c) All skew-symmetric matrices ($A^{\mathrm{T}} = -A$).

27 Construct six linearly independent 3 by 3 echelon matrices $U_1, \ldots, U_6$.

28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

29 What subspace of 3 by 3 matrices is spanned (take all combinations) by

(a) the invertible matrices?

(b) the rank one matrices?

(c) the identity matrix?

30 Find a basis for the space of 2 by 3 matrices whose nullspace contains $(2, 1, 1)$.

Questions 31–35 are about spaces where the "vectors" are functions.

31 (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

(b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

32 The cosine space $\mathbf{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace with $y(0) = 0$.

33 Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$

(b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

34 Suppose $y_1(x), y_2(x), y_3(x)$ are three different functions of x. The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

35 Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$.

36 Find a basis for the space $\mathbf{S}$ of vectors (a, b, c, d) with $a + c + d = 0$ and also for the space $\mathbf{T}$ with $a + b = 0$ and $c = 2d$. What is the dimension of the intersection $\mathbf{S} \cap \mathbf{T}$?

37 If $AS = SA$ for the *shift matrix* S, show that A must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

"The subspace of matrices that commute with the shift S has dimension _____."

38 Which of the following are bases for $\mathbf{R}^3$?

(a) $(1, 2, 0)$ and $(0, 1, -1)$

(b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$

(c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$

(d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$

39 Suppose A is 5 by 4 with rank 4. Show that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution when the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible. Show that $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $[\,A \;\; \boldsymbol{b}\,]$ is singular.

40 (a) Find a basis for all solutions to $d^4y/dx^4 = y(x)$.
(b) Find a particular solution to $d^4y/dx^4 = y(x) + 1$. Find the complete solution.

Challenge Problems

41 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives $c_1 P_1 + \cdots + c_5 P_5 =$ zero matrix, and check entries to prove c_i is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

42 Choose $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including $\boldsymbol{x}$ itself, span a subspace $\mathbf{S}$. Find specific vectors $\boldsymbol{x}$ so that the dimension of $\mathbf{S}$ is: (a) zero, (b) one, (c) three, (d) four.

43 Intersections and sums have $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$. Start with a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ for the intersection $\mathbf{V} \cap \mathbf{W}$. Extend with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ to a basis for $\mathbf{V}$, and separately with $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$ to a basis for $\mathbf{W}$. Prove that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s and $\boldsymbol{w}$'s together are ***independent***. The dimensions have $(r + s) + (r + t) = (r) + (r + s + t)$ as desired.

44 Mike Artin suggested a neat higher-level proof of that dimension formula in Problem 43. From all inputs $\boldsymbol{v}$ in $\mathbf{V}$ and $\boldsymbol{w}$ in $\mathbf{W}$, the "sum transformation" produces $\boldsymbol{v}+\boldsymbol{w}$. Those outputs fill the space $\mathbf{V}+\mathbf{W}$. The nullspace contains all pairs $\boldsymbol{v} = \boldsymbol{u}$, $\boldsymbol{w} = -\boldsymbol{u}$ for vectors $\boldsymbol{u}$ in $\mathbf{V} \cap \mathbf{W}$. (Then $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u} = \mathbf{0}$.) So $\dim(\mathbf{V}+\mathbf{W}) + \dim(\mathbf{V} \cap \mathbf{W})$ equals $\dim(\mathbf{V}) + \dim(\mathbf{W})$ (*input dimension from* $\mathbf{V}$ *and* $\mathbf{W}$) by the crucial formula

dimension of outputs + dimension of nullspace = dimension of inputs.

Problem For an m by n matrix of rank r, what are those 3 dimensions? Outputs = column space. This question will be answered in Section 3.6, can you do it now?

45 Inside $\mathbf{R}^n$, suppose dimension $(\mathbf{V})$ + dimension $(\mathbf{W}) > n$. Show that some nonzero vector is in both $\mathbf{V}$ and $\mathbf{W}$.

46 Suppose A is 10 by 10 and $A^2 = 0$ (zero matrix). This means that the column space of A is contained in the _____. If A has rank r, those subspaces have dimension $r \le 10 - r$. So the rank is $r \le 5$.

(This problem was added to the second printing: If $A^2 = 0$ it says that $r \le n/2$.)

3.6 Dimensions of the Four Subspaces

The main theorem in this chapter connects ***rank*** and ***dimension***. The ***rank*** of a matrix is the number of pivots. The ***dimension*** of a subspace is the number of vectors in a basis. We count pivots or we count basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A, and the other two from A^{T}:

Four Fundamental Subspaces

1. The ***row space*** is $C(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbf{R}^m$.
3. The ***nullspace*** is $N(A)$, a subspace of $\mathbf{R}^n$.
4. The ***left nullspace*** is $N(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^m$. This is our new space.

In this book the column space and nullspace came first. We know $C(A)$ and $N(A)$ pretty well. Now the other two subspaces come forward. The row space contains all combinations of the rows. *This is the column space of A^{T}.*

For the left nullspace we solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$—that system is n by m. *This is the nullspace of A^{T}.* The vectors $\boldsymbol{y}$ go on the *left* side of A when the equation is written as $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$. The matrices A and A^{T} are usually different. So are their column spaces and their nullspaces. But those spaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: ***The row space and column space have the same dimension*** r (the rank of the matrix). The other important fact involves the two nullspaces:

$N(A)$ and $N(A^{\mathrm{T}})$ have dimensions $n - r$ and $m - r$, to make up the full n and m.

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together (two in $\mathbf{R}^n$ and two in $\mathbf{R}^m$). That completes the "right way" to understand every $A\boldsymbol{x} = \boldsymbol{b}$. Stay with it—you are doing real mathematics.

The Four Subspaces for R

Suppose A is reduced to its row echelon form R. For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't change!) as we look back at A. The main point is that *the four dimensions are the same for A and R.*

As a specific 3 by 5 example, look at the four subspaces for the echelon matrix R:

$$\begin{matrix} m = 3 \\ n = 5 \\ r = 2 \end{matrix} \qquad \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textbf{pivot rows 1 and 2} \\ \\ \textbf{pivot columns 1 and 4} \end{matrix}$$

The rank of this matrix R is $r = 2$ (*two pivots*). Take the four subspaces in order.

1. The ***row space*** of R has dimension 2, matching the rank.

Reason: The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing new. So rows 1 and 2 span the row space $C(R^T)$.

The pivot rows 1 and 2 are independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows are a basis for the row space.

The dimension of the row space is the rank r. The nonzero rows of R form a basis.

2. The ***column space*** of R also has dimension $r = 2$.

Reason: The pivot columns 1 and 4 form a basis for $C(R)$. They are independent because they start with the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns. Actually the combinations we need are the three special solutions!

Column 2 is 3 (column 1). The special solution is $(-3, 1, 0, 0, 0)$.

Column 3 is 5 (column 1). The special solution is $(-5, 0, 1, 0, 0,)$.

Column 5 is 7 (column 1) + 2 (column 4). That solution is $(-7, 0, 0, -2, 1)$.

The pivot columns are independent, and they span, so they are a basis for $C(R)$.

The dimension of the column space is the rank r. The pivot columns form a basis.

3. The ***nullspace*** has dimension $n - r = 5 - 2$. There are $n - r = 3$ free variables. Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $Rx = \mathbf{0}$. Set a free variable to 1, and solve for x_1 and x_4:

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \begin{array}{l} Rx = \mathbf{0} \text{ has the} \\ \text{complete solution} \\ x = x_2 s_2 + x_3 s_3 + x_5 s_5 \end{array} .$$

There is a special solution for each free variable. With n variables and r pivot variables, that leaves $n - r$ free variables and special solutions. $N(R)$ has dimension $n - r$.

The nullspace has dimension $n - r$. The special solutions form a basis.

The special solutions are independent, because they contain the identity matrix in rows 2, 3, 5. All solutions are combinations of special solutions, $x = x_2 s_2 + x_3 s_3 + x_5 s_5$, because this puts x_2, x_3 and x_5 in the correct positions. Then the pivot variables x_1 and x_4 are totally determined by the equations $Rx = \mathbf{0}$.

4. The ***nullspace of R^{T}*** (***left nullspace of R***) has dimension $m - r = 3 - 2$.

Reason: The equation $R^{\mathrm{T}}y = \mathbf{0}$ looks for combinations of the columns of R^{T} (*the rows of R*) that produce zero. This equation $R^{\mathrm{T}}y = \mathbf{0}$ or $y^{\mathrm{T}}R = \mathbf{0}^{\mathrm{T}}$ is

$$\textbf{Left nullspace} \qquad \begin{array}{r} y_1\,[1,\ 3,\ 5,\ 0,\ 7] \\ +y_2\,[0,\ 0,\ 0,\ 1,\ 2] \\ +y_3\,[0,\ 0,\ 0,\ 0,\ 0] \\ \hline [0,\ 0,\ 0,\ 0,\ 0] \end{array} \tag{1}$$

The solutions y_1, y_2, y_3 are pretty clear. We need $y_1 = 0$ and $y_2 = 0$. The variable y_3 is free (it can be anything). The nullspace of R^{T} contains all vectors $y = (0, 0, y_3)$. It is the line of all multiples of the basis vector $(0, 0, 1)$.

In all cases R ends with $m - r$ zero rows. Every combination of these $m - r$ rows gives zero. These are the *only* combinations of the rows of R that give zero, because the pivot rows are linearly independent. The left nullspace of R contains all these solutions $y = (0, \cdots, 0, y_{r+1}, \cdots, y_m)$ to $R^{\mathrm{T}}y = 0$.

If A is m by n of rank r, its left nullspace has dimension $m - r$.

To produce a zero combination, y must start with r zeros. This leaves dimension $m - r$.

Why is this a "*left* nullspace"? The reason is that $R^{\mathrm{T}}y = \mathbf{0}$ can be transposed to $y^{\mathrm{T}}R = \mathbf{0}^{\mathrm{T}}$. Now y^{T} is a row vector to the *left* of R. You see the y's in equation (1) multiplying the rows. This subspace came fourth, and some linear algebra books omit it—but that misses the beauty of the whole subject.

In $\mathbf{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n).
In $\mathbf{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m).

So far this is proved for echelon matrices R. Figure 3.5 shows the same for A.

The Four Subspaces for A

We have a job still to do. ***The subspace dimensions for A are the same as for R.*** The job is to explain why. A is now any matrix that reduces to $R = \text{rref}(A)$.

$$\textbf{A reduces to R} \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \qquad \text{Notice } C(A) \neq C(R) \tag{2}$$

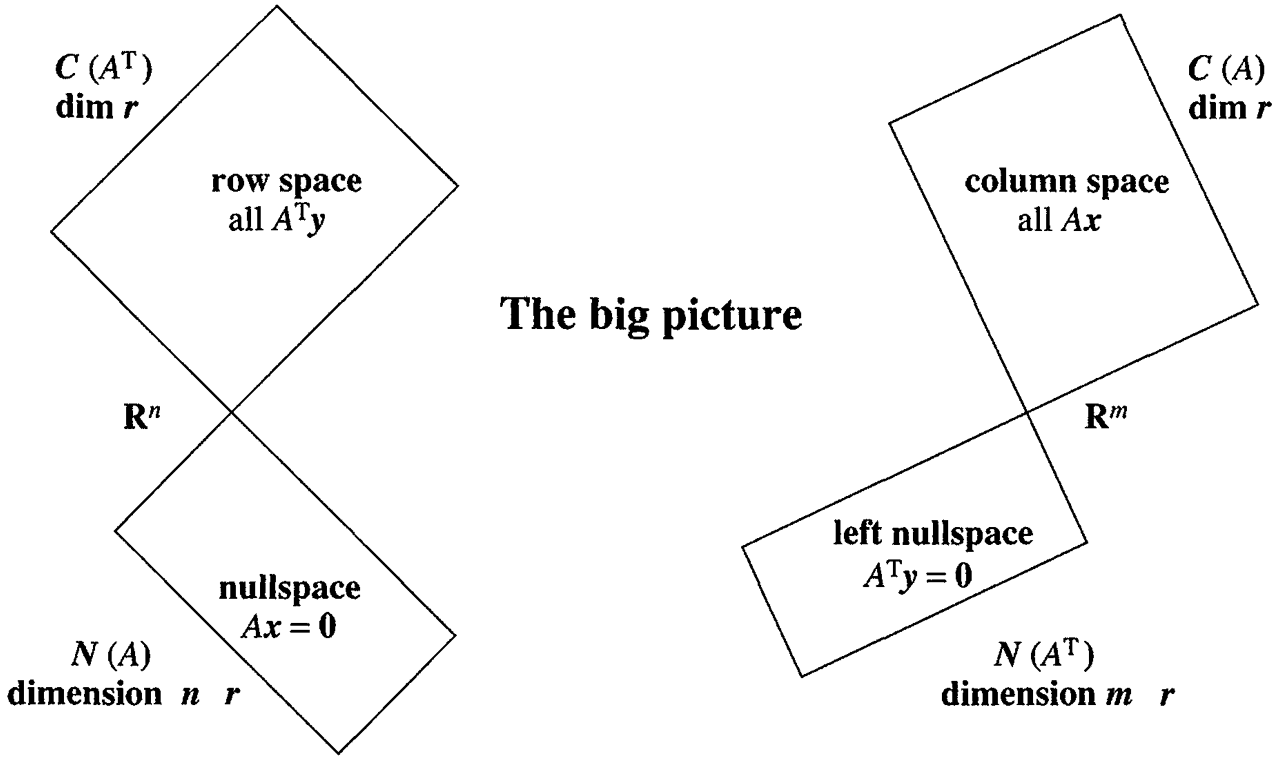


Figure 3.5: The dimensions of the Four Fundamental Subspaces (for R and for A).

An elimination matrix takes A to R. The big picture (Figure 3.5) applies to both. The invertible matrix E is the product of the elementary matrices that reduce A to R:

$$\textbf{A to R and back} \qquad EA = R \quad \text{and} \quad A = E^{-1}R \tag{3}$$

1 ***A has the same row space as R*. Same dimension r and same basis.**

Reason: Every row of A is a combination of the rows of R. Also every row of R is a combination of the rows of A. Elimination changes rows, but not row *spaces*.

Since A has the same row space as R, we can choose the first r rows of R as a basis. Or we could choose r suitable rows of the original A. They might not always be the *first* r rows of A, because those could be dependent. The good r rows of A are the ones that end up as pivot rows in R.

2 ***The column space of A has dimension r*.** For every matrix this is essential:

The number of independent columns equals the number of independent rows.

Wrong reason: "A and R have the same column space." This is false. The columns of R often end in zeros. The columns of A don't often end in zeros. The column spaces are different, but their *dimensions* are the same—equal to r.

Right reason: The ***same*** combinations of the columns are zero (or nonzero) for A and R. Say that another way: $Ax = \mathbf{0}$ *exactly when* $Rx = \mathbf{0}$. The r pivot columns (of both) are independent.

Conclusion The r pivot columns of A are a basis for *its* column space.

3 ***A has the same nullspace as R.* Same dimension $n - r$ and same basis.**

Reason: The elimination steps don't change the solutions. The special solutions are a basis for this nullspace (as we always knew). There are $n - r$ free variables, so the dimension of the nullspace is $n - r$. Notice that $r + (n - r)$ equals n:

$$\textbf{(dimension of column space)} + \textbf{(dimension of nullspace)} = \textbf{dimension of } \mathbf{R}^n.$$

4 ***The left nullspace of A*** (the nullspace of A^{T}) ***has dimension $m - r$.***

Reason: A^{T} is just as good a matrix as A. When we know the dimensions for every A, we also know them for A^{T}. Its column space was proved to have dimension r. Since A^{T} is n by m, the "whole space" is now $\mathbf{R}^m$. The counting rule for A was $r + (n - r) = n$. The counting rule for A^{T} is $r + (m - r) = m$. We now have all details of the main theorem:

***Fundamental Theorem of Linear Algebra,* Part 1**

The column space and row space both have dimension r.
The nullspaces have dimensions $n - r$ and $m - r$.

By concentrating on *spaces* of vectors, not on individual numbers or vectors, we get these clean rules. You will soon take them for granted—eventually they begin to look obvious. But if you write down an 11 by 17 matrix with 187 nonzero entries, I don't think most people would see why these facts are true:

Two key facts
$$\text{dimension of } \boldsymbol{C}(A) = \text{dimension of } \boldsymbol{C}(A^{\mathrm{T}}) = \text{rank of } A$$
$$\text{dimension of } \boldsymbol{C}(A) + \text{dimension of } \boldsymbol{N}(A) = 17.$$

Example 1 $A = [\,1 \ \ 2 \ \ 3\,]$ has $m = 1$ and $n = 3$ and rank $r = 1$.

The row space is a line in $\mathbf{R}^3$. The nullspace is the plane $A\boldsymbol{x} = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is $3 - 1$). The dimensions add to $\mathbf{1 + 2 = 3}$.

The columns of this 1 by 3 matrix are in $\mathbf{R}^1$! The column space is all of $\mathbf{R}^1$. The left nullspace contains only the zero vector. The only solution to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ is $\boldsymbol{y} = \mathbf{0}$, no other multiple of $[\,1 \ \ 2 \ \ 3\,]$ gives the zero row. Thus $\boldsymbol{N}(A^{\mathrm{T}})$ is $\mathbf{Z}$, the zero space with dimension 0 (which is $m - r$). In $\mathbf{R}^m$ the dimensions add to $\mathbf{1 + 0 = 1}$.

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ with $n = 3$ and rank $r = 1$.

The row space is the same line through $(1,2,3)$. The nullspace must be the same plane $x_1 + 2x_2 + 3x_3 = 0$. Their dimensions still add to $1 + 2 = 3$.

All columns are multiples of the first column $(1,2)$. Twice the first row minus the second row is the zero row. Therefore $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has the solution $\boldsymbol{y} = (2,-1)$. The column space and left nullspace are **perpendicular lines** in $\mathbf{R}^2$. Dimensions $1 + 1 = 2$.

$$\text{Column space} = \text{line through } \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Left nullspace} = \text{line through } \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

If A has three equal rows, its rank is _____. What are two of the $\boldsymbol{y}$'s in its left nullspace?

The y's in the left nullspace combine the rows to give the zero row.

Matrices of Rank One

That last example had rank $r = 1$—and rank one matrices are special. We can describe them all. You will see again that dimension of row space = dimension of column space. When $r = 1$, every row is a multiple of the same row:

$$A = uv^{\mathrm{T}} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{equals} \quad \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = v^{\mathrm{T}}.$$

A column times a row (4 by 1 times 1 by 3) produces a matrix (4 by 3). All rows are multiples of the row $(1,2,3)$. All columns are multiples of the column $(1,2,-3,0)$. The row space is a line in $\mathbf{R}^n$, and the column space is a line in $\mathbf{R}^m$.

Every rank one matrix has the special form $A = u v^{\mathrm{T}}$ = column times row.

The columns are multiples of u. The rows are multiples of v^{T}. *The nullspace is the plane perpendicular to v.* ($Ax = \mathbf{0}$ means that $u(v^{\mathrm{T}}x) = \mathbf{0}$ and then $v^{\mathrm{T}}x = 0$.) It is this **perpendicularity** of the subspaces that will be Part 2 of the Fundamental Theorem.

■ REVIEW OF THE KEY IDEAS ■

1. The r pivot rows of R are a basis for the row spaces of R and A (same space).
2. The r pivot columns of A (!) are a basis for its column space.
3. The $n - r$ special solutions are a basis for the nullspaces of A and R (same space).
4. The last $m - r$ rows of I are a basis for the left nullspace of R.
5. The last $m - r$ rows of E are a basis for the left nullspace of A.

Note about the four subspaces The Fundamental Theorem looks like pure algebra, but it has very important applications. My favorites are the networks in Chapter 8 (often I go there for my next lecture). The equation for y in the left nullspace is $A^{\mathrm{T}}y = 0$:

Flow into a node equals flow out. Kirchhoff's Current Law is the "balance equation".

This is (in my opinion) the most important equation in applied mathematics. All models in science and engineering and economics involve a balance—of force or heat flow or charge or momentum or money. That balance equation, plus Hooke's Law or Ohm's Law or some law connecting "potentials" to "flows", gives a clear framework for applied mathematics.

My textbook on *Computational Science and Engineering* develops that framework, together with algorithms to solve the equations: Finite differences, finite elements, spectral methods, iterative methods, and multigrid.

■ WORKED EXAMPLES ■

3.6 A Find bases and dimensions for all four fundamental subspaces if you know that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = LU = E^{-1}R.$$

By changing only *one* number in R, change the dimensions of all four subspaces.

Solution This matrix has pivots in columns 1 and 3. Its rank is $r = 2$.

Row space	Basis $(1,3,0,5)$ and $(0,0,1,6)$ from R. Dimension 2.
Column space	Basis $(1,2,5)$ and $(0,1,0)$ from E^{-1} (and A). Dimension 2.
Nullspace	Basis $(-3,1,0,0)$ and $(-5,0,-6,1)$ from R. Dimension 2.
Nullspace of A^{T}	Basis $(-5,0,1)$ from row 3 of E. Dimension $3-2=1$.

We need to comment on that left nullspace $N(A^{\mathrm{T}})$. $EA = R$ says that the last row of E combines the three rows of A into the zero row of R. So that last row of E is a basis vector for the left nullspace. If R had *two* zero rows, then the last *two* rows of E would be a basis. (Just like elimination, $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$ combines rows of A to give zero rows in R.)

To change all these dimensions we need to change the rank r. One way to do that is to change an entry (*any entry*) in the zero row of R.

3.6 B Put four 1's into a 5 by 6 matrix of zeros, keeping the dimension of its *row space* as small as possible. Describe all the ways to make the dimension of its *column space* as small as possible. Describe all the ways to make the dimension of its *nullspace* as small as possible. How to make the *sum of the dimensions of all four subspaces small*?

Solution The rank is 1 if the four 1's go into the same row, or into the same column. They can also go into *two rows and two columns* (so $a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$). Since the column space and row space always have the same dimensions, this answers the first two questions: Dimension 1.

The nullspace has its smallest possible dimension $6 - 4 = 2$ when the rank is $r = 4$. To achieve rank 4, the 1's must go into four different rows and columns.

You can't do anything about the sum $r + (n-r) + r + (m-r) = n + m$. It will be $6 + 5 = 11$ no matter how the 1's are placed. The sum is 11 even if there aren't any 1's...

If all the other entries of A are 2's instead of 0's, how do these answers change?

Problem Set 3.6

1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Left nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(e) Row space $=$ column space, nullspace $\neq$ left nullspace.

5 If $\mathbf{V}$ is the subspace spanned by $(1,1,1)$ and $(2,1,0)$, find a matrix A that has $\mathbf{V}$ as its row space. Find a matrix B that has $\mathbf{V}$ as its nullspace.

6 Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [\,A \quad A\,]$.

8 What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

9 Which subspaces are the same for these matrices of different sizes?

(a) $[\,A\,]$ and $\begin{bmatrix} A \\ A \end{bmatrix}$ (b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

Prove that all three of those matrices have the *same rank r*.

10 If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?

11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution.*

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

12 Construct a matrix with $(1,0,1)$ and $(1,2,0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

13 True or false (with a reason or a counterexample):

(a) If $m = n$ then the row space of A equals the column space.

(b) The matrices A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

14 Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

15 If you exchange the first two rows of A, which of the four subspaces stay the same? If $\boldsymbol{v} = (1,2,3,4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix.

16 Explain why $\boldsymbol{v} = (1,0,-1)$ *cannot be a row of A and also in the nullspace.*

17 Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

18 (Left nullspace) Add the extra column $\boldsymbol{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^{T} and which are in the nullspace of A?

19 Following the method of Problem 18, reduce A to echelon form and look at zero rows. The $\boldsymbol{b}$ column tells which combinations you have taken of the rows:

$$\text{(a)} \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$$

From the $\boldsymbol{b}$ column after elimination, read off $m-r$ basis vectors in the left nullspace. Those y's are combinations of rows that give zero rows.

20 (a) Check that the solutions to $A\boldsymbol{x} = \boldsymbol{0}$ are perpendicular to the rows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = ER.$$

(b) How many independent solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$? Why is $\boldsymbol{y}^{\mathrm{T}}$ the last row of E^{-1}?

21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if _____ or if _____.

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1,0,0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0,0,1)$.

22 Construct $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ whose column space has basis $(1,2,4), (2,2,1)$ and whose row space has basis $(1,0), (1,1)$. Write A as (3 by 2) times (2 by 2).

23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A cannot be invertible?

24 (Important) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when $\boldsymbol{d}$ is in which of the four subspaces? The solution $\boldsymbol{y}$ is unique when the _____ contains only the zero vector.

25 True or false (with a reason or a counterexample):

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

26 (***Rank of*** AB) If $AB = C$, the rows of C are combinations of the rows of _____. So the rank of C is not greater than the rank of _____. Since $B^{\mathrm{T}}A^{\mathrm{T}} = C^{\mathrm{T}}$, the rank of C is also not greater than the rank of _____.

27 If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

28 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \text{four zero rows} & & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

29 Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that rank $(A) = 2$ but neither side passed up a winning move?

Challenge Problems

30 If $A = \boldsymbol{uv}^{\mathrm{T}}$ is a 2 by 2 matrix of rank 1, redraw Figure 3.5 to show clearly the Four Fundamental Subspaces. If B produces those same four subspaces, what is the exact relation of B to A?

31 **M** is the space of 3 by 3 matrices. Multiply every matrix X in **M** by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to $AX =$ zero matrix?

(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of **M**? Why do the dimensions add to $(n - r) + r = 9$?

32 Suppose the m by n matrices A and B have *the same four subspaces*. If they are both in row reduced echelon form, prove that F must equal G:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}.$$

Chapter 4

Orthogonality

4.1 Orthogonality of the Four Subspaces

Two vectors are orthogonal when their dot product is zero: $v \cdot w = 0$ or $v^{\mathrm{T}} w = 0$. This chapter moves to **orthogonal subspaces** and **orthogonal bases** and **orthogonal matrices**. The vectors in two subspaces, and the vectors in a basis, and the vectors in the columns, all pairs will be orthogonal. Think of $a^2 + b^2 = c^2$ for a *right triangle* with sides v and w.

Orthogonal vectors $\quad v^{\mathrm{T}} w = 0 \quad$ and $\quad \|v\|^2 + \|w\|^2 = \|v + w\|^2.$

The right side is $(v + w)^{\mathrm{T}}(v + w)$. This equals $v^{\mathrm{T}} v + w^{\mathrm{T}} w$ when $v^{\mathrm{T}} w = w^{\mathrm{T}} v = 0$.

Subspaces entered Chapter 3 to throw light on $Ax = b$. Right away we needed the column space (for b) and the nullspace (for x). Then the light turned onto A^{T}, uncovering two more subspaces. Those four fundamental subspaces reveal what a matrix really does.

A matrix multiplies a vector: A *times* x. At the first level this is only numbers. At the second level Ax is a combination of column vectors. The third level shows subspaces. But I don't think you have seen the whole picture until you study Figure 4.2. It fits the subspaces together, to show the hidden reality of A times x. The 90° angles between subspaces are new—and we have to say what those right angles mean.

The row space is perpendicular to the nullspace. Every row of A is perpendicular to every solution of $Ax = \mathbf{0}$. That gives the 90° angle on the left side of the figure. This perpendicularity of subspaces is Part 2 of the Fundamental Theorem of Linear Algebra.

The column space is perpendicular to the nullspace of A^{T}. When b is outside the column space—when we want to solve $Ax = b$ and can't do it—then this nullspace of A^{T} comes into its own. It contains the error $e = b - Ax$ in the "least-squares" solution. Least squares is the key application of linear algebra in this chapter.

Part 1 of the Fundamental Theorem gave the dimensions of the subspaces. The row and column spaces have the same dimension r (they are drawn the same size). The two nullspaces have the remaining dimensions $n - r$ and $m - r$. Now we will show that ***the row space and nullspace are orthogonal subspaces inside*** $\mathbf{R}^n$.

DEFINITION Two subspaces V and W of a vector space are ***orthogonal*** if every vector v in V is perpendicular to every vector w in W:

Orthogonal subspaces $\quad v^{\mathrm{T}}w = 0$ ***for all*** v ***in*** V ***and all*** w ***in*** W.

Example 1 The floor of your room (extended to infinity) is a subspace V. The line where two walls meet is a subspace W (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector in the floor.

Example 2 Two walls look perpendicular but they are not orthogonal subspaces! The meeting line is in both V and W—and this line is not perpendicular to itself. Two planes (dimensions 2 and 2 in $\mathbf{R}^3$) cannot be orthogonal subspaces.

When a vector is in two orthogonal subspaces, it *must* be zero. It is perpendicular to itself. It is v and it is w, so $v^{\mathrm{T}}v = 0$. This has to be the zero vector.

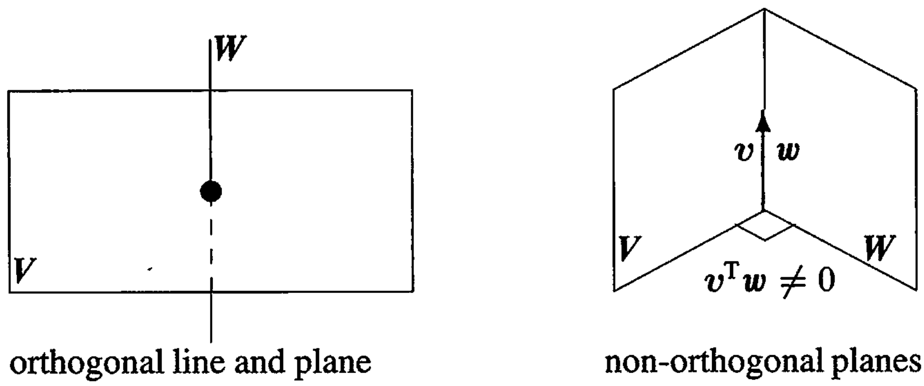


Figure 4.1: Orthogonality is impossible when dim V +dim W > dimension of whole space.

The crucial examples for linear algebra come from the fundamental subspaces. Zero is the only point where the nullspace meets the row space. More than that, the nullspace and row space of A meet at 90°. This key fact comes directly from $Ax = \mathbf{0}$:

Every vector x in the nullspace is perpendicular to every row of A, because $Ax = \mathbf{0}$. ***The nullspace*** $N(A)$ ***and the row space*** $C(A^{\mathrm{T}})$ ***are orthogonal subspaces of*** $\mathbf{R}^n$.

To see why x is perpendicular to the rows, look at $Ax = \mathbf{0}$. Each row multiplies x:

$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \longleftarrow & (\text{row } 1) \cdot x \text{ is zero} \\ \\ \longleftarrow & (\text{row } m) \cdot x \text{ is zero} \end{matrix} \qquad (1)$$

The first equation says that row 1 is perpendicular to x. The last equation says that row m is perpendicular to x. *Every row has a zero dot product with* x. Then x is also perpendicular to every *combination* of the rows. The whole row space $C(A^{\mathrm{T}})$ is orthogonal to $N(A)$.

Here is a second proof of that orthogonality for readers who like matrix shorthand. The vectors in the row space are combinations $A^{\mathrm{T}}y$ of the rows. Take the dot product of $A^{\mathrm{T}}y$ with any x in the nullspace. *These vectors are perpendicular*:

Nullspace and Row space $$x^{\mathrm{T}}(A^{\mathrm{T}}y) = (Ax)^{\mathrm{T}}y = \mathbf{0}^{\mathrm{T}}y = 0. \qquad (2)$$

We like the first proof. You can see those rows of A multiplying x to produce zeros in equation (1). The second proof shows why A and A^{T} are both in the Fundamental Theorem. A^{T} goes with y and A goes with x. At the end we used $Ax = \mathbf{0}$.

Example 3 The rows of A are perpendicular to $x = (1, 1, -1)$ in the nullspace:

$$Ax = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the dot products} \quad \begin{matrix} 1+3-4=0 \\ 5+2-7=0 \end{matrix}$$

Now we turn to the other two subspaces. In this example, the column space is all of $\mathbf{R}^2$. The nullspace of A^{T} is only the zero vector (orthogonal to every vector). The columns of A and nullspace of A^{T} are always orthogonal subspaces.

Every vector y in the nullspace of A^{T} is perpendicular to every column of A. ***The left nullspace*** $N(A^{\mathrm{T}})$ ***and the column space*** $C(A)$ ***are orthogonal in*** $\mathbf{R}^m$.

Apply the original proof to A^{T}. Its nullspace is orthogonal to its row space—and the row space of A^{T} is the column space of A. Q.E.D.

For a visual proof, look at $A^{\mathrm{T}}y = \mathbf{0}$. Each column of A multiplies y to give 0:

$$C(A) \perp N(A^{\mathrm{T}}) \qquad A^{\mathrm{T}}y = \begin{bmatrix} (\textbf{column } 1)^{\mathrm{T}} \\ \cdots \\ (\textbf{column } n)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \\ y \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix}. \qquad (3)$$

The dot product of y with every column of A is zero. Then y in the left nullspace is perpendicular to each column—and to the whole column space.

Orthogonal Complements

Important The fundamental subspaces are more than just orthogonal (in pairs). Their dimensions are also right. Two lines could be perpendicular in $\mathbf{R}^3$, but those lines *could not be* the row space and nullspace of a 3 by 3 matrix. The lines have dimensions 1 and 1, adding to 2. The correct dimensions r and $n - r$ must add to $n = 3$.

The fundamental subspaces have dimensions 2 and 1, or 3 and 0. Those subspaces are not only orthogonal, they are *orthogonal complements*.

DEFINITION The ***orthogonal complement*** of a subspace V contains ***every*** vector that is perpendicular to V. This orthogonal subspace is denoted by $V^{\perp}$ (pronounced "V perp").

By this definition, the nullspace is the orthogonal complement of the row space. ***Every*** x that is perpendicular to the rows satisfies $Ax = \mathbf{0}$.

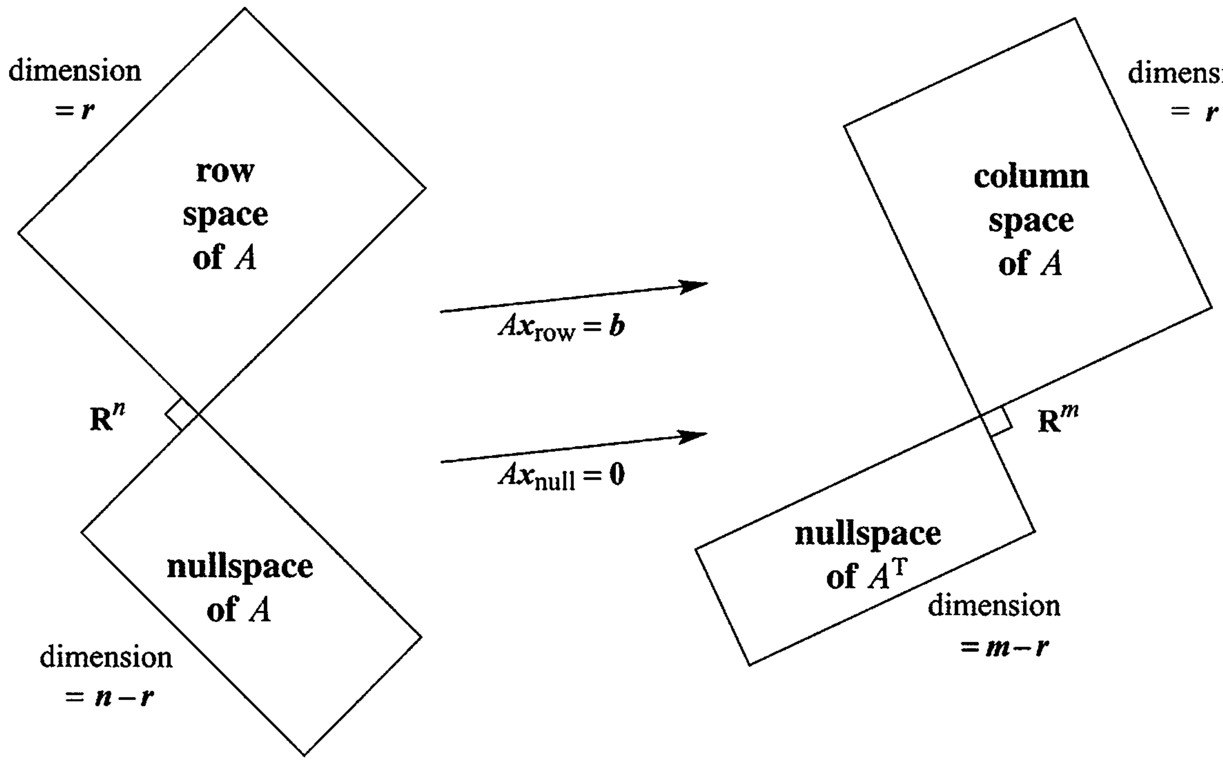


Figure 4.2: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

The reverse is also true. ***If v is orthogonal to the nullspace, it must be in the row space.*** Otherwise we could add this v as an extra row of the matrix, without changing its nullspace. The row space would grow, which breaks the law $r + (n - r) = n$. We conclude that the nullspace complement $N(A)^{\perp}$ is exactly the row space $C(A^{\mathrm{T}})$.

The left nullspace and column space are orthogonal in $\mathbf{R}^m$, and they are orthogonal complements. Their dimensions r and $m - r$ add to the full dimension m.

***Fundamental Theorem of Linear Algebra*, Part 2**

***$N(A)$ is the orthogonal complement of the row space $C(A^{\mathrm{T}})$* (in $\mathbf{R}^n$).**

***$N(A^{\mathrm{T}})$ is the orthogonal complement of the column space $C(A)$* (in $\mathbf{R}^m$).**

Part 1 gave the dimensions of the subspaces. Part 2 gives the 90° angles between them. The point of "complements" is that every x can be split into a *row space component* x_r and a *nullspace component* x_n. When A multiplies $x = x_r + x_n$, Figure 4.3 shows what happens:

The nullspace component goes to zero: $Ax_n = \mathbf{0}$.

The row space component goes to the column space: $Ax_r = Ax$.

Every vector goes to the column space! Multiplying by A cannot do anything else.

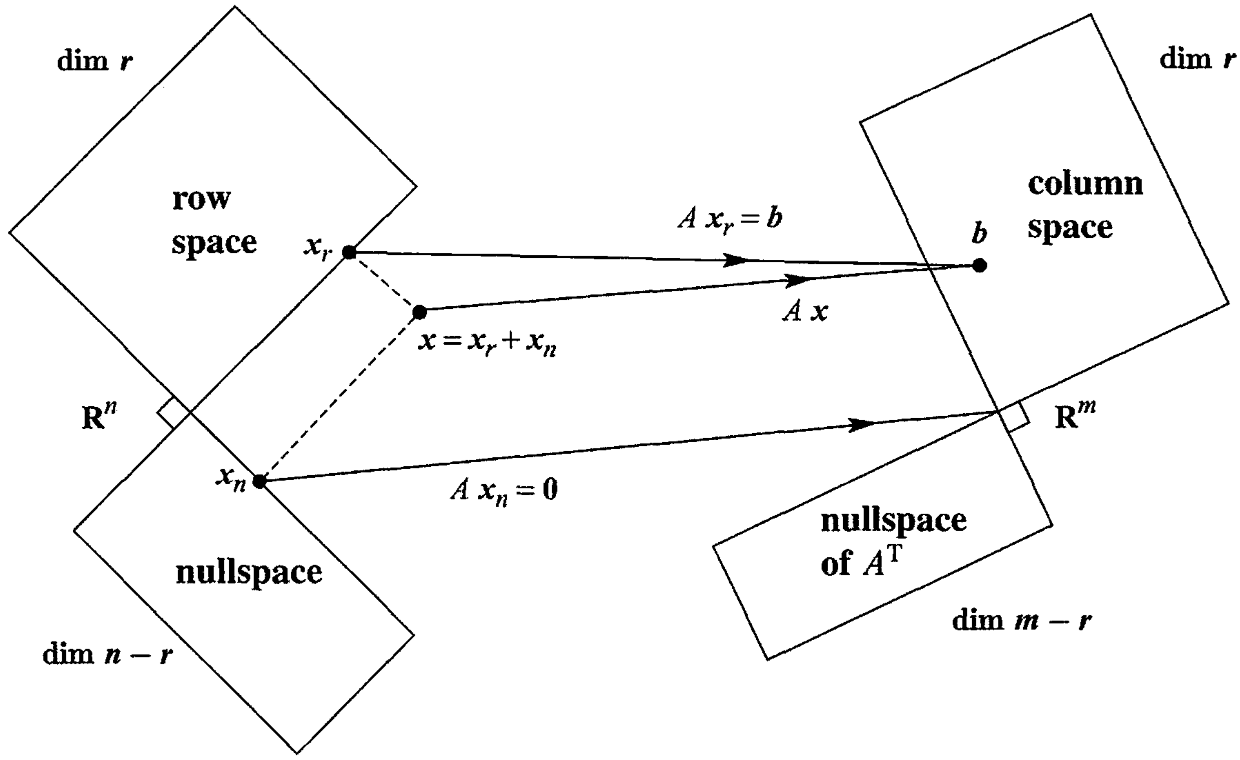


Figure 4.3: This update of Figure 4.2 shows the true action of A on $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$. Row space vector $\boldsymbol{x}_r$ to column space, nullspace vector $\boldsymbol{x}_n$ to zero.

More than that: *Every vector $\boldsymbol{b}$ in the column space comes from one and only one vector in the row space.* Proof: If $A\boldsymbol{x}_r = A\boldsymbol{x}'_r$, the difference $\boldsymbol{x}_r - \boldsymbol{x}'_r$ is in the nullspace. It is also in the row space, where $\boldsymbol{x}_r$ and $\boldsymbol{x}'_r$ came from. This difference must be the zero vector, because the nullspace and row space are perpendicular. Therefore $\boldsymbol{x}_r = \boldsymbol{x}'_r$.

There is an r by r invertible matrix hiding inside A, if we throw away the two nullspaces. ***From the row space to the column space, A is invertible.*** The "pseudoinverse" will invert it in Section 7.3.

Example 4 Every diagonal matrix has an r by r invertible submatrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{contains the submatrix} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The other eleven zeros are responsible for the nullspaces. The rank of B is also $r = 2$:

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \end{bmatrix} \quad \text{contains} \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{in the pivot rows and columns.}$$

Every A becomes a diagonal matrix, when we choose the right bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. This ***Singular Value Decomposition*** has become extremely important in applications.

Combining Bases from Subspaces

What follows are some valuable facts about bases. They were saved until now—when we are ready to use them. After a week you have a clearer sense of what a basis is (*linearly independent* vectors that *span* the space). Normally we have to check both of these properties. When the count is right, one property implies the other:

> Any n independent vectors in $\mathbf{R}^n$ must span $\mathbf{R}^n$. So they are a basis.
> Any n vectors that span $\mathbf{R}^n$ must be independent. So they are a basis.

Starting with the correct number of vectors, one property of a basis produces the other. This is true in any vector space, but we care most about $\mathbf{R}^n$. When the vectors go into the columns of an n by n *square* matrix A, here are the same two facts:

> If the n columns of A are independent, they span $\mathbf{R}^n$. So $A\boldsymbol{x} = \boldsymbol{b}$ is solvable.
> If the n columns span $\mathbf{R}^n$, they are independent. So $A\boldsymbol{x} = \boldsymbol{b}$ has only one solution.

Uniqueness implies existence and existence implies uniqueness. ***Then A is invertible.*** If there are no free variables, the solution $\boldsymbol{x}$ is unique. There must be n pivots. Then back substitution solves $A\boldsymbol{x} = \boldsymbol{b}$ (the solution exists).

Starting in the opposite direction, suppose $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for every $\boldsymbol{b}$ (*existence of solutions*). Then elimination produced no zero rows. There are n pivots and no free variables. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ (*uniqueness* of solutions).

With bases for the row space and the nullspace, we have $r + (n - r) = n$ vectors, This is the right number. Those n vectors are independent.[2] *Therefore they span* $\mathbf{R}^n$.

Each $\boldsymbol{x}$ is the sum $\boldsymbol{x}_r + \boldsymbol{x}_n$ of a row space vector $\boldsymbol{x}_r$ and a nullspace vector $\boldsymbol{x}_n$.

The splitting in Figure 4.3 shows the key point of orthogonal complements—the dimensions add to n and all vectors are fully accounted for.

Example 5 For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ split $\boldsymbol{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ into $\boldsymbol{x}_r + \boldsymbol{x}_n = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The vector $(2, 4)$ is in the row space. The orthogonal vector $(2, -1)$ is in the nullspace. The next section will compute this splitting for any A and $\boldsymbol{x}$, by a projection.

[2]If a combination of all n vectors gives $\boldsymbol{x}_r + \boldsymbol{x}_n = \mathbf{0}$, then $\boldsymbol{x}_r = -\boldsymbol{x}_n$ is in both subspaces. So $\boldsymbol{x}_r = \boldsymbol{x}_n = 0$. All coefficients of the row space basis and nullspace basis must be zero—which proves independence of the n vectors together.

■ REVIEW OF THE KEY IDEAS ■

1. Subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ are orthogonal if every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$.
2. $\boldsymbol{V}$ and $\boldsymbol{W}$ are "orthogonal complements" if $\boldsymbol{W}$ contains **all** vectors perpendicular to $\boldsymbol{V}$ (and vice versa). Inside $\mathbf{R}^n$, the dimensions of complements $\boldsymbol{V}$ and $\boldsymbol{W}$ add to n.
3. The nullspace $\boldsymbol{N}(A)$ and the row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements, from $A\boldsymbol{x} = \mathbf{0}$. Similarly $\boldsymbol{N}(A^{\mathrm{T}})$ and $\boldsymbol{C}(A)$ are orthogonal complements.
4. Any n independent vectors in $\mathbf{R}^n$ will span $\mathbf{R}^n$.
5. Every $\boldsymbol{x}$ in $\mathbf{R}^n$ has a nullspace component $\boldsymbol{x}_n$ and a row space component $\boldsymbol{x}_r$.

■ WORKED EXAMPLES ■

4.1 A Suppose $\boldsymbol{S}$ is a six-dimensional subspace of nine-dimensional space $\mathbf{R}^9$.

(a) What are the possible dimensions of subspaces orthogonal to $\boldsymbol{S}$?

(b) What are the possible dimensions of the orthogonal complement $\boldsymbol{S}^\perp$ of $\boldsymbol{S}$?

(c) What is the smallest possible size of a matrix A that has row space $\boldsymbol{S}$?

(d) What is the shape of its nullspace matrix N?

Solution

(a) If $\boldsymbol{S}$ is six-dimensional in $\mathbf{R}^9$, subspaces orthogonal to $\boldsymbol{S}$ can have dimensions 0, 1, 2, 3.

(b) The complement $\boldsymbol{S}^\perp$ is the largest orthogonal subspace, with dimension 3.

(c) The smallest matrix A is 6 by 9 (its six rows are a basis for $\boldsymbol{S}$).

(d) Its nullspace matrix N is 9 by 3. The columns of N contain a basis for $\boldsymbol{S}^\perp$.

If a new row 7 of B is a combination of the six rows of A, then B has the same row space as A. It also has the same nullspace matrix N. The special solutions $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$ will be the same. Elimination will change row 7 of B to all zeros.

4.1 B The equation $x - 3y - 4z = 0$ describes a plane $\boldsymbol{P}$ in $\mathbf{R}^3$ (actually a subspace).

(a) The plane $\boldsymbol{P}$ is the nullspace $\boldsymbol{N}(A)$ of what 1 by 3 matrix A?

(b) Find a basis $\boldsymbol{s}_1, \boldsymbol{s}_2$ of special solutions of $x - 3y - 4z = 0$ (these would be the columns of the nullspace matrix N).

(c) Also find a basis for the line $P^{\perp}$ that is perpendicular to P.

(d) Split $v = (6, 4, 5)$ into its nullspace component v_n in P and its row space component v_r in $P^{\perp}$.

Solution

(a) The equation $x - 3y - 4z = 0$ is $Ax = \mathbf{0}$ for the 1 by 3 matrix $A = [1 \;\; -3 \;\; -4]$.

(b) Columns 2 and 3 are free (the only pivot is 1). The special solutions with free variables 1 and 0 are $s_1 = (3, 1, 0)$ and $s_2 = (4, 0, 1)$ in the plane $P = N(A)$.

(c) The row space of A is the line $P^{\perp}$ in the direction of the row $z = (1, -3, -4)$.

(d) To split v into $v_n + v_r = (c_1 s_1 + c_2 s_2) + c_3 z$, solve for $c_1 = 1, c_2 = 1, c_3 = -1$.

$$\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{aligned} v_n &= s_1 + s_2 = (7,1,1) \quad \text{is in} \quad P = N(A) \\ v_r &= -s_3 = (-1,3,4) \quad \text{is in} \quad P^{\perp} = C(A^{\mathrm{T}}). \\ v &= (6,4,5) \quad \text{equals} \quad (7,1,1) + (-1,3,4) \end{aligned}$$

This method used a basis for each subspace combined into an overall basis s_1, s_2, z. Section 4.2 will also project v onto a subspace S. There we will not need a basis for the perpendicular subspace $S^{\perp}$.

Problem Set 4.1

Questions 1–12 grow out of Figures 4.2 and 4.3 with four subspaces.

1 Construct any 2 by 3 matrix of rank one. Copy Figure 4.2 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?

2 Redraw Figure 4.3 for a 3 by 2 matrix of rank $r = 2$. Which subspace is Z (zero vector only)? The nullspace part of any vector x in $\mathbf{R}^2$ is $x_n =$ _____.

3 Construct a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) Every row is orthogonal to every column (A is not the zero matrix)

(e) Columns add up to a column of zeros, rows add to a row of 1's.

4 If $AB = 0$ then the columns of B are in the _____ of A. The rows of A are in the _____ of B. Why can't A and B be 3 by 3 matrices of rank 2?

5 (a) If $Ax = b$ has a solution and $A^{\mathrm{T}}y = 0$, *is* $(y^{\mathrm{T}}x = 0)$ *or* $(y^{\mathrm{T}}b = 0)$?

(b) If $A^{\mathrm{T}}y = (1,1,1)$ has a solution and $Ax = 0$, then _____.

6 This system of equations $Ax = b$ has *no solution* (they lead to $0 = 1$):

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? Its dot product $y^{\mathrm{T}}b$ is 1, so no solution x.

7 Every system with no solution is like the one in Problem 6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called **Fredholm's Alternative**:

Exactly one of these problems has a solution

$$Ax = b \quad \textbf{OR} \quad A^{\mathrm{T}}y = 0 \text{ with } y^{\mathrm{T}}b = 1.$$

If b is not in the column space of A, it is not orthogonal to the nullspace of A^{T}. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

8 In Figure 4.3, how do we know that Ax_r is equal to Ax? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ what is x_r?

9 If $A^{\mathrm{T}}Ax = 0$ then $Ax = 0$. Reason: Ax is in the nullspace of A^{T} and also in the _____ of A and those spaces are _____. *Conclusion*: $A^{\mathrm{T}}A$ *has the same nullspace as* A. *This key fact is repeated in the next section.*

10 Suppose A is a symmetric matrix ($A^{\mathrm{T}} = A$).

(a) Why is its column space perpendicular to its nullspace?

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z? **Symmetric matrices have perpendicular eigenvectors** $x^{\mathrm{T}}z = 0$.

11 (Recommended) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

12 Find the pieces x_r and x_n and draw Figure 4.3 properly if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Questions 13–23 are about orthogonal subspaces.

13 Put bases for the subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ into the columns of matrices V and W. Explain why the test for orthogonal subspaces can be written $V^{\mathrm{T}}W =$ zero matrix. This matches $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$ for orthogonal vectors.

14 The floor $\boldsymbol{V}$ and the wall $\boldsymbol{W}$ are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet). No planes $\boldsymbol{V}$ and $\boldsymbol{W}$ in $\mathbf{R}^3$ can be orthogonal! Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector $A\boldsymbol{x}$ and also $B\widehat{\boldsymbol{x}}$. Think 3 by 4 with the matrix $[\,A \;\; B\,]$.

15 Extend Problem 14 to a p-dimensional subspace $\boldsymbol{V}$ and a q-dimensional subspace $\boldsymbol{W}$ of $\mathbf{R}^n$. What inequality on $p+q$ guarantees that $\boldsymbol{V}$ intersects $\boldsymbol{W}$ in a nonzero vector? These subspaces cannot be orthogonal.

16 Prove that every $\boldsymbol{y}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is perpendicular to every $A\boldsymbol{x}$ in the column space, using the matrix shorthand of equation (2). Start from $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

17 If $\boldsymbol{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\boldsymbol{S}^{\perp}$? If $\boldsymbol{S}$ is spanned by $(1,1,1)$, what is $\boldsymbol{S}^{\perp}$? If $\boldsymbol{S}$ is spanned by $(1,1,1)$ and $(1,1,-1)$, what is a basis for $\boldsymbol{S}^{\perp}$?

18 Suppose $\boldsymbol{S}$ only contains two vectors $(1,5,1)$ and $(2,2,2)$ (not a subspace). Then $\boldsymbol{S}^{\perp}$ is the nullspace of the matrix $A =$ _____. $\boldsymbol{S}^{\perp}$ is a subspace even if $\boldsymbol{S}$ is not.

19 Suppose $\boldsymbol{L}$ is a one-dimensional subspace (a line) in $\mathbf{R}^3$. Its orthogonal complement $\boldsymbol{L}^{\perp}$ is the _____ perpendicular to $\boldsymbol{L}$. Then $(\boldsymbol{L}^{\perp})^{\perp}$ is a _____ perpendicular to $\boldsymbol{L}^{\perp}$. In fact $(\boldsymbol{L}^{\perp})^{\perp}$ is the same as _____.

20 Suppose $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$. Then $\boldsymbol{V}^{\perp}$ contains only the vector _____. Then $(\boldsymbol{V}^{\perp})^{\perp}$ is _____. So $(\boldsymbol{V}^{\perp})^{\perp}$ is the same as _____.

21 Suppose $\boldsymbol{S}$ is spanned by the vectors $(1,2,2,3)$ and $(1,3,3,2)$. Find two vectors that span $\boldsymbol{S}^{\perp}$. This is the same as solving $A\boldsymbol{x} = \mathbf{0}$ for which A?

22 If $\boldsymbol{P}$ is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\boldsymbol{P}^{\perp}$. Construct a matrix that has $\boldsymbol{P}$ as its nullspace.

23 If a subspace $\boldsymbol{S}$ is contained in a subspace $\boldsymbol{V}$, prove that $\boldsymbol{S}^{\perp}$ contains $\boldsymbol{V}^{\perp}$.

Questions 24–30 are about perpendicular columns and rows.

24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

25 Find $A^{\mathrm{T}}A$ if the columns of A are unit vectors, all mutually perpendicular.

26 Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^{\mathrm{T}}A$. Why is it a diagonal matrix?

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are _____. They are the same line if _____. In that case (b_1, b_2) is perpendicular to the vector _____. The nullspace of the matrix is the line $3x + y =$ _____. One particular vector in that nullspace is _____.

28 Why is each of these statements false?

(a) $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

(b) The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.

(c) Two subspaces that meet only in the zero vector are orthogonal.

29 Find a matrix with $\boldsymbol{v} = (1, 2, 3)$ in the row space and column space. Find another matrix with $\boldsymbol{v}$ in the nullspace and column space. Which pairs of subspaces can $\boldsymbol{v}$ *not* be in?

Challenge Problems

30 Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. So $\boldsymbol{N}(A)$ contains $\boldsymbol{C}(B)$. Prove from the dimensions of $\boldsymbol{N}(A)$ and $\boldsymbol{C}(B)$ that $\operatorname{rank}(A) + \operatorname{rank}(B) \le 4$.

31 The command $N = \mathsf{null}(A)$ will produce a basis for the nullspace of A. Then the command $B = \mathsf{null}(N')$ will produce a basis for the _____ of A.

32 Suppose I give you four nonzero vectors $\boldsymbol{r}, \boldsymbol{n}, \boldsymbol{c}, \boldsymbol{l}$ in $\mathbf{R}^2$.

(a) What are the conditions for those to be bases for the four fundamental subspaces $\boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A), \boldsymbol{C}(A), \boldsymbol{N}(A^{\mathrm{T}})$ of a 2 by 2 matrix?

(b) What is one possible matrix A?

33 Suppose I give you eight vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{l}_1, \boldsymbol{l}_2$ in $\mathbf{R}^4$.

(a) What are the conditions for those pairs to be bases for the four fundamental subspaces of a 4 by 4 matrix?

(b) What is one possible matrix A?

4.2 Projections

May we start this section with two questions? (In addition to that one.) The first question aims to show that projections are easy to visualize. The second question is about "projection matrices"—symmetric matrices with $P^2 = P$. *The projection of $\boldsymbol{b}$ is $P\boldsymbol{b}$.*

1 What are the projections of $\boldsymbol{b} = (2,3,4)$ onto the z axis and the xy plane?

2 What matrices produce those projections onto a line and a plane?

When $\boldsymbol{b}$ is projected onto a line, ***its projection $\boldsymbol{p}$ is the part of $\boldsymbol{b}$ along that line.*** If $\boldsymbol{b}$ is projected onto a plane, $\boldsymbol{p}$ is the part in that plane. ***The projection $\boldsymbol{p}$ is $P\boldsymbol{b}$.***

The projection matrix P multiplies $\boldsymbol{b}$ to give $\boldsymbol{p}$. This section finds $\boldsymbol{p}$ and P.

The projection onto the z axis we call $\boldsymbol{p}_1$. The second projection drops straight down to the xy plane. The picture in your mind should be Figure 4.4. Start with $\boldsymbol{b} = (2,3,4)$. One projection gives $\boldsymbol{p}_1 = (0,0,4)$ and the other gives $\boldsymbol{p}_2 = (2,3,0)$. Those are the parts of $\boldsymbol{b}$ along the z axis and in the xy plane.

The projection matrices P_1 and P_2 are 3 by 3. They multiply $\boldsymbol{b}$ with 3 components to produce $\boldsymbol{p}$ with 3 components. Projection onto a line comes from a rank one matrix. Projection onto a plane comes from a rank two matrix:

$$\text{Onto the } z \text{ axis:} \quad P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Onto the } xy \text{ plane:} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

P_1 picks out the z component of every vector. P_2 picks out the x and y components. To find the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of $\boldsymbol{b}$, multiply $\boldsymbol{b}$ by P_1 and P_2 (small $\boldsymbol{p}$ for the vector, capital P for the matrix that produces it):

$$\boldsymbol{p}_1 = P_1\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \qquad \boldsymbol{p}_2 = P_2\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

In this case the projections p_1 and p_2 are perpendicular. The xy plane and the z axis are ***orthogonal subspaces***, like the floor of a room and the line between two walls.

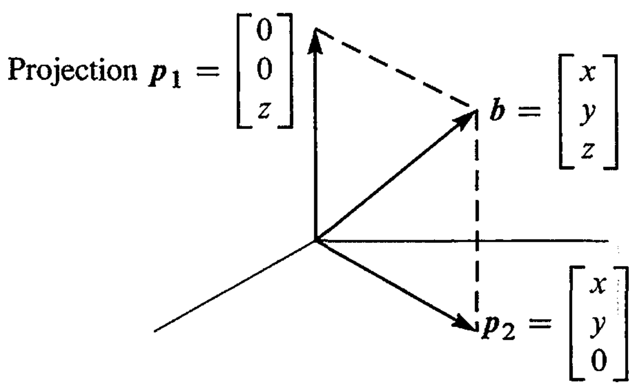


Figure 4.4: The projections $\boldsymbol{p}_1 = P_1\boldsymbol{b}$ and $\boldsymbol{p}_2 = P_2\boldsymbol{b}$ onto the z axis and the xy plane.

More than that, the line and plane are orthogonal ***complements***. Their dimensions add to $1+2=3$. Every vector $\boldsymbol{b}$ in the whole space is the sum of its parts in the two subspaces. The projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are exactly those parts:

$$\text{The vectors give } \boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{b}. \qquad \text{The matrices give } P_1 + P_2 = I. \tag{1}$$

This is perfect. Our goal is reached—for this example. We have the same goal for any line and any plane and any n-dimensional subspace. The object is to find the part $\boldsymbol{p}$ in each subspace, and the projection matrix P that produces that part $\boldsymbol{p} = P\boldsymbol{b}$. Every subspace of $\mathbf{R}^m$ has its own m by m projection matrix. To compute P, we absolutely need a good description of the subspace that it projects onto.

The best description of a subspace is a basis. We put the basis vectors into the columns of A. ***Now we are projecting onto the column space of A!*** Certainly the z axis is the column space of the 3 by 1 matrix A_1. The xy plane is the column space of A_2. That plane is also the column space of A_3 (a subspace has many bases):

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}.$$

Our problem is ***to project any b onto the column space of any m by n matrix.*** Start with a line (dimension $n=1$). The matrix A has only one column. Call it $\boldsymbol{a}$.

Projection Onto a Line

A line goes through the origin in the direction of $\boldsymbol{a} = (a_1, \ldots, a_m)$. Along that line, we want the point $\boldsymbol{p}$ closest to $\boldsymbol{b} = (b_1, \ldots, b_m)$. The key to projection is orthogonality: ***The line from b to p is perpendicular to the vector a.*** This is the dotted line marked $\boldsymbol{e}$ for error in Figure 4.5—which we now compute by algebra.

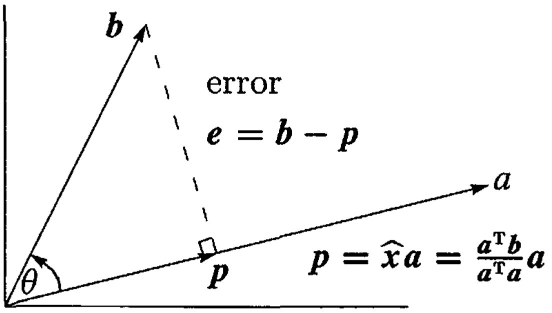
The projection $\boldsymbol{p}$ is some multiple of $\boldsymbol{a}$. Call it $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$ = "x hat" times $\boldsymbol{a}$. Computing this number $\widehat{x}$ will give the vector $\boldsymbol{p}$. Then from the formula for $\boldsymbol{p}$, we read off the projection matrix P. These three steps will lead to all projection matrices: *find $\widehat{x}$, then find the vector $\boldsymbol{p}$, then find the matrix P.*

The dotted line $\boldsymbol{b} - \boldsymbol{p}$ is $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$. It is perpendicular to $\boldsymbol{a}$—this will determine $\widehat{x}$. Use the fact that $\boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$ when their dot product is zero:

Projecting $\boldsymbol{b}$ onto $\boldsymbol{a}$, error $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$

$$\boldsymbol{a} \cdot (\boldsymbol{b} - \widehat{x}\boldsymbol{a}) = 0 \quad \text{or} \quad \boldsymbol{a} \cdot \boldsymbol{b} - \widehat{x}\boldsymbol{a} \cdot \boldsymbol{a} = 0 \qquad \widehat{x} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}. \tag{2}$$

The multiplication $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ is the same as $\boldsymbol{a} \cdot \boldsymbol{b}$. Using the transpose is better, because it applies also to matrices. Our formula $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ gives the projection $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$.



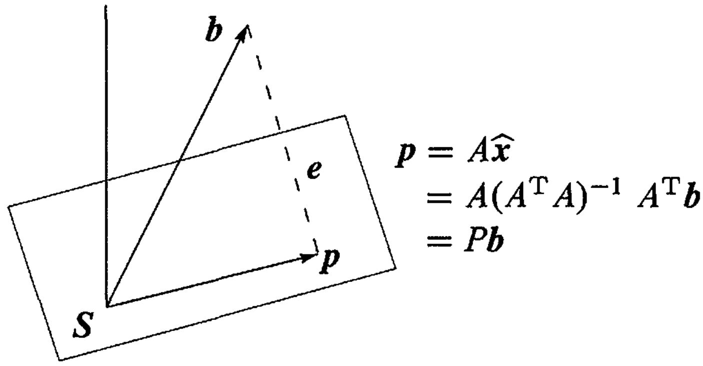


Figure 4.5: The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto S = column space of A.

The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector $\boldsymbol{p} = \widehat{x}\boldsymbol{a} = \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}$.

Special case 1: If $\boldsymbol{b} = \boldsymbol{a}$ then $\widehat{x} = 1$. The projection of $\boldsymbol{a}$ onto $\boldsymbol{a}$ is itself. $P\boldsymbol{a} = \boldsymbol{a}$.

Special case 2: If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$. The projection is $\boldsymbol{p} = \boldsymbol{0}$.

Example 1 Project $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ to find $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$ in Figure 4.5.

Solution The number $\widehat{x}$ is the ratio of $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 5$ to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$. So the projection is $\boldsymbol{p} = \frac{5}{9}\boldsymbol{a}$. The error vector between $\boldsymbol{b}$ and $\boldsymbol{p}$ is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. Those vectors $\boldsymbol{p}$ and $\boldsymbol{e}$ will add to $\boldsymbol{b} = (1, 1, 1)$:

$$\boldsymbol{p} = \frac{5}{9}\boldsymbol{a} = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right) \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \left(\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9}\right).$$

The error $\boldsymbol{e}$ should be perpendicular to $\boldsymbol{a} = (1, 2, 2)$ and it is: $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$.

Look at the right triangle of $\boldsymbol{b}$, $\boldsymbol{p}$, and $\boldsymbol{e}$. The vector $\boldsymbol{b}$ is split into two parts—its component along the line is $\boldsymbol{p}$, its perpendicular part is $\boldsymbol{e}$. Those two sides of a right triangle have length $\|\boldsymbol{b}\| \cos\theta$ and $\|\boldsymbol{b}\| \sin\theta$. Trigonometry matches the dot product:

$$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} \quad \text{has length} \quad \|\boldsymbol{p}\| = \frac{\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\cos\theta}{\|\boldsymbol{a}\|^2}\|\boldsymbol{a}\| = \|\boldsymbol{b}\|\cos\theta. \tag{3}$$

The dot product is a lot simpler than getting involved with $\cos\theta$ and the length of $\boldsymbol{b}$. The example has square roots in $\cos\theta = 5/3\sqrt{3}$ and $\|\boldsymbol{b}\| = \sqrt{3}$. There are no square roots in the projection $\boldsymbol{p} = 5\boldsymbol{a}/9$. The good way to $5/9$ is $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{a}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Now comes the ***projection matrix***. In the formula for $\boldsymbol{p}$, what matrix is multiplying $\boldsymbol{b}$? You can see the matrix better if the number $\widehat{x}$ is on the right side of $\boldsymbol{a}$:

Projection matrix P $\quad \boldsymbol{p} = \boldsymbol{a}\widehat{x} = \boldsymbol{a}\dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = P\boldsymbol{b}$ ***when the matrix is*** $\quad P = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$.

P is a column times a row! The column is $\boldsymbol{a}$, the row is $\boldsymbol{a}^{\mathrm{T}}$. Then divide by the number $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. The projection matrix P is m by m, but ***its rank is one***. We are projecting onto a one-dimensional subspace, the line through $\boldsymbol{a}$. That is the column space of P.

Example 2 Find the projection matrix $P = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$ onto the line through $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Solution Multiply column $\boldsymbol{a}$ times row $\boldsymbol{a}^{\mathrm{T}}$ and divide by $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$:

$$\text{Projection matrix} \qquad P = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

This matrix projects *any* vector $\boldsymbol{b}$ onto $\boldsymbol{a}$. Check $\boldsymbol{p} = P\boldsymbol{b}$ for $\boldsymbol{b} = (1, 1, 1)$ in Example 1:

$$\boldsymbol{p} = P\boldsymbol{b} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} \qquad \text{which is correct.}$$

If the vector $\boldsymbol{a}$ is doubled, the matrix P stays the same. It still projects onto the same line. If the matrix is squared, P^2 equals P. ***Projecting a second time doesn't change anything***, so $\boldsymbol{P^2 = P}$. The diagonal entries of P add up to $\frac{1}{9}(1 + 4 + 4) = 1$.

The matrix $I - P$ should be a projection too. It produces the other side $\boldsymbol{e}$ of the triangle—the perpendicular part of $\boldsymbol{b}$. Note that $(I - P)\boldsymbol{b}$ equals $\boldsymbol{b} - \boldsymbol{p}$ which is $\boldsymbol{e}$ in the left nullspace. ***When P projects onto one subspace, $I - P$ projects onto the perpendicular subspace***. Here $I - P$ projects onto the plane perpendicular to $\boldsymbol{a}$.

Now we move beyond projection onto a line. Projecting onto an n-dimensional subspace of $\mathbf{R}^m$ takes more effort. The crucial formulas will be collected in equations (5)–(6)–(7). Basically you need to remember those three equations.

Projection Onto a Subspace

Start with n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in $\mathbf{R}^m$. Assume that these $\boldsymbol{a}$'s are linearly independent.

Problem: Find the combination $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n$ closest to a given vector $\boldsymbol{b}$. We are projecting each $\boldsymbol{b}$ in $\mathbf{R}^m$ onto the subspace spanned by the $\boldsymbol{a}$'s, to get $\boldsymbol{p}$.

With $n = 1$ (only one vector $\boldsymbol{a}_1$) this is projection onto a line. The line is the column space of A, which has just one column. In general the matrix A has n columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

The combinations in $\mathbf{R}^m$ are the vectors $A\boldsymbol{x}$ in the column space. We are looking for the particular combination $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ (***the projection***) that is closest to $\boldsymbol{b}$. The hat over $\widehat{\boldsymbol{x}}$ indicates the *best* choice $\widehat{\boldsymbol{x}}$, to give the closest vector in the column space. That choice is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ when $n = 1$. For $n > 1$, the best $\widehat{\boldsymbol{x}}$ is to be found now.

We compute projections onto n-dimensional subspaces in three steps as before: ***Find the vector $\widehat{\boldsymbol{x}}$, find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$, find the matrix P.***

The key is in the geometry! The dotted line in Figure 4.5 goes from $\boldsymbol{b}$ to the nearest point $A\widehat{\boldsymbol{x}}$ in the subspace. ***This error vector $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to the subspace.***

The error $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ makes a right angle with all the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. The n right angles give the n equations for $\widehat{\boldsymbol{x}}$:

$$\begin{matrix} \boldsymbol{a}_1^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = 0 \\ \vdots \\ \boldsymbol{a}_n^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = 0 \end{matrix} \quad \text{or} \quad \begin{bmatrix} - \boldsymbol{a}_1^{\mathrm{T}} - \\ \vdots \\ - \boldsymbol{a}_n^{\mathrm{T}} - \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{b} - A\widehat{\boldsymbol{x}} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathbf{0} \\ \\ \end{bmatrix}. \tag{4}$$

The matrix with those rows $\boldsymbol{a}_i^{\mathrm{T}}$ is A^{T}. The n equations are exactly $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$.

Rewrite $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$ in its famous form $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. This is the equation for $\widehat{\boldsymbol{x}}$, and the coefficient matrix is $A^{\mathrm{T}}A$. Now we can find $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and P, in that order:

The combination $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n = A\widehat{\boldsymbol{x}}$ that is closest to $\boldsymbol{b}$ comes from

$$A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0} \quad \text{or} \quad \boxed{A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}.} \tag{5}$$

This symmetric matrix $A^{\mathrm{T}}A$ is n by n. It is invertible if the $\boldsymbol{a}$'s are independent. The solution is $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$. The ***projection*** of $\boldsymbol{b}$ onto the subspace is $\boldsymbol{p}$:

$$\boxed{\boldsymbol{p} = A\widehat{\boldsymbol{x}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}.} \tag{6}$$

This formula shows the n by n ***projection matrix*** that produces $\boldsymbol{p} = P\boldsymbol{b}$:

$$\boxed{P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.} \tag{7}$$

Compare with projection onto a line, when the matrix A has only one column $\boldsymbol{a}$:

For $n = 1$ $\quad \boxed{\widehat{x} = \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}}$ **and** $\boxed{\boldsymbol{p} = \boldsymbol{a}\dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}}$ **and** $\boxed{P = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.}$

Those formulas are identical with (5) and (6) and (7). The number $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ becomes the matrix $A^{\mathrm{T}}A$. When it is a number, we divide by it. When it is a matrix, we invert it. The new formulas contain $(A^{\mathrm{T}}A)^{-1}$ instead of $1/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. The linear independence of the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ will guarantee that this inverse matrix exists.

The key step was $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$. We used geometry ($\boldsymbol{e}$ is perpendicular to all the $\boldsymbol{a}$'s). Linear algebra gives this "normal equation" too, in a very quick way:

1. Our subspace is the column space of A.
2. The error vector $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to that column space.
3. Therefore $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is in the nullspace of A^{T}. This means $A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0}$.

The left nullspace is important in projections. That nullspace of A^{T} contains the error vector $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$. The vector $\boldsymbol{b}$ is being split into the projection $\boldsymbol{p}$ and the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. Projection produces a right triangle (Figure 4.5) with sides $\boldsymbol{p}$, $\boldsymbol{e}$, and $\boldsymbol{b}$.

Example 3 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ find $\widehat{x}$ and $\boldsymbol{p}$ and P.

Solution Compute the square matrix $A^{\mathrm{T}}A$ and also the vector $A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

Now solve the normal equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$ to find $\widehat{x}$:

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \text{gives} \quad \widehat{x} = \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ \mathbf{-3} \end{bmatrix}. \tag{8}$$

The combination $\boldsymbol{p} = A\widehat{x}$ is the projection of $\boldsymbol{b}$ onto the column space of A:

$$\boldsymbol{p} = \mathbf{5} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{-3} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}. \quad \text{The error is} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{9}$$

Two checks on the calculation. First, the error $\boldsymbol{e} = (1, -2, 1)$ is perpendicular to both columns $(1, 1, 1)$ and $(0, 1, 2)$. Second, the final P times $\boldsymbol{b} = (6, 0, 0)$ correctly gives $\boldsymbol{p} = (5, 2, -1)$. That solves the problem for one particular $\boldsymbol{b}$.

To find $\boldsymbol{p} = P\boldsymbol{b}$ for every $\boldsymbol{b}$, compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The determinant of $A^{\mathrm{T}}A$ is $15 - 9 = 6$; then $(A^{\mathrm{T}}A)^{-1}$ is easy. Multiply A times $(A^{\mathrm{T}}A)^{-1}$ times A^{T} to reach P:

$$(A^{\mathrm{T}}A)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \quad \text{and} \quad P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \tag{10}$$

We must have $P^2 = P$, because a second projection doesn't change the first projection.

Warning The matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is deceptive. You might try to split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$. If you make that mistake, and substitute it into P, you will find $P = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}}$. Apparently everything cancels. This looks like $P = I$, the identity matrix. We want to say why this is wrong.

The matrix A is rectangular. It has no inverse matrix. We cannot split $(A^{\mathrm{T}}A)^{-1}$ into A^{-1} times $(A^{\mathrm{T}})^{-1}$ because there is no A^{-1} in the first place.

In our experience, a problem that involves a rectangular matrix almost always leads to $A^{\mathrm{T}}A$. When A has independent columns, $A^{\mathrm{T}}A$ is invertible. This fact is so crucial that we state it clearly and give a proof.

$A^{\mathrm{T}}A$ is invertible if and only if A has linearly independent columns.

Proof $A^{\mathrm{T}}A$ is a square matrix (n by n). For every matrix A, we will now show that $A^{\mathrm{T}}A$ ***has the same nullspace as*** A. When the columns of A are linearly independent, its nullspace contains only the zero vector. Then $A^{\mathrm{T}}A$, with this same nullspace, is invertible.

Let A be any matrix. If $\boldsymbol{x}$ is in its nullspace, then $A\boldsymbol{x} = \mathbf{0}$. Multiplying by A^{T} gives $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{x}$ is also in the nullspace of $A^{\mathrm{T}}A$.

Now start with the nullspace of $A^{\mathrm{T}}A$. **From $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ we must prove $A\boldsymbol{x} = \mathbf{0}$.** We can't multiply by $(A^{\mathrm{T}})^{-1}$, which generally doesn't exist. Just multiply by $\boldsymbol{x}^{\mathrm{T}}$:

$$(\boldsymbol{x}^{\mathrm{T}})A^{\mathrm{T}}A\boldsymbol{x} = 0 \quad \text{or} \quad (A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = 0 \quad \text{or} \quad \|A\boldsymbol{x}\|^2 = 0.$$

This says: If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ has length zero. Therefore $A\boldsymbol{x} = \mathbf{0}$. Every vector $\boldsymbol{x}$ in one nullspace is in the other nullspace. If $A^{\mathrm{T}}A$ has dependent columns, so has A. If $A^{\mathrm{T}}A$ has independent columns, so has A. This is the good case:

When A has independent columns, $A^{\mathrm{T}}A$ is square, symmetric, and invertible.

To repeat for emphasis: $A^{\mathrm{T}}A$ is (n by m) times (m by n). Then $A^{\mathrm{T}}A$ is square (n by n). It is symmetric, because its transpose is $(A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}}$ which equals $A^{\mathrm{T}}A$. We just proved that $A^{\mathrm{T}}A$ is invertible—provided A has independent columns. Watch the difference between dependent and independent columns:

$$\underset{A^{\mathrm{T}}}{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}} \underset{\text{dependent}}{\overset{A}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}}} = \underset{\text{singular}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}}} \qquad \underset{A^{\mathrm{T}}}{\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}} \underset{\text{indep.}}{\overset{A}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}}} = \underset{\text{invertible}}{\overset{A^{\mathrm{T}}A}{\begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix}}}$$

Very brief summary To find the projection $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n$, solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. This gives $\widehat{\boldsymbol{x}}$. The projection is $A\widehat{\boldsymbol{x}}$ and the error is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$. The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ gives $\boldsymbol{p} = P\boldsymbol{b}$.

This matrix satisfies $P^2 = P$. ***The distance from $\boldsymbol{b}$ to the subspace is $\|\boldsymbol{e}\|$.***

■ REVIEW OF THE KEY IDEAS ■

1. The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is $\boldsymbol{p} = \boldsymbol{a}\widehat{x} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$.

2. The rank one projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ multiplies $\boldsymbol{b}$ to produce $\boldsymbol{p}$.

3. Projecting $\boldsymbol{b}$ onto a subspace leaves $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ perpendicular to the subspace.

4. When A has full rank n, the equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ leads to $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$.

5. The projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $P^{\mathrm{T}} = P$ and $P^2 = P$.

■ WORKED EXAMPLES ■

4.2 A Project the vector $\boldsymbol{b} = (3,4,4)$ onto the line through $\boldsymbol{a} = (2,2,1)$ and then onto the plane that also contains $\boldsymbol{a}^* = (1,0,0)$. Check that the first error vector $\boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

Find the 3 by 3 projection matrix P onto that plane of $\boldsymbol{a}$ and $\boldsymbol{a}^*$. Find a vector whose projection onto the plane is the zero vector.

Solution The projection of $\boldsymbol{b} = (3,4,4)$ onto the line through $\boldsymbol{a} = (2,2,1)$ is $\boldsymbol{p} = 2\boldsymbol{a}$:

Onto a line $$\boldsymbol{p} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = \frac{18}{9}(2,2,1) = (4,4,2).$$

The error vector $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = (-1,0,2)$ is perpendicular to $\boldsymbol{a}$. So $\boldsymbol{p}$ is correct.

The plane of $\boldsymbol{a} = (2,2,1)$ and $\boldsymbol{a}^* = (1,0,0)$ is the column space of $A = [\boldsymbol{a}\ \boldsymbol{a}^*]$:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \quad (A^{\mathrm{T}}A)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

Then $\boldsymbol{p}^* = P\boldsymbol{b} = (3, 4.8, 2.4)$. The error $\boldsymbol{e}^* = \boldsymbol{b} - \boldsymbol{p}^* = (0, -.8, 1.6)$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{a}^*$. This $\boldsymbol{e}^*$ is in the nullspace of P and its projection is zero! Note $P^2 = P$.

4.2 B Suppose your pulse is measured at $x = 70$ beats per minute, then at $x = 80$, then at $x = 120$. Those three equations $Ax = \boldsymbol{b}$ in one unknown have $A^{\mathrm{T}} = [1\ 1\ 1]$ and $\boldsymbol{b} = (70, 80, 120)$. ***The best $\widehat{x}$ is the* _____ *of* 70, 80, 120.** Use calculus and projection:

1. Minimize $E = (x-70)^2 + (x-80)^2 + (x-120)^2$ by solving $dE/dx = 0$.
2. Project $\boldsymbol{b} = (70, 80, 120)$ onto $\boldsymbol{a} = (1,1,1)$ to find $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Solution The closest horizontal line to the heights 70, 80, 120 is the *average* $\widehat{x} = 90$:

$$\frac{dE}{dx} = 2(x-70) + 2(x-80) + 2(x-120) = 0 \quad \text{gives} \quad \widehat{x} = \frac{70+80+120}{3}$$

$$\textbf{Projection}: \quad \widehat{x} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{(1,1,1)^{\mathrm{T}}(70,80,120)}{(1,1,1)^{\mathrm{T}}(1,1,1)} = \frac{70+80+120}{3} = 90.$$

4.2 C In *recursive* least squares, a fourth measurement 130 changes $\widehat{x}_{\text{old}}$ to $\widehat{x}_{\text{new}}$. Compute $\widehat{x}_{\text{new}}$ and verify the *update formula* $\widehat{x}_{\text{new}} = \widehat{x}_{\text{old}} + \frac{1}{4}(130 - \widehat{x}_{\text{old}})$.

Going from 999 to 1000 measurements, $\widehat{x}_{\text{new}} = \widehat{x}_{\text{old}} + \frac{1}{1000}(b_{1000} - \widehat{x}_{\text{old}})$ would only need $\widehat{x}_{\text{old}}$ and the latest value b_{1000}. We don't have to average all 1000 numbers!

Solution The new measurement $b_4 = 130$ adds a fourth equation and $\widehat{x}$ is updated to 100. You can average b_1, b_2, b_3, b_4 or combine the average of b_1, b_2, b_3 with b_4:

$$\frac{70+80+120+130}{4} = 100 \quad \text{is also} \quad \widehat{x}_{\text{old}} + \frac{1}{4}(b_4 - \widehat{x}_{\text{old}}) = 90 + \frac{1}{4}(40).$$

The update from 999 to 1000 measurements shows the "gain matrix" $\frac{1}{1000}$ in a **Kalman filter** multiplying the prediction error $b_{\text{new}} - \widehat{x}_{\text{old}}$. Notice $\frac{1}{1000} = \frac{1}{999} - \frac{1}{999000}$:

$$\widehat{x}_{\text{new}} = \frac{b_1 + \cdots + b_{1000}}{1000} = \frac{b_1 + \cdots + b_{999}}{999} + \frac{1}{1000}\left(b_{1000} - \frac{b_1 + \cdots + b_{999}}{999}\right).$$

Problem Set 4.2

Questions 1–9 ask for projections onto lines. Also errors $e = b - p$ and matrices P.

1 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix}1\\2\\2\end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \text{(b)}\ \boldsymbol{b} = \begin{bmatrix}1\\3\\1\end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix}-1\\-3\\-1\end{bmatrix}.$$

2 *Draw* the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$:

$$\text{(a)}\ \boldsymbol{b} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{(b)}\ \boldsymbol{b} = \begin{bmatrix}1\\1\end{bmatrix} \text{ and } \boldsymbol{a} = \begin{bmatrix}1\\-1\end{bmatrix}.$$

3 In Problem 1, find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the line through each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to compute the projection $\boldsymbol{p}$.

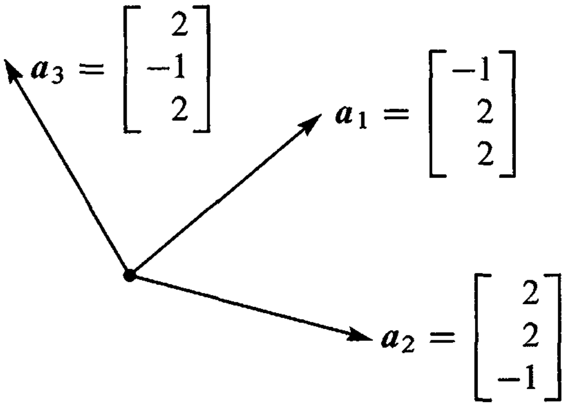
4 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 2. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

5 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is.

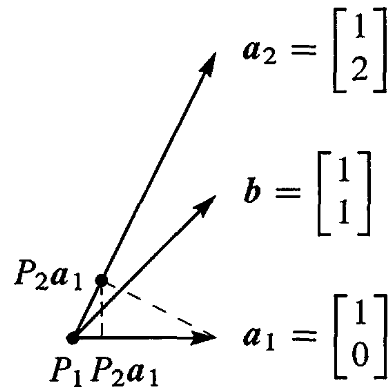
6 Project $\boldsymbol{b} = (1, 0, 0)$ onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in Problem 5 and also onto $\boldsymbol{a}_3 = (2, -1, 2)$. Add up the three projections $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$.

7 Continuing Problems 5–6, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal!

8 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.



Questions 5–6–7



Questions 8–9–10

9 In Problem 8, the projection of $\boldsymbol{b}$ onto the *plane* of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ will equal $\boldsymbol{b}$. Find $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ for $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$.

10 Project $\boldsymbol{a}_1 = (1, 0)$ onto $\boldsymbol{a}_2 = (1, 2)$. Then project the result back onto $\boldsymbol{a}_1$. Draw these projections and multiply the projection matrices $P_1 P_2$: Is this a projection?

Questions 11–20 ask for projections, and projection matrices, onto subspaces.

11 Project $\boldsymbol{b}$ onto the column space of A by solving $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$:

$$\text{(a)}\ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. It should be perpendicular to the columns of A.

12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 11. Verify that $P_1\boldsymbol{b}$ gives the first projection $\boldsymbol{p}_1$. Also verify $P_2^2 = P_2$.

13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

14 Suppose $\boldsymbol{b}$ equals 2 times the first column of A. What is the projection of $\boldsymbol{b}$ onto the column space of A? Is $P = I$ for sure in this case? Compute $\boldsymbol{p}$ and P when $\boldsymbol{b} = (0, 2, 4)$ and the columns of A are $(0, 1, 2)$ and $(1, 2, 0)$.

15 If A is doubled, then $P = 2A(4A^{\mathrm{T}}A)^{-1}2A^{\mathrm{T}}$. This is the same as $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The column space of $2A$ is the same as _____. Is $\widehat{\boldsymbol{x}}$ the same for A and $2A$?

16 What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\boldsymbol{b} = (2, 1, 1)$?

17 (*Important*) If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the _____.

18 (a) If P is the 2 by 2 projection matrix onto the line through $(1, 1)$, then $I - P$ is the projection matrix onto ____.

(b) If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto ____.

19 To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane should be the column space. Then compute $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

20 To find the projection matrix P onto the same plane $x - y - 2z = 0$, write down a vector $\boldsymbol{e}$ that is perpendicular to that plane. Compute the projection $Q = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ and then $P = I - Q$.

Questions 21–26 show that projection matrices satisfy $P^2 = P$ and $P^{\mathrm{T}} = P$.

21 Multiply the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is ____.

22 Prove that $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is symmetric by computing P^{T}. Remember that the inverse of a symmetric matrix is symmetric.

23 If A is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. It is true that $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$. *When A is invertible, why is $P = I$? **What is the error $\boldsymbol{e}$?***

24 The nullspace of A^{T} is ____ to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} =$ ____. Check that $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ gives this answer.

25 The projection matrix P onto an n-dimensional subspace has rank $r = n$. ***Reason:*** The projections $P\boldsymbol{b}$ fill the subspace S. So S is the ____ of P.

26 If an m by m matrix has $A^2 = A$ and its rank is m, prove that $A = I$.

27 The important fact that ends the section is this: ***If $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x} = \boldsymbol{0}$.*** *New Proof*: The vector $A\boldsymbol{x}$ is in the nullspace of ____. $A\boldsymbol{x}$ is always in the column space of ____. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

28 Use $P^{\mathrm{T}} = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry P_{22}. This number is $\frac{2}{6} = \frac{4}{36} + \frac{4}{36} + \frac{4}{36}$ for

$$P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

29 If B has rank m (full row rank, independent rows) show that BB^{T} is invertible.

Challenge Problems

30 (a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3 by 3 projection matrix P_R onto the row space of A. Multiply $B = P_C A P_R$. Your answer B should be a little surprising—can you explain it?

31 In $\mathbf{R}^m$, suppose I give you $\boldsymbol{b}$ and $\boldsymbol{p}$, and $\boldsymbol{p}$ is a combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. How would you test to see if $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the subspace spanned by the $\boldsymbol{a}$'s?

32 Suppose P_1 is the projection matrix onto the 1-dimensional subspace spanned by the first column of A. Suppose P_2 is the projection matrix onto the 2-dimensional column space of A. After thinking a little, compute the product $P_2 P_1$.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

33 P_1 and P_2 are projections onto subspaces $\boldsymbol{S}$ and $\boldsymbol{T}$. What is the requirement on those subspaces to have $P_1 P_2 = P_2 P_1$?

34 **If A has r independent columns and B has r independent rows, AB is invertible.**

Proof: When A is m by r with independent columns, we know that $A^{\mathrm{T}} A$ is invertible. If B is r by n with independent rows, show that BB^{T} is invertible. (Take $A = B^{\mathrm{T}}$.)

Now show that AB has rank r. Hint: Why does $A^{\mathrm{T}} A B B^{\mathrm{T}}$ have rank r? That matrix multiplication by A^{T} and B^{T} cannot increase the rank of AB, by Problem 3.6:26.

4.3 Least Squares Approximations

It often happens that $Ax = b$ has no solution. The usual reason is: *too many equations.* The matrix has more rows than columns. There are more equations than unknowns (m is greater than n). The n columns span a small part of m-dimensional space. Unless all measurements are perfect, b is outside that column space. Elimination reaches an impossible equation and stops. But we can't stop just because measurements include noise.

To repeat: We cannot always get the error $e = b - Ax$ down to zero. When e is zero, x is an exact solution to $Ax = b$. *When the length of e is as small as possible, $\widehat{x}$ is a least squares solution.* Our goal in this section is to compute $\widehat{x}$ and use it. These are real problems and they need an answer.

The previous section emphasized p (the projection). This section emphasizes $\widehat{x}$ (the least squares solution). They are connected by $p = A\widehat{x}$. The fundamental equation is still $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Here is a short unofficial way to reach this equation:

When $Ax = b$ has no solution, multiply by A^{T} and solve $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

Example 1 A crucial application of least squares is fitting a straight line to m points. Start with three points: *Find the closest line to the points* $(0,6)$, $(1,0)$, *and* $(2,0)$.

No straight line $b = C + Dt$ goes through those three points. We are asking for two numbers C and D that satisfy three equations. Here are the equations at $t = 0, 1, 2$ to match the given values $b = 6, 0, 0$:

$t = 0$	The first point is on the line $b = C + Dt$ if	$C + D \cdot 0 = 6$
$t = 1$	The second point is on the line $b = C + Dt$ if	$C + D \cdot 1 = 0$
$t = 2$	The third point is on the line $b = C + Dt$ if	$C + D \cdot 2 = 0$.

This 3 by 2 system has *no solution*: $b = (6,0,0)$ is not a combination of the columns $(1,1,1)$ and $(0,1,2)$. Read off A, x, and b from those equations:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad Ax = b \text{ is } \textit{not} \text{ solvable.}$$

The same numbers were in Example 3 in the last section. We computed $\widehat{x} = (5, -3)$. Those numbers are the best C and D, so $5 - 3t$ will be the best line for the 3 points. We must connect projections to least squares, by explaining why $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

In practical problems, there could easily be $m = 100$ points instead of $m = 3$. They don't exactly match any straight line $C + Dt$. Our numbers $6, 0, 0$ exaggerate the error so you can see e_1, e_2, and e_3 in Figure 4.6.

Minimizing the Error

How do we make the error $e = b - Ax$ as small as possible? This is an important question with a beautiful answer. The best x (called $\widehat{x}$) can be found by geometry or algebra or calculus: 90° angle or project using P or set the derivative of the error to zero.

By geometry Every Ax lies in the plane of the columns $(1,1,1)$ and $(0,1,2)$. In that plane, we look for the point closest to $\boldsymbol{b}$. *The nearest point is the projection* $\boldsymbol{p}$.

The best choice for $A\widehat{x}$ is $\boldsymbol{p}$. The smallest possible error is $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. The three points at heights (p_1, p_2, p_3) *do lie on a line*, because $\boldsymbol{p}$ is in the column space. In fitting a straight line, $\widehat{x}$ gives the best choice for (C, D).

By algebra Every vector $\boldsymbol{b}$ splits into two parts. The part in the column space is $\boldsymbol{p}$. The perpendicular part in the nullspace of A^{T} is $\boldsymbol{e}$. There is an equation we cannot solve ($A\boldsymbol{x} = \boldsymbol{b}$). There is an equation $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ we do solve (by removing $\boldsymbol{e}$):

$$A\boldsymbol{x} = \boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e} \quad \text{is impossible;} \qquad A\widehat{\boldsymbol{x}} = \boldsymbol{p} \quad \text{is solvable.} \tag{1}$$

The solution to $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ leaves the least possible error (which is $\boldsymbol{e}$):

$$\textbf{Squared length for any } \boldsymbol{x} \qquad \|A\boldsymbol{x} - \boldsymbol{b}\|^2 = \|A\boldsymbol{x} - \boldsymbol{p}\|^2 + \|\boldsymbol{e}\|^2. \tag{2}$$

This is the law $c^2 = a^2 + b^2$ for a right triangle. The vector $A\boldsymbol{x} - \boldsymbol{p}$ in the column space is perpendicular to $\boldsymbol{e}$ in the left nullspace. We reduce $A\boldsymbol{x} - \boldsymbol{p}$ to zero by choosing $\boldsymbol{x}$ to be $\widehat{\boldsymbol{x}}$. That leaves the smallest possible error $\boldsymbol{e} = (e_1, e_2, e_3)$.

Notice what "smallest" means. The *squared length* of $A\boldsymbol{x} - \boldsymbol{b}$ is minimized:

The least squares solution $\widehat{x}$ makes $E = \|A\boldsymbol{x} - \boldsymbol{b}\|^2$ as small as possible.

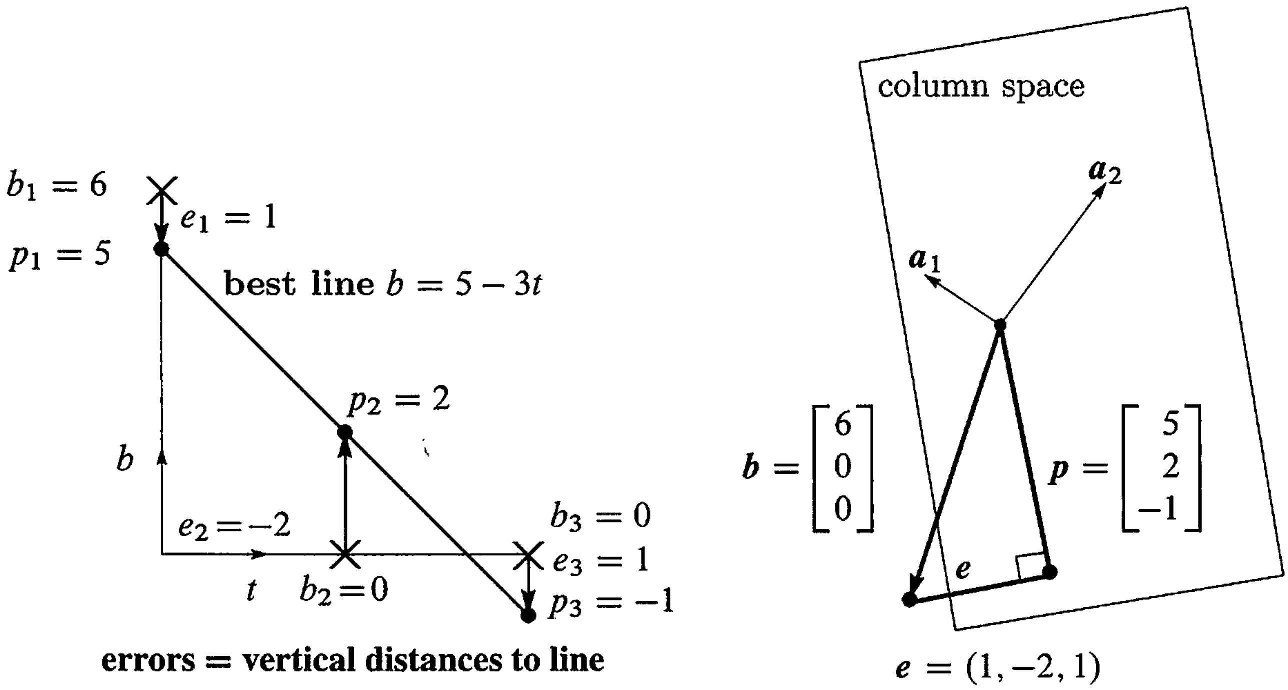


Figure 4.6: **Best line and projection: Two pictures, same problem.** The line has heights $\boldsymbol{p} = (5, 2, -1)$ with errors $\boldsymbol{e} = (1, -2, 1)$. The equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ give $\widehat{\boldsymbol{x}} = (5, -3)$. The best line is $b = 5 - 3t$ and the projection is $\boldsymbol{p} = 5\boldsymbol{a}_1 - 3\boldsymbol{a}_2$.

Figure 4.6a shows the closest line. It misses by distances $e_1, e_2, e_3 = 1, -2, 1$. *Those are vertical distances.* The least squares line minimizes $E = e_1^2 + e_2^2 + e_3^2$.

Figure 4.6b shows the same problem in 3-dimensional space ($\boldsymbol{b}\ \boldsymbol{p}\ \boldsymbol{e}$ space). The vector $\boldsymbol{b}$ is not in the column space of A. That is why we could not solve $A\boldsymbol{x} = \boldsymbol{b}$. No line goes through the three points. The smallest possible error is the perpendicular vector $\boldsymbol{e}$. This is $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$, the vector of errors $(1, -2, 1)$ in the three equations. Those are the distances from the best line. Behind both figures is the fundamental equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Notice that the errors $1, -2, 1$ add to zero. The error $\boldsymbol{e} = (e_1, e_2, e_3)$ is perpendicular to the first column $(1, 1, 1)$ in A. The dot product gives $e_1 + e_2 + e_3 = 0$.

By calculus Most functions are minimized by calculus! The graph bottoms out and the derivative in every direction is zero. Here the error function E to be minimized is a *sum of squares* $e_1^2 + e_2^2 + e_3^2$ (the square of the error in each equation):

$$E = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 = (C + D \cdot 0 - 6)^2 + (C + D \cdot 1)^2 + (C + D \cdot 2)^2. \tag{3}$$

The unknowns are C and D. With two unknowns there are *two derivatives*—both zero at the minimum. They are "partial derivatives" because $\partial E/\partial C$ treats D as constant and $\partial E/\partial D$ treats C as constant:

$$\begin{aligned} \partial E/\partial C &= 2(C + D \cdot 0 - 6) \quad + 2(C + D \cdot 1) \quad + 2(C + D \cdot 2) \quad = 0 \\ \partial E/\partial D &= 2(C + D \cdot 0 - 6)(\mathbf{0}) + 2(C + D \cdot 1)(\mathbf{1}) + 2(C + D \cdot 2)(\mathbf{2}) = 0. \end{aligned}$$

$\partial E/\partial D$ contains the extra factors $\mathbf{0}, \mathbf{1}, \mathbf{2}$ from the chain rule. (The last derivative from $(C + 2\mathrm{D})^2$ was 2 times $C + 2\mathrm{D}$ times that extra 2.) In the C derivative the corresponding factors are $1, 1, 1$, because C is always multiplied by 1. It is no accident that 1, 1, 1 and 0, 1, 2 are the columns of A.

Now cancel 2 from every term and collect all C's and all D's:

$$\begin{array}{ll} \text{The } C \text{ derivative is zero:} & 3C + 3D = 6 \\ \text{The } D \text{ derivative is zero:} & 3C + 5D = 0 \end{array} \qquad \textbf{This matrix} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \textbf{ is } A^{\mathrm{T}}A \tag{4}$$

These equations are identical with $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. The best C and D are the components of $\widehat{\boldsymbol{x}}$. The equations from calculus are the same as the "normal equations" from linear algebra. These are the key equations of least squares:

The partial derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ ***are zero when*** $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

The solution is $C = 5$ and $D = -3$. Therefore $b = 5 - 3t$ is the best line—it comes closest to the three points. At $t = 0, 1, 2$ this line goes through $\boldsymbol{p} = 5, 2, -1$. It could not go through $\boldsymbol{b} = 6, 0, 0$. The errors are $1, -2, 1$. This is the vector $\boldsymbol{e}$!

The Big Picture

The key figure of this book shows the four subspaces and the true action of a matrix. The vector $\boldsymbol{x}$ on the left side of Figure 4.3 went to $\boldsymbol{b} = A\boldsymbol{x}$ on the right side. In that figure $\boldsymbol{x}$ was split into $\boldsymbol{x}_r + \boldsymbol{x}_n$. There were many solutions to $A\boldsymbol{x} = \boldsymbol{b}$.

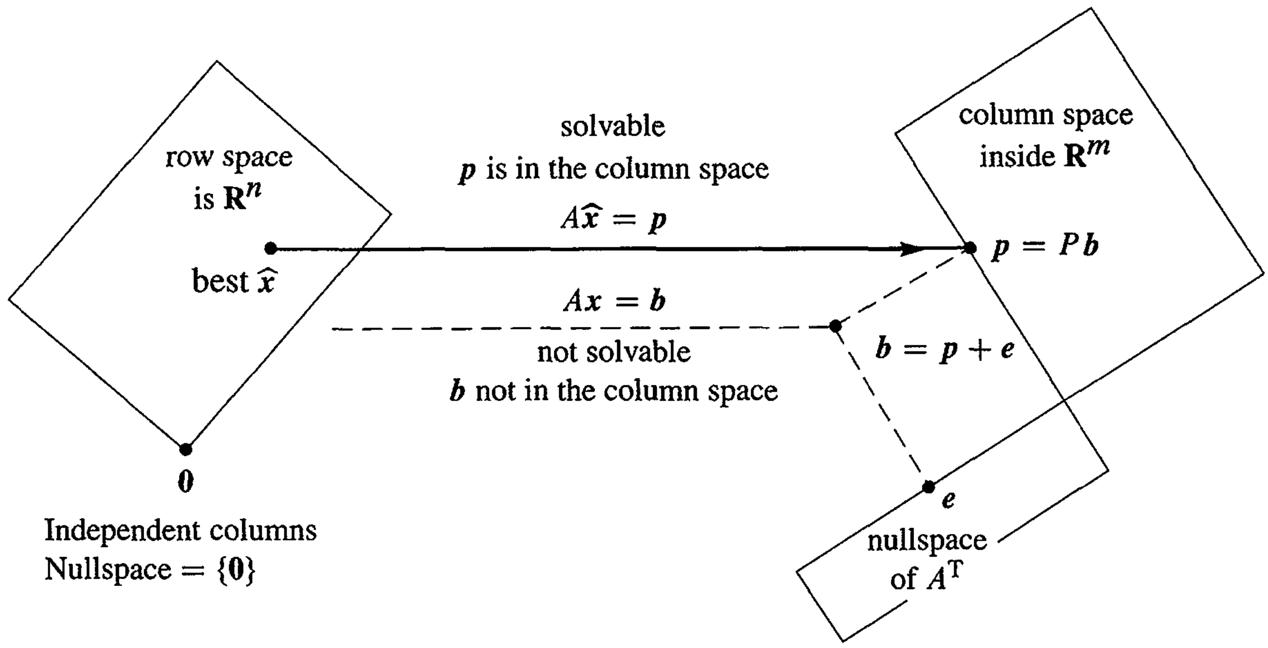


Figure 4.7: The projection $p = A\widehat{x}$ is closest to b, so $\widehat{x}$ minimizes $E = \|b - Ax\|^2$.

In this section the situation is just the opposite. There are *no* solutions to $Ax = b$. *Instead of splitting up x we are splitting up b.* Figure 4.3 shows the big picture for least squares. Instead of $Ax = b$ we solve $A\widehat{x} = p$. The error $e = b - p$ is unavoidable.

Notice how the nullspace $N(A)$ is very small—just one point. With independent columns, the only solution to $Ax = 0$ is $x = 0$. Then $A^{\mathrm{T}}A$ is invertible. The equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ fully determines the best vector $\widehat{x}$. The error has $A^{\mathrm{T}}e = 0$.

Chapter 7 will have the complete picture—all four subspaces included. Every x splits into $x_r + x_n$, and every b splits into $p + e$. The best solution is $\widehat{x}_r$ in the row space. We can't help e and we don't want x_n—this leaves $A\widehat{x} = p$.

Fitting a Straight Line

Fitting a line is the clearest application of least squares. It starts with $m > 2$ points, hopefully near a straight line. At times $t_1, \ldots, t_m$ those m points are at heights $b_1, \ldots, b_m$. The best line $C + Dt$ misses the points by vertical distances $e_1, \ldots, e_m$. No line is perfect, and the least squares line minimizes $E = e_1^2 + \cdots + e_m^2$.

The first example in this section had three points in Figure 4.6. Now we allow m points (and m can be large). The two components of $\widehat{x}$ are still C and D.

A line goes through the m points when we exactly solve $Ax = b$. Generally we can't do it. Two unknowns C and D determine a line, so A has only $n = 2$ columns. To fit the m points, we are trying to solve m equations (and we only want two!):

$$Ax = b \quad \text{is} \quad \begin{array}{c} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{array} \quad \text{with} \quad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \tag{5}$$

The column space is so thin that almost certainly $\boldsymbol{b}$ is outside of it. When $\boldsymbol{b}$ happens to lie in the column space, the points happen to lie on a line. In that case $\boldsymbol{b} = \boldsymbol{p}$. Then $A\boldsymbol{x} = \boldsymbol{b}$ is solvable and the errors are $\boldsymbol{e} = (0, \ldots, 0)$.

The closest line $C + Dt$ has heights $p_1, \ldots, p_m$ with errors $e_1, \ldots, e_m$.

Solve $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ for $\widehat{x} = (C, D)$. The errors are $e_i = b_i - C - Dt_i$.

Fitting points by a straight line is so important that we give the two equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, once and for all. The two columns of A are independent (unless all times t_i are the same). So we turn to least squares and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Dot-product matrix $$A^{\mathrm{T}}A = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}. \tag{6}$$

On the right side of the normal equation is the 2 by 1 vector $A^{\mathrm{T}}\boldsymbol{b}$:

$$A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \tag{7}$$

In a specific problem, these numbers are given. The best $\widehat{\boldsymbol{x}} = (C, D)$ is in equation (9).

The line $C + Dt$ minimizes $e_1^2 + \cdots + e_m^2 = \|A\boldsymbol{x} - \boldsymbol{b}\|^2$ when $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$:

$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \tag{8}$$

The vertical errors at the m points on the line are the components of $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. This error vector (the *residual*) $\boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is perpendicular to the columns of A (geometry). The error is in the nullspace of A^{T} (linear algebra). The best $\widehat{\boldsymbol{x}} = (C, D)$ minimizes the total error E, the sum of squares:

$$E(x) = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 = (C + Dt_1 - b_1)^2 + \cdots + (C + Dt_m - b_m)^2.$$

When calculus sets the derivatives $\partial E/\partial C$ and $\partial E/\partial D$ to zero, it produces $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

Other least squares problems have more than two unknowns. Fitting by the best parabola has $n = 3$ coefficients C, D, E (see below). In general we are fitting m data points by n parameters $x_1, \ldots, x_n$. The matrix A has n columns and $n < m$. The derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ give the n equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. **The derivative of a square is linear.** This is why the method of least squares is so popular.

Example 2 A has *orthogonal columns* when the measurement times t_i add to zero.

Suppose $b = 1, 2, 4$ at times $t = -2, 0, 2$. Those times add to zero. The columns of A have *zero dot product*:

$$\begin{aligned} C + D(-2) &= 1 \\ C + \quad D(0) &= 2 \\ C + \quad D(2) &= 4 \end{aligned} \qquad \text{or} \qquad Ax = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Look at the zeros in $A^{\mathrm{T}}A$:

$$A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b} \qquad \text{is} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

Main point: Now $A^{\mathrm{T}}A$ *is diagonal.* We can solve separately for $C = \frac{7}{3}$ and $D = \frac{6}{8}$. The zeros in $A^{\mathrm{T}}A$ are dot products of perpendicular columns in A. The diagonal matrix $A^{\mathrm{T}}A$, with entries $m = 3$ and $t_1^2 + t_2^2 + t_3^2 = 8$, is virtually as good as the identity matrix.

Orthogonal columns are so helpful that it is worth moving the time origin to produce them. To do that, subtract away the average time $\widehat{t} = (t_1 + \cdots + t_m)/m$. The shifted times $T_i = t_i - \widehat{t}$ add to $\sum T_i = m\widehat{t} - m\widehat{t} = 0$. With the columns now orthogonal, $A^{\mathrm{T}}A$ is diagonal. Its entries are m and $T_1^2 + \cdots + T_m^2$. The best C and D have direct formulas:

$$\boldsymbol{T} \textbf{ is } \boldsymbol{t - \widehat{t}} \qquad C = \frac{b_1 + \cdots + b_m}{m} \qquad \text{and} \qquad D = \frac{b_1T_1 + \cdots + b_mT_m}{T_1^2 + \cdots + T_m^2}. \tag{9}$$

The best line is $\boldsymbol{C + DT}$ ***or*** $\boldsymbol{C + D(t - \widehat{t})}$. The time shift that makes $A^{\mathrm{T}}A$ diagonal is an example of the Gram-Schmidt process: *orthogonalize the columns in advance.*

Fitting by a Parabola

If we throw a ball, it would be crazy to fit the path by a straight line. A parabola $b = C + Dt + Et^2$ allows the ball to go up and come down again (b is the height at time t). The actual path is not a perfect parabola, but the whole theory of projectiles starts with that approximation.

When Galileo dropped a stone from the Leaning Tower of Pisa, it accelerated. The distance contains a quadratic term $\frac{1}{2}gt^2$. (Galileo's point was that the stone's mass is not involved.) Without that t^2 term we could never send a satellite into the right orbit. But even with a nonlinear function like t^2, the unknowns C, D, E appear linearly! Choosing the best parabola is still a problem in linear algebra.

Problem Fit heights $b_1, \ldots, b_m$ at times $t_1, \ldots, t_m$ by a parabola $C + Dt + Et^2$.

Solution With $m > 3$ points, the m equations for an exact fit are generally unsolvable:

$$\begin{aligned} C + Dt_1 + Et_1^2 &= b_1 \\ &\vdots \\ C + Dt_m + Et_m^2 &= b_m \end{aligned} \qquad \text{has the } m \text{ by 3 matrix} \qquad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}. \tag{10}$$

Least squares The closest parabola $C + Dt + Et^2$ chooses $\widehat{x} = (C, D, E)$ to satisfy the three normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$.

May I ask you to convert this to a problem of projection? The column space of A has dimension _____. The projection of $\boldsymbol{b}$ is $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$, which combines the three columns using the coefficients C, D, E. The error at the first data point is $e_1 = b_1 - C - Dt_1 - Et_1^2$. The total squared error is $e_1^2 +$ _____. If you prefer to minimize by calculus, take the partial derivatives of E with respect to _____, _____, _____. These three derivatives will be zero when $\widehat{\boldsymbol{x}} = (C, D, E)$ solves the 3 by 3 system of equations _____.

Section 8.5 has more least squares applications. The big one is Fourier series—approximating functions instead of vectors. The function to be minimized changes from a sum of squared errors $e_1^2 + \cdots + e_m^2$ to an integral of the squared error.

Example 3 For a parabola $b = C + Dt + Et^2$ to go through the three heights $b = 6, 0, 0$ when $t = 0, 1, 2$, the equations are

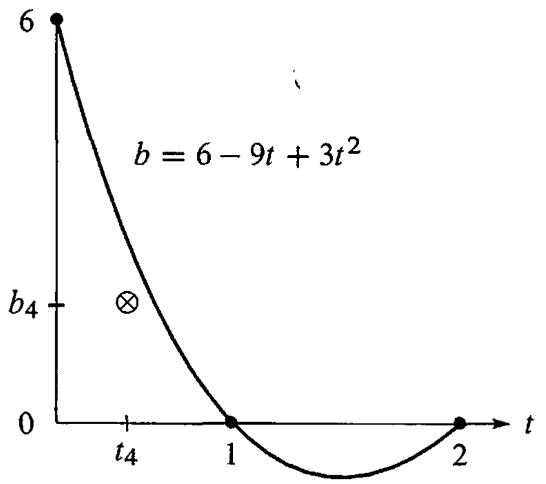
$$\begin{aligned} C + D \cdot 0 + E \cdot 0^2 &= 6 \\ C + D \cdot 1 + E \cdot 1^2 &= 0 \\ C + D \cdot 2 + E \cdot 2^2 &= 0. \end{aligned} \tag{11}$$

This is $A\boldsymbol{x} = \boldsymbol{b}$. We can solve it exactly. Three data points give three equations and a square matrix. The solution is $\boldsymbol{x} = (C, D, E) = (6, -9, 3)$. The parabola through the three points in Figure 4.8a is $b = 6 - 9t + 3t^2$.

What does this mean for projection? The matrix has three columns, which span the whole space $\mathbf{R}^3$. The projection matrix is the identity. The projection of $\boldsymbol{b}$ is $\boldsymbol{b}$. The error is zero. We didn't need $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$, because we solved $A\boldsymbol{x} = \boldsymbol{b}$. Of course we could multiply by A^{T}, but there is no reason to do it.

Figure 4.8 also shows a fourth point b_4 at time t_4. If that falls on the parabola, the new $A\boldsymbol{x} = \boldsymbol{b}$ (four equations) is still solvable. When the fourth point is not on the parabola, we turn to $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Will the least squares parabola stay the same, with all the error at the fourth point? Not likely!

The smallest error vector (e_1, e_2, e_3, e_4) is perpendicular to $(1, 1, 1, 1)$, the first column of A. Least squares balances out the four errors, and they add to zero.



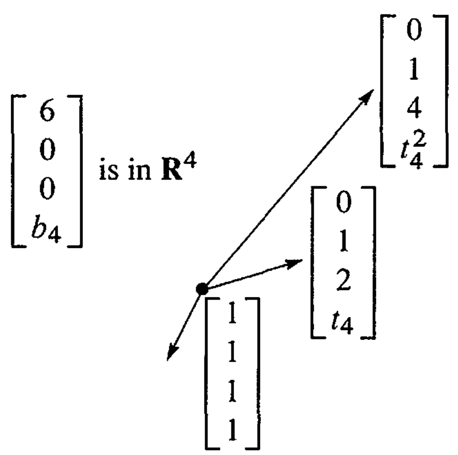


Figure 4.8: From Example 3: An exact fit of the parabola at $t = 0, 1, 2$ means that $\boldsymbol{p} = \boldsymbol{b}$ and $\boldsymbol{e} = \boldsymbol{0}$. The point b_4 off the parabola makes $m > n$ and we need least squares.

■ REVIEW OF THE KEY IDEAS ■

1. The least squares solution $\widehat{x}$ minimizes $E = \|Ax - b\|^2$. This is the sum of squares of the errors in the m equations $(m > n)$.

2. The best $\widehat{x}$ comes from the normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

3. To fit m points by a line $b = C + Dt$, the normal equations give C and D.

4. The heights of the best line are $p = (p_1, \ldots, p_m)$. The vertical distances to the data points are the errors $e = (e_1, \ldots, e_m)$.

5. If we try to fit m points by a combination of $n < m$ functions, the m equations $Ax = b$ are generally unsolvable. The n equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ give the least squares solution—the combination with smallest MSE (mean square error).

■ WORKED EXAMPLES ■

4.3 A Start with nine measurements b_1 to b_9, *all zero*, at times $t = 1, \ldots, 9$. The tenth measurement $b_{10} = 40$ is an outlier. Find the best *horizontal line* $y = C$ to fit the ten points $(1,0), (2,0), \ldots, (9,0), (10,40)$ using three measures for the error E:
(1) Least squares $E_2 = e_1^2 + \cdots + e_{10}^2$ (then the normal equation for C is linear)
(2) Least maximum error $E_\infty = |e_{\max}|$ (3) Least sum of errors $E_1 = |e_1| + \cdots + |e_{10}|$.

Solution (1) The least squares fit to $0, 0, \ldots, 0, 40$ by a horizontal line is $C = 4$:

$$A = \text{column of 1's} \quad A^{\mathrm{T}}A = 10 \quad A^{\mathrm{T}}b = \text{sum of } b_i = 40. \quad \text{So } 10C = 40.$$

(2) The least maximum error requires $C = 20$, halfway between 0 and 40.

(3) The least sum requires $C = 0$ (!!). The sum of errors $9|C| + |40 - C|$ would increase if C moves up from zero.

The least sum comes from the *median* measurement (the median of $0, \ldots, 0, 40$ is zero). Many statisticians feel that the least squares solution is too heavily influenced by outliers like $b_{10} = 40$, and they prefer least sum. But the equations become nonlinear.

Now find the least squares straight line $C + Dt$ through those ten points.

$$A^{\mathrm{T}}A = \begin{bmatrix} 10 & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} = \begin{bmatrix} 10 & 55 \\ 55 & 385 \end{bmatrix} \qquad A^{\mathrm{T}}b = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix} = \begin{bmatrix} 40 \\ 400 \end{bmatrix}$$

Those come from equation (8). Then $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ gives $C = -8$ and $D = 24/11$.

What happens to C and D if you multiply the b_i by 3 and then add 30 to get $b_{\text{new}} = (30, 30, \ldots, 150)$? Linearity allows us to rescale $b = (0, 0, \ldots, 40)$. Multiplying b by 3 will multiply C and D by 3. Adding 30 to all b_i will add 30 to C.

4.3 B Find the parabola $C+Dt+Et^2$ that comes closest (least squares error) to the values $\boldsymbol{b}=(0,0,1,0,0)$ at the times $t=-2,-1,0,1,2$. First write down the five equations $A\boldsymbol{x}=\boldsymbol{b}$ in three unknowns $\boldsymbol{x}=(C,D,E)$ for a parabola to go through the five points. No solution because no such parabola exists. Solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$.

I would predict $D=0$. Why should the best parabola be symmetric around $t=0$? In $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$, equation 2 for D should uncouple from equations 1 and 3.

Solution The five equations $A\boldsymbol{x}=\boldsymbol{b}$ have a rectangular "Vandermonde" matrix A:

$$\begin{array}{l} C+D\,(-2)+E\,(-2)^2=0\\ C+D\,(-1)+E\,(-1)^2=0\\ C+D\quad(0)+E\quad(0)^2=1\\ C+D\quad(1)+E\quad(1)^2=0\\ C+D\quad(2)+E\quad(2)^2=0 \end{array} \qquad A=\begin{bmatrix}1&-2&4\\1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{bmatrix} \qquad A^{\mathrm{T}}A=\begin{bmatrix}5&\mathbf{0}&10\\\mathbf{0}&10&\mathbf{0}\\10&\mathbf{0}&34\end{bmatrix}$$

Those zeros in $A^{\mathrm{T}}A$ mean that column 2 of A is orthogonal to columns 1 and 3. We see this directly in A (the times $-2,-1,0,1,2$ are symmetric). The best C,D,E in the parabola $C+Dt+Et^2$ come from $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$, and D is uncoupled:

$$\begin{bmatrix}5&0&10\\0&10&0\\10&0&34\end{bmatrix}\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{leads to} \quad \begin{array}{l}C=34/70\\D=0\ \text{ as predicted}\\E=-10/70\end{array}$$

Problem Set 4.3

Problems 1–11 use four data points $\boldsymbol{b}=(0,8,8,20)$ to bring out the key ideas.

1 With $b=0,8,8,20$ at $t=0,1,3,4$, set up and solve the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line in Figure 4.9a, find its four heights p_i and four errors e_i. What is the minimum value $E=e_1^2+e_2^2+e_3^2+e_4^2$?

2 (Line $C+Dt$ does go through p's) With $b=0,8,8,20$ at times $t=0,1,3,4$, write down the four equations $A\boldsymbol{x}=\boldsymbol{b}$ (unsolvable). Change the measurements to $p=1,5,13,17$ and find an exact solution to $A\widehat{\boldsymbol{x}}=\boldsymbol{p}$.

3 Check that $\boldsymbol{e}=\boldsymbol{b}-\boldsymbol{p}=(-1,3,-5,3)$ is perpendicular to both columns of the same matrix A. What is the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{b}$ to the column space of A?

4 (By calculus) Write down $E=\|A\boldsymbol{x}-\boldsymbol{b}\|^2$ as a sum of four squares—the last one is $(C+4D-20)^2$. Find the derivative equations $\partial E/\partial C=0$ and $\partial E/\partial D=0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$.

5 Find the height C of the best *horizontal line* to fit $\boldsymbol{b}=(0,8,8,20)$. An exact fit would solve the unsolvable equations $C=0$, $C=8$, $C=8$, $C=20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$. Draw the horizontal line at height $\widehat{x}=C$ and the four errors in $\boldsymbol{e}$.

6 Project $\boldsymbol{b} = (0, 8, 8, 20)$ onto the line through $\boldsymbol{a} = (1, 1, 1, 1)$. Find $\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ and the projection $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$. Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and find the shortest distance $\|\boldsymbol{e}\|$ from $\boldsymbol{b}$ to the line through $\boldsymbol{a}$.

7 Find the closest line $b = Dt$, *through the origin*, to the same four points. An exact fit would solve $D \cdot 0 = 0, D \cdot 1 = 8, D \cdot 3 = 8, D \cdot 4 = 20$. Find the 4 by 1 matrix and solve $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$. Redraw Figure 4.9a showing the best line $b = Dt$ and the e's.

8 Project $\boldsymbol{b} = (0, 8, 8, 20)$ onto the line through $\boldsymbol{a} = (0, 1, 3, 4)$. Find $\widehat{x} = D$ and $\boldsymbol{p} = \widehat{x}\boldsymbol{a}$. The best C in Problems 5–6 and the best D in Problems 7–8 do *not* agree with the best (C, D) in Problems 1–4. That is because $(1, 1, 1, 1)$ and $(0, 1, 3, 4)$ are _____ perpendicular.

9 For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $Ax = \boldsymbol{b}$ in three unknowns $\boldsymbol{x} = (C, D, E)$. Set up the three normal equations $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$ (solution not required). In Figure 4.9a you are now fitting a parabola to 4 points—what is happening in Figure 4.9b?

10 For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write down the four equations $A\boldsymbol{x} = \boldsymbol{b}$. Solve them by elimination. In Figure 4.9a this cubic now goes exactly through the points. What are $\boldsymbol{p}$ and $\boldsymbol{e}$?

11 The average of the four times is $\widehat{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. The average of the four b's is $\widehat{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

(a) Verify that the best line goes through the center point $(\widehat{t}, \widehat{b}) = (2, 9)$.

(b) Explain why $C + D\widehat{t} = \widehat{b}$ comes from the first equation in $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$.

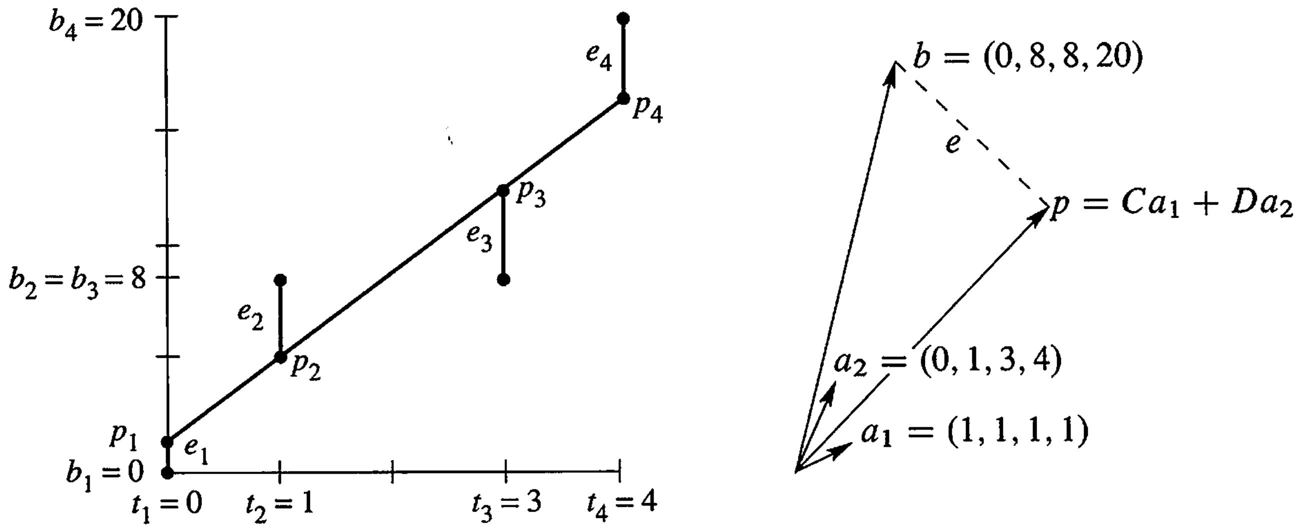


Figure 4.9: **Problems 1–11**: The closest line $C + Dt$ matches $C\boldsymbol{a}_1 + D\boldsymbol{a}_2$ in $\mathbf{R}^4$.

Questions 12–16 introduce basic ideas of statistics—the foundation for least squares.

12 (Recommended) This problem projects $\boldsymbol{b} = (b_1, \ldots, b_m)$ onto the line through $\boldsymbol{a} = (1, \ldots, 1)$. We solve m equations $\boldsymbol{a}x = \boldsymbol{b}$ in 1 unknown (by least squares).

(a) Solve $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}\widehat{x} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ to show that $\widehat{x}$ is the *mean* (the average) of the $\boldsymbol{b}$'s.

(b) Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{a}\widehat{x}$ and the *variance* $\|\boldsymbol{e}\|^2$ and the *standard deviation* $\|\boldsymbol{e}\|$.

(c) The horizontal line $\widehat{\boldsymbol{b}} = 3$ is closest to $\boldsymbol{b} = (1, 2, 6)$. Check that $\boldsymbol{p} = (3, 3, 3)$ is perpendicular to $\boldsymbol{e}$ and find the 3 by 3 projection matrix P.

13 First assumption behind least squares: $A\boldsymbol{x} = \boldsymbol{b} -$ ***(noise e with mean zero)***. Multiply the error vectors $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ by $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ to get $\widehat{\boldsymbol{x}} - \boldsymbol{x}$ on the right. The estimation errors $\widehat{\boldsymbol{x}} - \boldsymbol{x}$ also average to zero. The estimate $\widehat{\boldsymbol{x}}$ is *unbiased.*

14 Second assumption behind least squares: The m errors $\boldsymbol{e}_i$ are independent with variance σ^2, so the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$. Multiply on the left by $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and on the right by $A(A^{\mathrm{T}}A)^{-1}$ to show that the average matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2(A^{\mathrm{T}}A)^{-1}$. This is the ***covariance matrix*** P in section 8.6.

15 A doctor takes 4 readings of your heart rate. The best solution to $x = b_1, \ldots, x = b_4$ is the average $\widehat{x}$ of $b_1, \ldots, b_4$. The matrix A is a column of 1's. Problem 14 gives the expected error $(\widehat{x} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} =$ _____. ***By averaging, the variance drops from*** σ^2 ***to*** $\sigma^2/4$.

16 If you know the average $\widehat{x}_9$ of 9 numbers $b_1, \ldots, b_9$, how can you quickly find the average $\widehat{x}_{10}$ with one more number b_{10}? The idea of *recursive* least squares is to avoid adding 10 numbers. What number multiplies $\widehat{x}_9$ in computing $\widehat{x}_{10}$?

$$\widehat{x}_{10} = \tfrac{1}{10}b_{10} + \underline{\qquad}\,\widehat{x}_9 = \tfrac{1}{10}(b_1 + \cdots + b_{10}) \quad \text{as in Worked Example 4.2 C.}$$

Questions 17–24 give more practice with $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and $\boldsymbol{e}$.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{x}} = (C, D)$ and draw the closest line.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \boldsymbol{0}$?

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{x}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{x}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____.

21 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains $\boldsymbol{p}$? Which contains $\widehat{\boldsymbol{x}}$? What is the nullspace of A?

22 Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

23 Is the error vector $\boldsymbol{e}$ orthogonal to $\boldsymbol{b}$ or $\boldsymbol{p}$ or $\boldsymbol{e}$ or $\widehat{\boldsymbol{x}}$? Show that $\|\boldsymbol{e}\|^2$ equals $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{b}$ which equals $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{b}$. This is the smallest total error E.

24 The partial derivatives of $\|A\boldsymbol{x}\|^2$ with respect to $x_1, \ldots, x_n$ fill the vector $2A^{\mathrm{T}}A\boldsymbol{x}$. The derivatives of $2\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{x}$ fill the vector $2A^{\mathrm{T}}\boldsymbol{b}$. So the derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ are zero when ______.

Challenge Problems

25 *What condition on $(t_1, b_1), (t_2, b_2), (t_3, b_3)$ puts those three points onto a straight line?* A column space answer is: (b_1, b_2, b_3) must be a combination of $(1, 1, 1)$ and (t_1, t_2, t_3). Try to reach a specific equation connecting the t's and b's. I should have thought of this question sooner!

26 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those 4 points are $A\boldsymbol{x} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{x} = (C, D, E)$. What is A? At the center $(0, 0)$ of the square, show that $C + Dx + Ey =$ average of the b's.

27 (Distance between lines) The points $P = (x, x, x)$ and $Q = (y, 3y, -1)$ are on two lines in space that don't meet. Choose x and y to minimize the squared distance $\|P - Q\|^2$. The line connecting the closest P and Q is perpendicular to ______.

28 Suppose the columns of A are not independent. How could you find a matrix B so that $P = B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ does give the projection onto the column space of A? (The usual formula will fail when $A^{\mathrm{T}}A$ is not invertible.)

29 Usually there will be exactly one hyperplane in $\mathbf{R}^n$ that contains the n given points $\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$. (Example for $n = 3$: There will be one plane containing $\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless ______.) What is the test to have exactly one plane in $\mathbf{R}^n$?

4.4 Orthogonal Bases and Gram-Schmidt

This section has two goals. The first is to see how orthogonality makes it easy to find $\widehat{x}$ and $\boldsymbol{p}$ and P. Dot products are zero—so $A^{\mathrm{T}}A$ becomes a diagonal matrix. ***The second goal is to construct orthogonal vectors.*** We will pick combinations of the original vectors to produce right angles. Those original vectors are the columns of A, probably *not* orthogonal. ***The orthogonal vectors will be the columns of a new matrix Q.***

From Chapter 3, a basis consists of independent vectors that span the space. The basis vectors could meet at any angle (except 0° and 180°). But every time we visualize axes, they are perpendicular. *In our imagination, the coordinate axes are practically always orthogonal.* This simplifies the picture and it greatly simplifies the computations.

The vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are ***orthogonal*** when their dot products $\boldsymbol{q}_i \cdot \boldsymbol{q}_j$ are zero. More exactly $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ whenever $i \neq j$. With one more step—just divide each vector by its length—the vectors become ***orthogonal unit vectors***. Their lengths are all 1. Then the basis is called ***orthonormal***.

DEFINITION The vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are ***orthonormal*** if

$$\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = \begin{cases} 0 & \text{when } i \neq j \quad (\textbf{\textit{orthogonal}} \text{ vectors}) \\ 1 & \text{when } i = j \quad (\textbf{\textit{unit}} \text{ vectors: } \|\boldsymbol{q}_i\| = 1) \end{cases}$$

A matrix with orthonormal columns is assigned the special letter Q.

The matrix Q is easy to work with because $Q^{\mathrm{T}}Q = I$. This repeats in matrix language that the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are orthonormal. Q is not required to be square.

A matrix Q with orthonormal columns satisfies $\boxed{Q^{\mathrm{T}}Q = I}$ **:**

$$\boxed{Q^{\mathrm{T}}Q} = \begin{bmatrix} -\boldsymbol{q}_1^{\mathrm{T}}- \\ -\boldsymbol{q}_2^{\mathrm{T}}- \\ \\ -\boldsymbol{q}_n^{\mathrm{T}}- \end{bmatrix} \begin{bmatrix} | & | & & | \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & & \boldsymbol{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \boxed{I}. \quad (1)$$

When row i of Q^{T} multiplies column j of Q, the dot product is $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j$. Off the diagonal $(i \neq j)$ that dot product is zero by orthogonality. On the diagonal $(i = j)$ the unit vectors give $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = \|\boldsymbol{q}_i\|^2 = 1$. Often Q is rectangular $(m > n)$. Sometimes $m = n$.

When Q is square, $Q^{\mathrm{T}}Q = I$ means that $Q^{\mathrm{T}} = Q^{-1}$: transpose = inverse.

If the columns are only orthogonal (not unit vectors), dot products still give a diagonal matrix (not the identity matrix). But this matrix is almost as good. The important thing is orthogonality—then it is easy to produce unit vectors.

To repeat: $Q^{\mathrm{T}}Q = I$ even when Q is rectangular. In that case Q^{T} is only an inverse from the left. For square matrices we also have $QQ^{\mathrm{T}} = I$, so Q^{T} is the two-sided inverse of Q. The rows of a square Q are orthonormal like the columns. ***The inverse is the transpose.*** In this square case we call Q an ***orthogonal matrix***.[1]

Here are three examples of orthogonal matrices—rotation and permutation and reflection. The quickest test is to check $Q^{\mathrm{T}}Q = I$.

Example 1 **(Rotation)** Q rotates every vector in the plane clockwise by the angle θ:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q^{\mathrm{T}} = Q^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

The columns of Q are orthogonal (take their dot product). They are unit vectors because $\sin^2\theta + \cos^2\theta = 1$. Those columns give an ***orthonormal basis*** for the plane $\mathbf{R}^2$. The standard basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are rotated through θ (see Figure 4.10a). Q^{-1} rotates vectors back through $-\theta$. It agrees with Q^{T}, because the cosine of $-\theta$ is the cosine of θ, and $\sin(-\theta) = -\sin\theta$. We have $Q^{\mathrm{T}}Q = I$ and $QQ^{\mathrm{T}} = I$.

Example 2 **(Permutation)** These matrices change the order to (y, z, x) and (y, x):

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

All columns of these Q's are unit vectors (their lengths are obviously 1). They are also orthogonal (the 1's appear in different places). *The inverse of a permutation matrix is its transpose.* The inverse puts the components back into their original order:

$$\textbf{Inverse = transpose:} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Every permutation matrix is an orthogonal matrix.

Example 3 **(Reflection)** If $\boldsymbol{u}$ is any unit vector, set $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. Notice that $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a matrix while $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ is the number $\|\boldsymbol{u}\|^2 = 1$. Then Q^{T} and Q^{-1} both equal Q:

$$Q^{\mathrm{T}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = Q \quad \text{and} \quad Q^{\mathrm{T}}Q = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} + 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = I. \tag{2}$$

Reflection matrices $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ are symmetric and also orthogonal. If you square them, you get the identity matrix: $Q^2 = Q^{\mathrm{T}}Q = I$. Reflecting twice through a mirror brings back the original. Notice $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ inside $4\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ in equation (2).

[1]"Orthonormal matrix" would have been a better name for Q, but it's not used. Any matrix with orthonormal columns has the letter Q, but we only call it an *orthogonal matrix* when it is square.

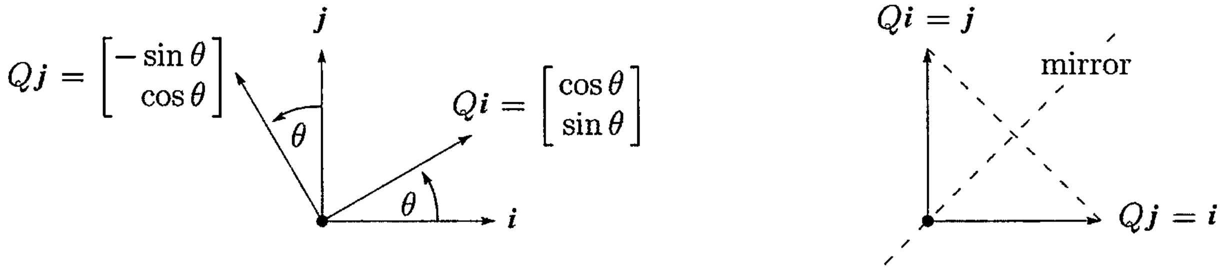


Figure 4.10: Rotation by $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ and reflection across 45° by $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

As examples choose two unit vectors, $\boldsymbol{u} = (1,0)$ and then $\boldsymbol{u} = (1/\sqrt{2}, -1/\sqrt{2})$. Compute $2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ (column times row) and subtract from I to get reflections Q_1 and Q_2:

$$Q_1 = I - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = I - 2\begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Q_1 reflects $(x, 0)$ across the y axis to $(-x, 0)$. Every vector (x, y) goes into its image $(-x, y)$, and the y axis is the mirror. Q_2 is reflection across the 45° line:

$$\textbf{Reflections} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

When (x, y) goes to (y, x), a vector like $(3, 3)$ doesn't move. It is on the mirror line. Figure 4.10b shows the 45° mirror.

Rotations preserve the length of a vector. So do reflections. So do permutations. So does multiplication by any orthogonal matrix—***lengths and angles don't change***.

If Q has orthonormal columns ($Q^{\mathrm{T}}Q = I$), it leaves lengths unchanged:

Same length $\quad \|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ **for every vector** $\boldsymbol{x}$. (3)

Q also preserves dot products: $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$. Just use $Q^{\mathrm{T}}Q = I$!

Proof $\|Q\boldsymbol{x}\|^2$ equals $\|\boldsymbol{x}\|^2$ because $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}Q^{\mathrm{T}}Q\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}I\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. Orthogonal matrices are excellent for computations—numbers can never grow too large when lengths of vectors are fixed. Stable computer codes use Q's as much as possible.

Projections Using Orthogonal Bases: Q Replaces A

This chapter is about projections onto subspaces. We developed the equations for $\widehat{\boldsymbol{x}}$ and $\boldsymbol{p}$ and the matrix P. When the columns of A were a basis for the subspace, all formulas involved $A^{\mathrm{T}}A$. The entries of $A^{\mathrm{T}}A$ are the dot products $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{a}_j$.

Suppose the basis vectors are actually orthonormal. The $\boldsymbol{a}$'s become $\boldsymbol{q}$'s. Then $A^{\mathrm{T}}A$ simplifies to $Q^{\mathrm{T}}Q = I$. Look at the improvements in $\widehat{x}$ and p and P. Instead of $Q^{\mathrm{T}}Q$ we print a blank for the identity matrix:

$$\underline{\qquad}\,\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b} \quad \text{and} \quad \boldsymbol{p} = Q\widehat{x} \quad \text{and} \quad P = Q\,\underline{\qquad}\,Q^{\mathrm{T}}. \tag{4}$$

The least squares solution of $Qx = b$ is $\widehat{x} = Q^{\mathrm{T}}b$. The projection matrix is $P = QQ^{\mathrm{T}}$.

There are no matrices to invert. This is the point of an orthonormal basis. The best $\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$ just has dot products of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ with $\boldsymbol{b}$. We have n 1-dimensional projections! The "coupling matrix" or "correlation matrix" $A^{\mathrm{T}}A$ is now $Q^{\mathrm{T}}Q = I$. There is no coupling. When A is Q, with orthonormal columns, here is $\boldsymbol{p} = Q\widehat{x} = QQ^{\mathrm{T}}\boldsymbol{b}$:

Projection onto q's

$$\boldsymbol{p} = \begin{bmatrix} | & & | \\ \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ \vdots \\ \boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b} \end{bmatrix} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{5}$$

Important case: When Q is square and $m = n$, the subspace is the whole space. Then $Q^{\mathrm{T}} = Q^{-1}$ and $\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$ is the same as $\boldsymbol{x} = Q^{-1}\boldsymbol{b}$. The solution is exact! The projection of $\boldsymbol{b}$ onto the whole space is $\boldsymbol{b}$ itself. In this case $P = QQ^{\mathrm{T}} = I$.

You may think that projection onto the whole space is not worth mentioning. But when $\boldsymbol{p} = \boldsymbol{b}$, our formula assembles $\boldsymbol{b}$ out of its 1-dimensional projections. If $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ is an orthonormal basis for the whole space, so Q is square, then every $\boldsymbol{b} = QQ^{\mathrm{T}}\boldsymbol{b}$ is the sum of its components along the $\boldsymbol{q}$'s:

$$\boldsymbol{b} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \boldsymbol{q}_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{6}$$

That is $QQ^{\mathrm{T}} = I$. It is the foundation of Fourier series and all the great "transforms" of applied mathematics. They break vectors or functions into perpendicular pieces. Then by adding the pieces, the inverse transform puts the function back together.

Example 4 The columns of this orthogonal Q are orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$:

$$Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{has} \quad Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I.$$

The separate projections of $\boldsymbol{b} = (0, 0, 1)$ onto $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$:

$$\boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) = \tfrac{2}{3}\boldsymbol{q}_1 \quad \text{and} \quad \boldsymbol{q}_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b}) = \tfrac{2}{3}\boldsymbol{q}_2 \quad \text{and} \quad \boldsymbol{q}_3(\boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{b}) = -\tfrac{1}{3}\boldsymbol{q}_3.$$

The sum of the first two is the projection of $\boldsymbol{b}$ onto the *plane* of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The sum of all three is the projection of $\boldsymbol{b}$ onto the *whole space*—which is $\boldsymbol{b}$ itself:

Reconstruct
$\boldsymbol{b} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$

$$\tfrac{2}{3}\boldsymbol{q}_1 + \tfrac{2}{3}\boldsymbol{q}_2 - \tfrac{1}{3}\boldsymbol{q}_3 = \tfrac{1}{9}\begin{bmatrix} -2+4-2 \\ 4-2-2 \\ 4+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{b}.$$

The Gram-Schmidt Process

The point of this section is that "orthogonal is good." Projections and least squares always involve $A^{\mathrm{T}}A$. When this matrix becomes $Q^{\mathrm{T}}Q = I$, the inverse is no problem. The one-dimensional projections are uncoupled. The best $\widehat{x}$ is $Q^{\mathrm{T}}\boldsymbol{b}$ (just n separate dot products). For this to be true, we had to say "*If* the vectors are orthonormal". ***Now we find a way to create orthonormal vectors.***

Start with three independent vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We intend to construct three orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. Then (at the end is easiest) we divide $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by their lengths. That produces three orthonormal vectors $\boldsymbol{q}_1 = \boldsymbol{A}/\|\boldsymbol{A}\|$, $\boldsymbol{q}_2 = \boldsymbol{B}/\|\boldsymbol{B}\|$, $\boldsymbol{q}_3 = \boldsymbol{C}/\|\boldsymbol{C}\|$.

Gram-Schmidt Begin by choosing $\boldsymbol{A} = \boldsymbol{a}$. This first direction is accepted. The next direction $\boldsymbol{B}$ must be perpendicular to $\boldsymbol{A}$. ***Start with b and subtract its projection along A.*** This leaves the perpendicular part, which is the orthogonal vector $\boldsymbol{B}$:

First Gram-Schmidt step $$\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A}. \tag{7}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal in Figure 4.11. Take the dot product with $\boldsymbol{A}$ to verify that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = 0$. This vector $\boldsymbol{B}$ is what we have called the error vector $\boldsymbol{e}$, perpendicular to $\boldsymbol{A}$. Notice that $\boldsymbol{B}$ in equation (7) is not zero (otherwise $\boldsymbol{a}$ and $\boldsymbol{b}$ would be dependent). The directions $\boldsymbol{A}$ and $\boldsymbol{B}$ are now set.

The third direction starts with $\boldsymbol{c}$. This is not a combination of $\boldsymbol{A}$ and $\boldsymbol{B}$ (because $\boldsymbol{c}$ is not a combination of $\boldsymbol{a}$ and $\boldsymbol{b}$). But most likely $\boldsymbol{c}$ is not perpendicular to $\boldsymbol{A}$ and $\boldsymbol{B}$. So subtract off its components in those two directions to get $\boldsymbol{C}$:

Next Gram-Schmidt step $$\boldsymbol{C} = \boldsymbol{c} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} - \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}. \tag{8}$$

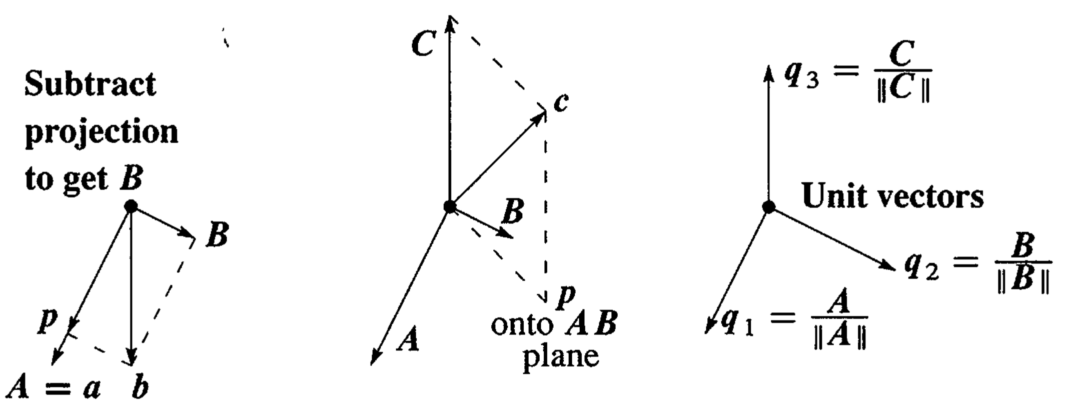


Figure 4.11: First project $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ and find the orthogonal $\boldsymbol{B}$ as $\boldsymbol{b} - \boldsymbol{p}$. Then project $\boldsymbol{c}$ onto the $\boldsymbol{AB}$ plane and find $\boldsymbol{C}$ as $\boldsymbol{c} - \boldsymbol{p}$. Divide by $\|\boldsymbol{A}\|$, $\|\boldsymbol{B}\|$, $\|\boldsymbol{C}\|$.

This is the one and only idea of the Gram-Schmidt process. ***Subtract from every new vector its projections in the directions already set.*** That idea is repeated at every step.[2] If we had a fourth vector $\boldsymbol{d}$, we would subtract three projections onto $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ to get $\boldsymbol{D}$. At the end, *or immediately when each one is found*, divide the orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ by their lengths. The resulting vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$ are orthonormal.

Example 5 Suppose the independent non-orthogonal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

Then $\boldsymbol{A} = \boldsymbol{a}$ has $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = 2$. Subtract from $\boldsymbol{b}$ its projection along $\boldsymbol{A} = (1, -1, 0)$:

First step $$\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} = \boldsymbol{b} - \tfrac{2}{2}\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Check: $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = 0$ as required. Now subtract two projections from $\boldsymbol{c}$ to get $\boldsymbol{C}$:

Next step $$\boldsymbol{C} = \boldsymbol{c} - \frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}}\boldsymbol{A} - \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B} = \boldsymbol{c} - \tfrac{6}{2}\boldsymbol{A} + \tfrac{6}{6}\boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Check: $\boldsymbol{C} = (1, 1, 1)$ is perpendicular to $\boldsymbol{A}$ and $\boldsymbol{B}$. Finally convert $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ to unit vectors (length 1, orthonormal). The lengths of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $\sqrt{2}$ and $\sqrt{6}$ and $\sqrt{3}$. Divide by those lengths, for an orthonormal basis:

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Usually $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ contain fractions. Almost always $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ contain square roots.

The Factorization $A = QR$

We started with a matrix A, whose columns were $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. We ended with a matrix Q, whose columns are $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. How are those matrices related? Since the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are combinations of the $\boldsymbol{q}$'s (and vice versa), there must be a third matrix connecting A to Q. This third matrix is the triangular R in $A = QR$.

The first step was $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\|$ (other vectors not involved). The second step was equation (7), where $\boldsymbol{b}$ is a combination of $\boldsymbol{A}$ and $\boldsymbol{B}$. At that stage $\boldsymbol{C}$ and $\boldsymbol{q}_3$ were not involved. This non-involvement of later vectors is the key point of Gram-Schmidt:

[2]I think Gram had the idea. I don't really know where Schmidt came in.

- The vectors $\boldsymbol{a}$ and $\boldsymbol{A}$ and $\boldsymbol{q}_1$ are all along a single line.
- The vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{q}_1, \boldsymbol{q}_2$ are all in the same plane.
- The vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are in one subspace (dimension 3).

At every step $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are combinations of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k$. Later $\boldsymbol{q}$'s are not involved. The connecting matrix R is ***triangular***, and we have $A = QR$:

$$\begin{bmatrix} & & \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a} & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{c} \\ & \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b} & \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{c} \\ & & \boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{c} \end{bmatrix} \quad \text{or} \quad \boxed{A = QR.} \tag{9}$$

$A = QR$ is Gram-Schmidt in a nutshell. Multiply by Q^{T} to see why $R = Q^{\mathrm{T}}A$.

(Gram-Schmidt) From independent vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, Gram-Schmidt constructs orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. The matrices with these columns satisfy $A = QR$. Then $R = Q^{\mathrm{T}}A$ is ***upper triangular*** because later $\boldsymbol{q}$'s are orthogonal to earlier $\boldsymbol{a}$'s.

Here are the $\boldsymbol{a}$'s and $\boldsymbol{q}$'s from the example. The i, j entry of $R = Q^{\mathrm{T}}A$ is row i of Q^{T} times column j of A. This is the dot product of $\boldsymbol{q}_i$ with $\boldsymbol{a}_j$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ \mathbf{0} & \sqrt{6} & -\sqrt{6} \\ \mathbf{0} & \mathbf{0} & \sqrt{3} \end{bmatrix} = QR.$$

The lengths of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are the numbers $\sqrt{2}$, $\sqrt{6}$, $\sqrt{3}$ on the diagonal of R. Because of the square roots, QR looks less beautiful than LU. Both factorizations are absolutely central to calculations in linear algebra.

Any m by n matrix A with independent columns can be factored into QR. The m by n matrix Q has orthonormal columns, and the square matrix R is upper triangular with positive diagonal. We must not forget why this is useful for least squares: $\boldsymbol{A^{\mathrm{T}}A}$ ***equals*** $\boldsymbol{R^{\mathrm{T}}Q^{\mathrm{T}}QR = R^{\mathrm{T}}R}$. The least squares equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$ simplifies to $Rx = Q^{\mathrm{T}}\boldsymbol{b}$:

$$\textbf{Least squares} \qquad R^{\mathrm{T}}R\widehat{x} = R^{\mathrm{T}}Q^{\mathrm{T}}\boldsymbol{b} \text{ or } R\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b} \text{ or } \boxed{\widehat{x} = R^{-1}Q^{\mathrm{T}}\boldsymbol{b}} \tag{10}$$

Instead of solving $Ax = \boldsymbol{b}$, which is impossible, we solve $R\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$ by back substitution—which is very fast. The real cost is the mn^2 multiplications in the Gram-Schmidt process, which are needed to construct the orthogonal Q and the triangular R.

Below is an informal code. It executes equations (11) and (12), for $k = 1$ then $k = 2$ and eventually $k = n$. The last line of that code normalizes to unit vectors $\boldsymbol{q}_j$:

$$\begin{array}{l}\textbf{Divide by length} \\ q_j \textbf{ = unit vector}\end{array} \qquad r_{jj} = \left(\sum_{i=1}^{m} v_{ij}^2\right)^{1/2} \quad \textbf{and} \quad q_{ij} = \frac{v_{ij}}{r_{jj}} \quad \textbf{for} \quad i = 1, \ldots, m. \tag{11}$$

The important lines subtract from $\boldsymbol{v} = \boldsymbol{a}_j$ its projection onto each $\boldsymbol{q}_i$:

$$r_{kj} = \sum_{i=1}^{m} q_{ik} v_{ij} \qquad \text{and} \qquad v_{ij} = v_{ij} - q_{ik} r_{kj}. \tag{12}$$

Starting from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} = \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ this code will construct $\boldsymbol{q}_1, \boldsymbol{B}, \boldsymbol{q}_2, \boldsymbol{C}, \boldsymbol{q}_3$:

$$\boldsymbol{q}_1 = \boldsymbol{a}_1/\|\boldsymbol{a}_1\| \qquad \boldsymbol{B} = \boldsymbol{a}_2 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_2)\boldsymbol{q}_1 \qquad \boldsymbol{q}_2 = \boldsymbol{B}/\|\boldsymbol{B}\|$$

$$\boldsymbol{C}^* = \boldsymbol{a}_3 - (\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_3)\boldsymbol{q}_1 \quad \boldsymbol{C} = \boldsymbol{C}^* - (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 \quad \boldsymbol{q}_3 = \boldsymbol{C}/\|\boldsymbol{C}\|$$

Equation (12) subtracts off projections as soon as the new vector $\boldsymbol{q}_k$ is found. This change to "subtract one projection at a time" is called ***modified Gram-Schmidt***. That is numerically more stable than equation (8) which subtracts all projections at once.

```
for j = 1:n                     % modified Gram-Schmidt
  v = A(:, j);                  % v begins as column j of A
  for i = 1:j-1                 % columns up to j-1, already settled in Q
    R(i, j) = Q(:, i)'*v;       % compute r_ij = q_i^T a_j which is q_i^T v
    v = v-R(i, j)*Q(:, i);      % subtract the projection (q_i^T a_j)q_i = (q_i^T v)q_i
  end                           % v is now perpendicular to all of q_1, ..., q_{j-1}
  R(j, j) = norm(v);            % diagonal entries of R
  Q(:, j) = v/R(j, j);          % normalize v to be the next unit vector q_j
end
```

To recover column j of A, undo the last step and the middle steps of the code:

$$R(j,j)\boldsymbol{q}_j = (\boldsymbol{v} \textbf{\textit{ minus its projections}}) = (\text{column } j \text{ of } A) - \sum_{i=1}^{j-1} R(i,j)\boldsymbol{q}_i\,. \tag{13}$$

Moving the sum to the far left, this is column j in the multiplication $A = QR$.

Confession Good software like LAPACK, used in good systems like MATLAB and Octave and Python, will not use this Gram-Schmidt code. There is now a better way. "Householder reflections" produce the upper triangular R, one column at a time, exactly as elimination produces the upper triangular U.

Those reflection matrices $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ will be described in Chapter 9 on numerical linear algebra. If A is tridiagonal we can simplify even more to use 2 by 2 rotations. The result is always $A = QR$ and the MATLAB command is $[Q, R] = \text{qr}(A)$. I believe that Gram-Schmidt is still the good process to understand, even if the reflections or rotations lead to a more perfect Q.

■ REVIEW OF THE KEY IDEAS ■

1. If the orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are the columns of Q, then $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ and $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = 1$ translate into $Q^{\mathrm{T}}Q = I$.
2. If Q is square (an ***orthogonal matrix***) then $Q^{\mathrm{T}} = Q^{-1}$: ***transpose = inverse***.
3. The length of $Q\boldsymbol{x}$ equals the length of $\boldsymbol{x}$: $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$.
4. The projection onto the column space spanned by the $\boldsymbol{q}$'s is $P = QQ^{\mathrm{T}}$.
5. If Q is square then $P = I$ and every $\boldsymbol{b} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b})$.
6. Gram-Schmidt produces orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ from independent $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. In matrix form this is the factorization $A = QR =$ (orthogonal Q)(triangular R).

■ WORKED EXAMPLES ■

4.4 A Add two more columns with all entries 1 or -1, so the columns of this 4 by 4 "Hadamard matrix" are orthogonal. How do you turn H_4 into an *orthogonal matrix* Q?

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 1 & 1 & x & x \\ 1 & -1 & x & x \\ 1 & 1 & x & x \\ 1 & -1 & x & x \end{bmatrix} \quad \text{and} \quad Q_4 = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

The block matrix $H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$ is the next Hadamard matrix with 1's and -1's. What is the product $H_8^{\mathrm{T}} H_8$?

The projection of $\boldsymbol{b} = (6, 0, 0, 2)$ onto the first column of H_4 is $\boldsymbol{p}_1 = (2, 2, 2, 2)$. The projection onto the second column is $\boldsymbol{p}_2 = (1, -1, 1, -1)$. What is the projection $\boldsymbol{p}_{1,2}$ of $\boldsymbol{b}$ onto the 2-dimensional space spanned by the first two columns?

Solution H_4 can be built from H_2 just as H_8 is built from H_4:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ has orthogonal columns.}$$

Then $Q = H/2$ has orthonormal columns. Dividing by 2 gives unit vectors in Q. Orthogonality for 5 by 5 is impossible because the dot product of columns would have five 1's and/or -1's and could not add to zero. H_8 has orthogonal columns of length $\sqrt{8}$.

$$H_8^{\mathrm{T}} H_8 = \begin{bmatrix} H^{\mathrm{T}} & H^{\mathrm{T}} \\ H^{\mathrm{T}} & -H^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \begin{bmatrix} 2H^{\mathrm{T}}H & 0 \\ 0 & 2H^{\mathrm{T}}H \end{bmatrix} = \begin{bmatrix} 8I & 0 \\ 0 & 8I \end{bmatrix}. \; Q_8 = \frac{H_8}{\sqrt{8}}$$

Key point of orthogonal columns: We can project $(6,0,0,2)$ onto $(1,1,1,1)$ and $(1,-1,1,-1)$ and **add**. There is no $A^{\mathrm{T}}A$ matrix to invert:

Add p's Projection $\boldsymbol{p}_{1,2} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = (2,2,2,2) + (1,-1,1,-1) = (3,1,3,1)$.

Check that columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of H are perpendicular to the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}_1 - \boldsymbol{p}_2$:

$$\boldsymbol{e} = \boldsymbol{b} - \frac{\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1}\boldsymbol{a}_1 - \frac{\boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{a}_2}\boldsymbol{a}_2 \quad \text{and} \quad \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{e} = \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} - \frac{\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1}\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1 = 0 \quad \text{and also} \quad \boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{e} = 0.$$

So $\boldsymbol{p}_1 + \boldsymbol{p}_2$ is in the space of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and *its error $\boldsymbol{e}$ is perpendicular to that space.*

The Gram-Schmidt process on those orthogonal columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ would not change their directions. It would only divide by their lengths. *But if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not orthogonal, the projection $\boldsymbol{p}_{1,2}$ is not generally $\boldsymbol{p}_1 + \boldsymbol{p}_2$.* For example, if $\boldsymbol{b}$ is the same as $\boldsymbol{a}_1$, then $\boldsymbol{p}_1 = \boldsymbol{b}$ and $\boldsymbol{p}_{1,2} = \boldsymbol{b}$ but $\boldsymbol{p}_2 \neq \boldsymbol{0}$.

Problem Set 4.4

Problems 1–12 are about orthogonal vectors and orthogonal matrices.

1 Are these pairs of vectors orthonormal or only orthogonal or only independent?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} .6 \\ .8 \end{bmatrix}$ and $\begin{bmatrix} .4 \\ -.3 \end{bmatrix}$ (c) $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

Change the second vector when necessary to produce orthonormal vectors.

2 The vectors $(2,2,-1)$ and $(-1,2,2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Put those into the columns of Q and multiply $Q^{\mathrm{T}}Q$ and QQ^{T}.

3 (a) If A has three orthogonal columns each of length 4, what is $A^{\mathrm{T}}A$?

(b) If A has three orthogonal columns of lengths $1,2,3$, what is $A^{\mathrm{T}}A$?

4 Give an example of each of the following:

(a) A matrix Q that has orthonormal columns but $QQ^{\mathrm{T}} \neq I$.

(b) Two orthogonal vectors that are not linearly independent.

(c) An orthonormal basis for $\mathbf{R}^3$, including the vector $\boldsymbol{q}_1 = (1,1,1)/\sqrt{3}$.

5 Find two orthogonal vectors in the plane $x + y + 2z = 0$. Make them orthonormal.

6 If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix. (Use $Q^{\mathrm{T}}Q = I$.)

7 If Q has orthonormal columns, what is the least squares solution $\widehat{x}$ to $Qx = b$?

8 If q_1 and q_2 are orthonormal vectors in $\mathbf{R}^5$, what combination ____ $q_1 +$ ____ q_2 is closest to a given vector b?

9 (a) Compute $P = QQ^{\mathrm{T}}$ when $q_1 = (.8, .6, 0)$ and $q_2 = (-.6, .8, 0)$. Verify that $P^2 = P$.

(b) Prove that always $(QQ^{\mathrm{T}})^2 = QQ^{\mathrm{T}}$ by using $Q^{\mathrm{T}}Q = I$. Then $P = QQ^{\mathrm{T}}$ is the projection matrix onto the column space of Q.

10 Orthonormal vectors are automatically linearly independent.

(a) Vector proof: When $c_1q_1+c_2q_2+c_3q_3 = \mathbf{0}$, what dot product leads to $c_1 = 0$? Similarly $c_2 = 0$ and $c_3 = 0$. Thus the q's are independent.

(b) Matrix proof: Show that $Qx = \mathbf{0}$ leads to $x = \mathbf{0}$. Since Q may be rectangular, you can use Q^{T} but not Q^{-1}.

11 (a) Gram-Schmidt: Find orthonormal vectors q_1 and q_2 in the plane spanned by $a = (1, 3, 4, 5, 7)$ and $b = (-6, 6, 8, 0, 8)$.

(b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

12 If a_1, a_2, a_3 is a basis for $\mathbf{R}^3$, any vector b can be written as

$$b = x_1a_1 + x_2a_2 + x_3a_3 \qquad \text{or} \qquad \begin{bmatrix} & & \\ a_1 & a_2 & a_3 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b.$$

(a) Suppose the a's are orthonormal. Show that $x_1 = a_1^{\mathrm{T}}b$.

(b) Suppose the a's are orthogonal. Show that $x_1 = a_1^{\mathrm{T}}b / a_1^{\mathrm{T}}a_1$.

(c) If the a's are independent, x_1 is the first component of ____ times b.

Problems 13–25 are about the Gram-Schmidt process and $A = QR$.

13 What multiple of $a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $b = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result B orthogonal to a? Sketch a figure to show a, b, and B.

14 Complete the Gram-Schmidt process in Problem 13 by computing $q_1 = a/\|a\|$ and $q_2 = B/\|B\|$ and factoring into QR:

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \|a\| & ? \\ 0 & \|B\| \end{bmatrix}.$$

15 (a) Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

(b) Which of the four fundamental subspaces contains q_3?

(c) Solve $Ax = (1, 2, 7)$ by least squares.

16 What multiple of $a = (4, 5, 2, 2)$ is closest to $b = (1, 2, 0, 0)$? Find orthonormal vectors q_1 and q_2 in the plane of a and b.

17 Find the projection of b onto the line through a:

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad p = ? \quad \text{and} \quad e = b - p = ?$$

Compute the orthonormal vectors $q_1 = a/\|a\|$ and $q_2 = e/\|e\|$.

18 (Recommended) Find orthogonal vectors A, B, C by Gram-Schmidt from a, b, c:

$$a = (1, -1, 0, 0) \qquad b = (0, 1, -1, 0) \qquad c = (0, 0, 1, -1).$$

A, B, C and a, b, c are bases for the vectors perpendicular to $d = (1, 1, 1, 1)$.

19 If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}R =$ _____ triangular times _____ triangular. *Gram-Schmidt on A corresponds to elimination on $A^{\mathrm{T}}A$.* The pivots for $A^{\mathrm{T}}A$ must be the squares of diagonal entries of R. Find Q and R by Gram-Schmidt for this A:

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}}A = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & \\ & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

20 True or false (give an example in either case):

(a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

(b) If Q (3 by 2) has orthonormal columns then $\|Qx\|$ always equals $\|x\|$.

21 Find an orthonormal basis for the column space of A:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Then compute the projection of b onto that column space.

22 Find orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by Gram-Schmidt from

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

23 Find $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

24 (a) Find a basis for the subspace $\boldsymbol{S}$ in $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) Find a basis for the orthogonal complement $\boldsymbol{S}^{\perp}$.

(c) Find $\boldsymbol{b}_1$ in $\boldsymbol{S}$ and $\boldsymbol{b}_2$ in $\boldsymbol{S}^{\perp}$ so that $\boldsymbol{b}_1 + \boldsymbol{b}_2 = \boldsymbol{b} = (1, 1, 1, 1)$.

25 If $ad - bc > 0$, the entries in $A = QR$ are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} a & -c \\ c & a \end{bmatrix}}{\sqrt{a^2 + c^2}} \frac{\begin{bmatrix} a^2 + c^2 & ab + cd \\ 0 & ad - bc \end{bmatrix}}{\sqrt{a^2 + c^2}}.$$

Write $A = QR$ when $a, b, c, d = 2, 1, 1, 1$ and also $1, 1, 1, 1$. Which entry of R becomes zero when the columns are dependent and Gram-Schmidt breaks down?

Problems 26–29 use the QR code in equations (11–12). It executes Gram-Schmidt.

26 Show why $\boldsymbol{C}$ (found via $\boldsymbol{C}^*$ in the steps after (12)) is equal to $\boldsymbol{C}$ in equation (8).

27 Equation (8) subtracts from $\boldsymbol{c}$ its components along $\boldsymbol{A}$ and $\boldsymbol{B}$. Why not subtract the components along $\boldsymbol{a}$ and along $\boldsymbol{b}$?

28 Where are the mn^2 multiplications in equations (11) and (12)?

29 Apply the MATLAB qr code to $\boldsymbol{a} = (2, 2, -1)$, $\boldsymbol{b} = (0, -3, 3)$, $\boldsymbol{c} = (1, 0, 0)$. What are the $\boldsymbol{q}$'s?

Problems 30–35 involve orthogonal matrices that are special.

30 The first four ***wavelets*** are in the columns of this wavelet matrix W:

$$W = \frac{1}{2} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}.$$

What is special about the columns? Find the inverse wavelet transform W^{-1}.

31 (a) Choose c so that Q is an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Project $\boldsymbol{b} = (1, 1, 1, 1)$ onto the first column. Then project $\boldsymbol{b}$ onto the plane of the first two columns.

32 If $\boldsymbol{u}$ is a unit vector, then $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a reflection matrix (Example 3). Find Q_1 from $\boldsymbol{u} = (0, 1)$ and Q_2 from $\boldsymbol{u} = (0, \sqrt{2}/2, \sqrt{2}/2)$. Draw the reflections when Q_1 and Q_2 multiply the vectors $(1, 2)$ and $(1, 1, 1)$.

33 Find all matrices that are both orthogonal and lower triangular.

34 $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is a reflection matrix when $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$. Two reflections give $Q^2 = I$.

(a) Show that $Q\boldsymbol{u} = -\boldsymbol{u}$. The mirror is perpendicular to $\boldsymbol{u}$.

(b) Find $Q\boldsymbol{v}$ when $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$. The mirror contains $\boldsymbol{v}$. It reflects to itself.

Challenge Problems

35 (MATLAB) Factor $[\,Q, R\,] = \mathbf{qr}(A)$ for $A = \mathbf{eye}(4) - \mathbf{diag}([1 \;\; 1 \;\; 1], -1)$. You are orthogonalizing the columns $(1, -1, 0, 0)$ and $(0, 1, -1, 0)$ and $(0, 0, 1, -1)$ and $(0, 0, 0, 1)$ of A. Can you scale the orthogonal columns of Q to get nice integer components?

36 If A is m by n with rank n, $\mathbf{qr}(A)$ produces a *square* Q and zeros below R:

The factors from MATLAB are $(m \text{ by } m)(m \text{ by } n)$ $\quad A = [Q_1 \;\; Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}.$

The n columns of Q_1 are an orthonormal basis for which fundamental subspace? The $m - n$ columns of Q_2 are an orthonormal basis for which fundamental subspace?

37 We know that $P = QQ^{\mathrm{T}}$ is the projection onto the column space of $Q(m \text{ by } n)$. Now add another column $\boldsymbol{a}$ to produce $A = [Q \;\; \boldsymbol{a}]$. What is the new orthonormal vector $\boldsymbol{q}$ from Gram-Schmidt: start with $\boldsymbol{a}$, subtract _____, divide by _____.

Chapter 5

Determinants

5.1 The Properties of Determinants

The determinant of a square matrix is a single number. That number contains an amazing amount of information about the matrix. It tells immediately whether the matrix is invertible. ***The determinant is zero when the matrix has no inverse.*** When A is invertible, the determinant of A^{-1} is $1/(\det A)$. If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$. In fact the determinant leads to a formula for every entry in A^{-1}.

This is one use for determinants—to find formulas for inverse matrices and pivots and solutions $A^{-1}\boldsymbol{b}$. For a large matrix we seldom use those formulas, because elimination is faster. For a 2 by 2 matrix with entries a, b, c, d, its determinant $ad - bc$ shows how A^{-1} changes as A changes:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

Multiply those matrices to get I. When the determinant is $ad - bc = 0$, we are asked to divide by zero and we can't—then A has no inverse. (The rows are parallel when $a/c = b/d$. This gives $ad = bc$ and $\det A = 0$). Dependent rows always lead to $\det A = 0$.

The determinant is also connected to the pivots. For a 2 by 2 matrix the pivots are a and $d - (c/a)b$. ***The product of the pivots is the determinant:***

$$\textbf{Product of pivots} \qquad a\left(d - \frac{c}{a}b\right) = ad - bc \quad \textbf{which is} \quad \det A.$$

After a row exchange the pivots change to c and $b - (a/c)d$. Those new pivots multiply to give $bc - ad$. The row exchange to $\left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right]$ reversed the sign of the determinant.

Looking ahead The determinant of an n by n matrix can be found in three ways:

1 Multiply the n pivots (times 1 or -1)	This is the **pivot formula.**
2 Add up $n!$ terms (times 1 or -1)	This is the **"big" formula.**
3 Combine n smaller determinants (times 1 or -1)	This is the **cofactor formula.**

You see that *plus or minus signs*—the decisions between 1 and -1—play a big part in determinants. That comes from the following rule for n by n matrices:

The determinant changes sign when two rows* (or two columns) *are exchanged.

The identity matrix has determinant $+1$. Exchange two rows and $\det P = -1$. Exchange two more rows and the new permutation has $\det P = +1$. Half of all permutations are *even* ($\det P = 1$) and half are *odd* ($\det P = -1$). Starting from I, half of the P's involve an even number of exchanges and half require an odd number. In the 2 by 2 case, ad has a plus sign and bc has minus—coming from the row exchange:

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \quad \text{and} \quad \det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

The other essential rule is linearity—but a warning comes first. Linearity does not mean that $\det(A+B) = \det A + \det B$. ***This is absolutely false.*** That kind of linearity is not even true when $A = I$ and $B = I$. The false rule would say that $\det(I + I) = 1 + 1 = 2$. The true rule is $\det 2I = 2^n$. Determinants are multiplied by 2^n (not just by 2) when matrices are multiplied by 2.

We don't intend to define the determinant by its formulas. It is better to start with its properties—*sign reversal and linearity*. The properties are simple (Section 5.1). They prepare for the formulas (Section 5.2). Then come the applications, including these three:

(1) Determinants give A^{-1} and $A^{-1}\boldsymbol{b}$ (this formula is called **Cramer's Rule**).

(2) When the edges of a box are the rows of A, the **volume** is $|\det A|$.

(3) For n special numbers λ, called **eigenvalues**, the determinants of $A - \lambda I$ is zero. This is a truly important application and it fills Chapter 6.

The Properties of the Determinant

Determinants have three basic properties (rules 1, 2, 3). By using those rules we can compute the determinant of any square matrix A. ***This number is written in two ways,*** $\det A$ ***and*** $|A|$. Notice: Brackets for the matrix, straight bars for its determinant. When A is a 2 by 2 matrix, the three properties lead to the answer we expect:

$$\text{The determinant of} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The last rules are $\det(AB) = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$. We will check all rules with the 2 by 2 formula, but do not forget: The rules apply to any n by n matrix. We will show how rules $4 - 10$ always follow from $1 - 3$.

Rule 1 (the easiest) matches $\det I = 1$ with the volume $= 1$ for a unit cube.

1 ***The determinant of the n by n identity matrix is*** **1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1.$$

2 ***The determinant changes sign when two rows are exchanged*** (sign reversal):

$$\text{Check:} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{(both sides equal } bc - ad\text{).}$$

Because of this rule, we can find $\det P$ for any permutation matrix. Just exchange rows of I until you reach P. Then $\det P = +1$ for an ***even*** number of row exchanges and $\det P = -1$ for an ***odd*** number.

The third rule has to make the big jump to the determinants of all matrices.

3 ***The determinant is a linear function of each row separately*** (all other rows stay fixed). If the first row is multiplied by t, the determinant is multiplied by t. If first rows are added, determinants are added. This rule only applies when the other rows do not change! Notice how c and d stay the same:

multiply row 1 by any number t $$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

add row 1 of A to row 1 of A': $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

In the first case, both sides are $tad - tbc$. Then t factors out. In the second case, both sides are $ad + a'd - bc - b'c$. These rules still apply when A is n by n, and the last $n-1$ rows don't change. May we emphasize rule 3 with numbers:

$$\begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}.$$

By itself, rule 3 does not say what those determinants are (the first one is 4).

Combining multiplication and addition, we get any linear combination in one row (the other rows must stay the same). Any row can be the one that changes, since rule 2 for row exchanges can put it up into the first row and back again.

This rule does not mean that $\det 2I = 2 \det I$. To obtain $2I$ we have to multiply *both* rows by 2, and the factor 2 comes out both times:

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4 \qquad \text{and} \qquad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2.$$

This is just like area and volume. Expand a rectangle by 2 and its area increases by 4. Expand an n-dimensional box by t and its volume increases by t^n. The connection is no accident—we will see how *determinants equal volumes*.

Pay special attention to rules 1–3. They completely determine the number $\det A$. We could stop here to find a formula for n by n determinants. (a little complicated) We prefer to go gradually, with other properties that follow directly from the first three. These extra rules $4 - 10$ make determinants much easier to work with.

4 ***If two rows of A are equal, then* $\det A = 0$.**

Equal rows Check 2 by 2 : $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$

Rule 4 follows from rule 2. (Remember we must use the rules and not the 2 by 2 formula.) *Exchange the two equal rows.* The determinant D is supposed to change sign. But also D has to stay the same, because the matrix is not changed. The only number with $-D = D$ is $D = 0$—this must be the determinant. (Note: In Boolean algebra the reasoning fails, because $-1 = 1$. Then D is defined by rules $1, 3, 4$.)

A matrix with two equal rows has no inverse. Rule 4 makes $\det A = 0$. But matrices can be singular and determinants can be zero without having equal rows! Rule 5 will be the key. We can do row operations without changing $\det A$.

5 ***Subtracting a multiple of one row from another row leaves* $\det A$ *unchanged.***

ℓ times row 1 from row 2 $\begin{vmatrix} a & b \\ c - \ell a & d - \ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$

Rule 3 (linearity) splits the left side into the right side plus another term $-\ell \begin{vmatrix} a & b \\ a & b \end{vmatrix}$. This extra term is zero by rule 4. Therefore rule 5 is correct (not just 2 by 2).

Conclusion *The determinant is not changed by the usual elimination steps from A to U.* Thus $\det A$ *equals* $\det U$. If we can find determinants of triangular matrices U, we can find determinants of all matrices A. Every row exchange reverses the sign, so always $\det A = \pm \det U$. Rule 5 has narrowed the problem to triangular matrices.

6 ***A matrix with a row of zeros has* $\det A = 0$.**

Row of zeros $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$ and $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$

For an easy proof, add some other row to the zero row. The determinant is not changed (rule 5). But the matrix now has two equal rows. So $\det A = 0$ by rule 4.

7 ***If A is triangular then* $\det A = a_{11}a_{22}\cdots a_{nn}$ *= product of diagonal entries.***

Triangular $\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad$ and also $\begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$

Suppose all diagonal entries of A are nonzero. Eliminate the off-diagonal entries by the usual steps. (If A is lower triangular, subtract multiples of each row from lower rows. If A

is upper triangular, subtract from higher rows.) By rule 5 the determinant is not changed—and now the matrix is diagonal:

$$\textbf{Diagonal matrix} \qquad \det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Factor a_{11} from the first row by rule 3. Then factor a_{22} from the second row. Eventually factor a_{nn} from the last row. The determinant is a_{11} times a_{22} times $\cdots$ times a_{nn} times $\det I$. Then rule 1 (used at last!) is $\det I = 1$.

What if a diagonal entry a_{ii} is zero? Then the triangular A is singular. Elimination produces a *zero row*. By rule 5 the determinant is unchanged, and by rule 6 a zero row means $\det A = 0$. Triangular matrices have easy determinants.

8 ***If A is singular then* $\det A = 0$. *If A is invertible then* $\det A \neq 0$.**

$$\textbf{Singular} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is singular if and only if } \quad ad - bc = 0.$$

Proof Elimination goes from A to U. If A is singular then U has a zero row. The rules give $\det A = \det U = 0$. If A is invertible then U has the pivots along its diagonal. The product of nonzero pivots (using rule 7) gives a nonzero determinant:

$$\textbf{Multiply pivots} \qquad \det A = \pm \det U = \pm \text{ (product of the pivots).} \qquad (2)$$

The pivots of a 2 by 2 matrix (if $a \neq 0$) are a and $d - (bc/a)$:

$$\textbf{The determinant is} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (bc/a) \end{vmatrix} = ad - bc.$$

This is the first formula for the determinant. MATLAB uses it to find $\det A$ from the pivots. The sign in $\pm \det u$ depends on whether the number of row exchanges is even or odd. In other words, $+1$ or -1 is the determinant of the permutation matrix P that exchanges rows. With no row exchanges, the number zero is even and $P = I$ and $\det A = \det U =$ *product of pivots*. Always $\det L = 1$, because L is triangular with 1's on the diagonal. What we have is this:

$$\text{If} \quad PA = LU \quad \text{then} \quad \det P \ \det A = \det L \ \det U. \qquad (3)$$

Again, $\det P = \pm 1$ and $\det A = \pm \det U$. Equation (3) is our first case of rule 9.

9 ***The determinant of AB is* $\det A$ *times* $\det B$: $|AB| = |A|\,|B|$.**

$$\textbf{Product rule} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}.$$

When the matrix B is A^{-1}, *this rule says that the determinant of* A^{-1} is $\mathbf{1/\det A}$:

$$\textbf{A times } A^{-1} \qquad \boldsymbol{AA^{-1} = I} \quad \textbf{so} \quad (\det A)(\det A^{-1}) = \det I = 1.$$

This product rule is the most intricate so far. Even the 2 by 2 case needs some algebra:

$$|A|\,|B| = (ad - bc)(ps - qr) = (ap + br)(cq + ds) - (aq + bs)(cp + dr) = |AB|.$$

For the n by n case, here is a snappy proof that $|AB| = |A|\,|B|$. When $|B|$ is not zero, consider the ratio $D(A) = |AB|/|B|$. *Check that this ratio has properties* 1,2,3. Then $D(A)$ has to be the determinant and we have $|A| = |AB|/|B|$: good.

Property 1 *(Determinant of I)* If $A = I$ then the ratio becomes $|B|/|B| = 1$.

Property 2 *(Sign reversal)* When two rows of A are exchanged, so are the same two rows of AB. Therefore $|AB|$ changes sign and so does the ratio $|AB|/|B|$.

Property 3 *(Linearity)* When row 1 of A is multiplied by t, so is row 1 of AB. This multiplies $|AB|$ by t and multiplies the ratio by t—as desired.
Add row 1 of A to row 1 of A'. Then row 1 of AB adds to row 1 of $A'B$. By rule 3, determinants add. After dividing by $|B|$, the ratios add—as desired.

Conclusion This ratio $|AB|/|B|$ has the same three properties that define $|A|$. Therefore it equals $|A|$. This proves the product rule $|AB| = |A|\,|B|$. The case $|B| = 0$ is separate and easy, because AB is singular when B is singular. Then $|AB| = |A|\,|B|$ is $0 = 0$.

10 ***The transpose A^{T} has the same determinant as A.***

$$\textbf{Transpose} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \quad \text{since both sides equal} \quad ad - bc.$$

The equation $|A^{\mathrm{T}}| = |A|$ becomes $0 = 0$ when A is singular (we know that A^{T} is also singular). Otherwise A has the usual factorization $PA = LU$. Transposing both sides gives $A^{\mathrm{T}}P^{\mathrm{T}} = U^{\mathrm{T}}L^{\mathrm{T}}$. The proof of $|A| = |A^{\mathrm{T}}|$ comes by using rule 9 for products:

$$\text{Compare} \quad \det P \det A = \det L \det U \quad \text{with} \quad \det A^{\mathrm{T}} \det P^{\mathrm{T}} = \det U^{\mathrm{T}} \det L^{\mathrm{T}}.$$

First, $\det L = \det L^{\mathrm{T}} = 1$ (both have 1's on the diagonal). Second, $\det U = \det U^{\mathrm{T}}$ (those triangular matrices have the same diagonal). Third, $\det P = \det P^{\mathrm{T}}$ (permutations have $P^{\mathrm{T}}P = I$, so $|P^{\mathrm{T}}|\,|P| = 1$ by rule 9; thus $|P|$ and $|P^{\mathrm{T}}|$ both equal 1 or both equal -1). So L, U, P have the same determinants as $L^{\mathrm{T}}, U^{\mathrm{T}}, P^{\mathrm{T}}$ and this leaves $\det A = \det A^{\mathrm{T}}$.

Important comment on columns Every rule for the rows can apply to the columns (just by transposing, since $|A| = |A^{\mathrm{T}}|$). The determinant changes sign when two columns are exchanged. *A zero column or two equal columns will make the determinant zero.* If a column is multiplied by t, so is the determinant. The determinant is a linear function of each column separately.

It is time to stop. The list of properties is long enough. Next we find and use an explicit formula for the determinant.

■ REVIEW OF THE KEY IDEAS ■

1. The determinant is defined by $\det I = 1$, sign reversal, and linearity in each row.
2. After elimination $\det A$ is $\pm$ (product of the pivots).
3. The determinant is zero exactly when A is not invertible.
4. Two remarkable properties are $\det AB = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$.

■ WORKED EXAMPLES ■

5.1 A Apply these operations to A and find the determinants of M_1, M_2, M_3, M_4:
In M_1, multiplying each a_{ij} by $(-1)^{i+j}$ gives a checkerboard sign pattern.
In M_2, rows $1, 2, 3$ of A are *subtracted* from rows $2, 3, 1$.
In M_3, rows $1, 2, 3$ of A are *added* to rows $2, 3, 1$.
How are the determinants of M_1, M_2, M_3 related to the determinant of A?

$$\begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - \text{row } 1 \\ \text{row } 3 - \text{row } 2 \end{bmatrix} \quad \begin{bmatrix} \text{row } 1 + \text{row } 3 \\ \text{row } 2 + \text{row } 1 \\ \text{row } 3 + \text{row } 2 \end{bmatrix}$$

Solution The three determinants are $\det A$, 0, and $2 \det A$. Here are reasons:

$$M_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \text{so } \det M_1 = (-1)(\det A)(-1).$$

M_2 is singular because its rows add to the zero row. Its determinant is zero.
M_3 can be split into *eight matrices* by Rule 3 (linearity in each row seperately):

$$\begin{vmatrix} \text{row } 1 + \text{row } 3 \\ \text{row } 2 + \text{row } 1 \\ \text{row } 3 + \text{row } 3 \end{vmatrix} = \begin{vmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} + \begin{vmatrix} \text{row } 3 \\ \text{row } 2 \\ \text{row } 3 \end{vmatrix} + \begin{vmatrix} \text{row } 1 \\ \text{row } 1 \\ \text{row } 3 \end{vmatrix} + \cdots + \begin{vmatrix} \text{row } 3 \\ \text{row } 1 \\ \text{row } 2 \end{vmatrix}.$$

All but the first and last have repeated rows and zero determinant. The first is A and the last has *two* row exchanges. So $\det M_3 = \det A + \det A$. (Try $A = I$.)

5.1 B Explain how to reach this determinant by row operations:

$$\det\begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} = a^2(3-a). \tag{4}$$

Solution Subtract row 3 from row 1 and then from row 2. This leaves

$$\det\begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}.$$

Now add column 1 to column 3, and also column 2 to column 3. This leaves a lower triangular matrix with $-a, -a, 3-a$ on the diagonal: $\det = (-a)(-a)(3-a)$.

The determinant is zero if $a = 0$ or $a = 3$. For $a = 0$ we have the *all-ones matrix*—certainly singular. For $a = 3$, each row adds to zero - again singular. Those numbers 0 and 3 are the eigenvalues of the all-ones matrix. This example is revealing and important, leading toward Chapter 6.

Problem Set 5.1

Questions 1–12 are about the rules for determinants.

1 If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

2 If a 3 by 3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

(b) The determinant of ABC is $|A|\,|B|\,|C|$.

(c) The determinant of $4A$ is $4|A|$.

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

4 Which row exchanges show that these "reverse identity matrices" J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \quad \text{but} \quad \det\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1.$$

5 For $n = 5, 6, 7$, count the row exchanges to permute the reverse identity J_n to the identity matrix I_n. Propose a rule for every size n and predict whether J_{101} has determinant $+1$ or -1.

6 Show how Rule 6 (determinant $= 0$ if a row is all zero) comes from Rule 3.

7 Find the determinants of rotations and reflections:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1-2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{bmatrix}.$$

8 Prove that every orthogonal matrix ($Q^{\mathrm{T}}Q = I$) has determinant 1 or -1.

(a) Use the product rule $|AB| = |A|\,|B|$ and the transpose rule $|Q| = |Q^{\mathrm{T}}|$.

(b) Use only the product rule. If $|\det Q| > 1$ then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n?

9 Do these matrices have determinant 0, 1, 2, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

10 If the entries in every row of A add to zero, solve $Ax = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

11 Suppose that $CD = -DC$ and find the flaw in this reasoning: Taking determinants gives $|C|\,|D| = -|D|\,|C|$. Therefore $|C| = 0$ or $|D| = 0$. One or both of the matrices must be singular. (That is not true.)

12 The inverse of a 2 by 2 matrix seems to have determinant $= 1$:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

Questions 13–27 use the rules to compute specific determinants.

13 Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

14 By applying row operations to produce an upper triangular U, compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

15 Use row operations to simplify and compute these determinants:

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \quad\text{and}\quad \det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}.$$

16 Find the determinants of a rank one matrix and a skew-symmetric matrix:

$$A = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}\begin{bmatrix}1 & -4 & 5\end{bmatrix} \quad\text{and}\quad K = \begin{bmatrix}0 & 1 & 3\\ -1 & 0 & 4\\ -3 & -4 & 0\end{bmatrix}.$$

17 A skew-symmetric matrix has $K^{\mathrm{T}} = -K$. Insert a, b, c for $1, 3, 4$ in Question 16 and show that $|K| = 0$. Write down a 4 by 4 example with $|K| = 1$.

18 Use row operations to show that the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix} = (b-a)(c-a)(c-b).$$

19 Find the determinants of U and U^{-1} and U^2:

$$U = \begin{bmatrix}1 & 4 & 6\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix} \quad\text{and}\quad U = \begin{bmatrix}a & b\\ 0 & d\end{bmatrix}.$$

20 Suppose you do two row operations at once, going from

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad\text{to}\quad \begin{bmatrix}a-Lc & b-Ld\\ c-la & d-lb\end{bmatrix}.$$

Find the second determinant. Does it equal $ad - bc$?

21 *Row exchange*: Add row 1 of A to row 2, then subtract row 2 from row 1. Then add row 1 to row 2 and multiply row 1 by -1 to reach B. Which rules show

$$\det B = \begin{vmatrix}c & d\\ a & b\end{vmatrix} \quad\text{equals}\quad -\det A = -\begin{vmatrix}a & b\\ c & d\end{vmatrix}?$$

Those rules could replace Rule 2 in the definition of the determinant.

22 From $ad - bc$, find the determinants of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad\text{and}\quad A^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \quad\text{and}\quad A - \lambda I = \begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}.$$

Which two numbers λ lead to $\det(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those numbers λ—it should not be invertible.

23 From $A = \left[\begin{smallmatrix} 4 & 1 \\ 2 & 3 \end{smallmatrix}\right]$ find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $\det(A - \lambda I) = 0$?

24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

25 If the i, j entry of A is i times j, show that $\det A = 0$. (Exception when $A = [\,1\,]$.)

26 If the i, j entry of A is $i + j$, show that $\det A = 0$. (Exception when $n = 1$ or 2.)

27 Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.

(b) The determinant of A is always the product of its pivots.

(c) The determinant of $A - B$ equals $\det A - \det B$.

(d) AB and BA have the same determinant.

29 What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \qquad \text{so} \qquad |P| = |A| \frac{1}{|A^{\mathrm{T}}||A|} |A^{\mathrm{T}}| = 1.$$

30 (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1}!

$$f(a,b,c,d) = \ln(ad - bc) \qquad \text{leads to} \qquad \begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = A^{-1}.$$

31 (MATLAB) The Hilbert matrix **hilb**(n) has i, j entry equal to $1/(i + j - 1)$. Print the determinants of **hilb**(1), **hilb**$(2), \ldots,$ **hilb**(10). Hilbert matrices are hard to work with! What are the pivots of **hilb** (5)?

32 (MATLAB) What is a typical determinant (experimentally) of **rand**(n) and **randn**(n) for $n = 50, 100, 200, 400$? (And what does "Inf" mean in MATLAB?)

33 (MATLAB) Find the largest determinant of a 6 by 6 matrix of 1's and -1's.

34 If you know that $\det A = 6$, what is the determinant of B?

$$\text{From } \det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \text{ find } \det B = \begin{vmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{vmatrix}.$$

5.2 Permutations and Cofactors

A computer finds the determinant from the pivots. This section explains two other ways to do it. There is a "big formula" using all $n!$ permutations. There is a "cofactor formula" using determinants of size $n-1$. The best example is my favorite 4 by 4 matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad \det A = 5.$$

We can find this determinant in all three ways: ***pivots, big formula, cofactors.***

1. The product of the pivots is $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4}$. Cancellation produces 5.
2. The "big formula" in equation (8) has $4! = 24$ terms. Only five terms are nonzero:

$$\det A = 16 - 4 - 4 - 4 + 1 = 5.$$

   The 16 comes from $2 \cdot 2 \cdot 2 \cdot 2$ on the diagonal of A. Where do -4 and $+1$ come from? When you can find those five terms, you have understood formula (8).

3. The numbers $2, -1, 0, 0$ in the first row multiply their cofactors $4, 3, 2, 1$ from the other rows. That gives $2 \cdot 4 - 1 \cdot 3 = 5$. Those cofactors are 3 by 3 determinants. Cofactors use the rows and columns that are *not* used by the entry in the first row. ***Every term in a determinant uses each row and column once!***

The Pivot Formula

Elimination leaves the pivots $d_1, \ldots, d_n$ on the diagonal of the upper triangular U. If no row exchanges are involved, ***multiply those pivots*** to find the determinant:

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n). \tag{1}$$

This formula for $\det A$ appeared in the previous section, with the further possibility of row exchanges. The permutation matrix in $PA = LU$ has determinant -1 or $+1$. This factor $\det P = \pm 1$ enters the determinant of A:

$$(\det P)(\det A) = (\det L)(\det U) \quad \text{gives} \quad \det A = \pm(d_1 d_2 \cdots d_n). \tag{2}$$

When A has fewer than n pivots, $\det A = 0$ by Rule 8. The matrix is singular.

Example 1 A row exchange produces pivots 4, 2, 1 and that important minus sign:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det A = -(4)(2)(1) = -8.$$

The odd number of row exchanges (namely one exchange) means that $\det P = -1$.

The next example has no row exchanges. It may be the first matrix we factored into LU (when it was 3 by 3). What is remarkable is that we can go directly to n by n. Pivots give the determinant. We will also see how determinants give the pivots.

Example 2 The first pivots of this tridiagonal matrix A are $2, \frac{3}{2}, \frac{4}{3}$. The next are $\frac{5}{4}$ and $\frac{6}{5}$ and eventually $\frac{n+1}{n}$. Factoring this n by n matrix reveals its determinant:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \cdot & \cdot & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \cdot & \cdot \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

The pivots are on the diagonal of U (the last matrix). When 2 and $\frac{3}{2}$ and $\frac{4}{3}$ and $\frac{5}{4}$ are multiplied, the fractions cancel. The determinant of the 4 by 4 matrix is 5. The 3 by 3 determinant is 4. *The n by n determinant is $n+1$:*

$$\mathbf{-1, 2, -1\ matrix} \qquad \det A = (2)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\cdots\left(\tfrac{n+1}{n}\right) = n+1.$$

Important point: The first pivots depend only on the *upper left corner* of the original matrix A. This is a rule for all matrices without row exchanges.

> The first k pivots come from the k by k matrix A_k in the top left corner of A.
> ***The determinant of that corner submatrix A_k is $d_1 d_2 \cdots d_k$.***

The 1 by 1 matrix A_1 contains the very first pivot d_1. This is $\det A_1$. The 2 by 2 matrix in the corner has $\det A_2 = d_1 d_2$. Eventually the n by n determinant uses the product of all n pivots to give $\det A_n$ which is $\det A$.

Elimination deals with the corner matrix A_k while starting on the whole matrix. We assume no row exchanges—then $A = LU$ and $A_k = L_k U_k$. Dividing one determinant by the previous determinant ($\det A_k$ divided by $\det A_{k-1}$) cancels everything but the latest pivot d_k. ***This gives a ratio of determinants formula for the pivots*:**

$$\textbf{Pivots from determinants} \qquad \textbf{The } k\textbf{th pivot is } d_k = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = \frac{\det A_k}{\det A_{k-1}}. \tag{3}$$

In the $-1, 2, -1$ matrices this ratio correctly gives the pivots $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}$. The Hilbert matrices in Problem 5.1.31 also build from the upper left corner.

We don't need row exchanges when all these corner submatrices have $\det A_k \neq 0$.

The Big Formula for Determinants

Pivots are good for computing. They concentrate a lot of information—enough to find the determinant. But it is hard to connect them to the original a_{ij}. That part will be clearer if we go back to rules 1-2-3, linearity and sign reversal and $\det I = 1$. We want to derive a single explicit formula for the determinant, directly from the entries a_{ij}.

The formula has $n!$ terms. Its size grows fast because $n! = 1, 2, 6, 24, 120, \ldots$. For $n = 11$ there are about forty million terms. For $n = 2$, the two terms are ad and bc. Half the terms have minus signs (as in $-bc$). The other half have plus signs (as in ad). For $n = 3$ there are $3! = (3)(2)(1)$ terms. Here are those six terms:

$$\text{3 by 3 determinant} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{matrix} \tag{4}$$

Notice the pattern. Each product like $a_{11}a_{23}a_{32}$ has ***one entry from each row***. It also has ***one entry from each column***. The column order 1, 3, 2 means that this particular term comes with a minus sign. The column order 3, 1, 2 in $a_{13}a_{21}a_{32}$ has a plus sign. It will be "permutations" that tell us the sign.

The next step ($n = 4$) brings $4! = 24$ terms. There are 24 ways to choose one entry from each row and column. Down the main diagonal, $a_{11}a_{22}a_{33}a_{44}$ with column order $1, 2, 3, 4$ always has a plus sign. That is the "identity permutation".

To derive the big formula I start with $n = 2$. The goal is to reach $ad - bc$ in a systematic way. Break each row into two simpler rows:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

Now apply linearity, first in row 1 (with row 2 fixed) and then in row 2 (with row 1 fixed):

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\ &= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}. \end{aligned} \tag{5}$$

The last line has $2^2 = 4$ determinants. The first and fourth are zero because their rows are dependent—one row is a multiple of the other row. We are left with $2! = 2$ determinants to compute:

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc.$$

The splitting led to permutation matrices. Their determinants give a plus or minus sign. The 1's are multiplied by numbers that come from A. The permutation tells the column sequence, in this case $(1, 2)$ or $(2, 1)$.

Now try $n = 3$. Each row splits into 3 simpler rows like $[a_{11} \;\; 0 \;\; 0]$. Using linearity in each row, $\det A$ splits into $3^3 = 27$ simple determinants. If a column choice is repeated—for example if we also choose $[a_{21} \;\; 0 \;\; 0]$—then the simple determinant is zero. We pay attention only when ***the nonzero terms come from different columns***.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$\textbf{\textit{Six terms}} \quad + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

There are $3! = 6$ ***ways to order the columns, so six determinants.*** The six permutations of $(1,2,3)$ include the identity permutation $(1,2,3)$ from $P = I$:

$$\textbf{Column numbers} = (1,2,3),(2,3,1),(3,1,2),(1,3,2),(2,1,3),(3,2,1). \tag{6}$$

The last three are *odd permutations* (one exchange). The first three are *even permutations* (0 or 2 exchanges). When the column sequence is (α,β,ω), we have chosen the entries $a_{1\alpha}a_{2\beta}a_{3\omega}$—and the column sequence comes with a plus or minus sign. The determinant of A is now split into six simple terms. Factor out the a_{ij}:

$$\begin{aligned}\det A = a_{11}a_{22}a_{33}\begin{vmatrix}1&&\\&1&\\&&1\end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix}&1&\\&&1\\1&&\end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix}&&1\\1&&\\&1&\end{vmatrix}\\ + a_{11}a_{23}a_{32}\begin{vmatrix}1&&\\&&1\\&1&\end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix}&1&\\1&&\\&&1\end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix}&&1\\&1&\\1&&\end{vmatrix}.\end{aligned} \tag{7}$$

The first three (even) permutations have $\det P = +1$, the last three (odd) permutations have $\det P = -1$. We have proved the 3 by 3 formula in a systematic way.

Now you can see the n by n formula. There are $n!$ orderings of the columns. The columns $(1,2,\ldots,n)$ go in each possible order $(\alpha,\beta,\ldots,\omega)$. Taking $a_{1\alpha}$ from row 1 and $a_{2\beta}$ from row 2 and eventually $a_{n\omega}$ from row n, the determinant contains the product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ times $+1$ or -1. Half the column orderings have sign -1.

The complete determinant of A is the sum of these $n!$ simple determinants, times 1 or -1. The simple determinants $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ choose ***one entry from every row and column***:

$$\begin{aligned}\det A &= \text{sum over all } \mathbf{n}! \text{ column permutations } P = (\alpha,\beta,\ldots,\omega)\\ &= \sum(\det P)a_{1\alpha}a_{2\beta}\cdots a_{n\omega} = \textbf{BIG FORMULA}.\end{aligned} \tag{8}$$

The 2 by 2 case is $+a_{11}a_{22} - a_{12}a_{21}$ (which is $ad - bc$). Here P is $(1,2)$ or $(2,1)$.

The 3 by 3 case has three products "down to the right" (see Problem 28) and three products "down to the left". Warning: Many people believe they should follow this pattern in the 4 by 4 case. They only take 8 products—but we need 24.

Example 3 (Determinant of U) When U is upper triangular, only one of the $n!$ products can be nonzero. This one term comes from the diagonal: $\det U = +u_{11}u_{22}\cdots u_{nn}$. All other column orderings pick at least one entry below the diagonal, where U has zeros. As soon as we pick a number like $u_{21} = 0$ from below the diagonal, that term in equation (8) is sure to be zero.

Of course $\det I = 1$. The only nonzero term is $+(1)(1)\cdots(1)$ from the diagonal.

Example 4 Suppose Z is the identity matrix except for column 3. Then

$$\text{determinant of } Z = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = c. \tag{9}$$

The term $(1)(1)(c)(1)$ comes from the main diagonal with a plus sign. There are 23 other products (choosing one factor from each row and column) but they are all zero. Reason: If we pick a, b, or d from column 3, that column is used up. Then the only available choice from row 3 is zero.

Here is a different reason for the same answer. If $c = 0$, then Z has a row of zeros and $\det Z = c = 0$ is correct. If c is not zero, *use elimination.* Subtract multiples of row 3 from the other rows, to knock out a, b, d. That leaves a diagonal matrix and $\det Z = c$.

This example will soon be used for "Cramer's Rule". If we move a, b, c, d into the first column of Z, the determinant is $\det Z = a$. (*Why*?) Changing one column of I leaves Z with an easy determinant, coming from its main diagonal only.

Example 5 Suppose A has 1's just above and below the main diagonal. Here $n = 4$:

$$A_4 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad P_4 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{have determinant 1.}$$

The only nonzero choice in the first row is column 2. The only nonzero choice in row 4 is column 3. Then rows 2 and 3 must choose columns 1 and 4. In other words P_4 is the only permutation that picks out nonzeros in A_4. The determinant of P_4 is $+1$ (two exchanges to reach $2, 1, 4, 3$). Therefore $\det A_4 = +1$.

Determinant by Cofactors

Formula (8) is a direct definition of the determinant. It gives you everything at once—but you have to digest it. Somehow this sum of $n!$ terms must satisfy rules 1-2-3 (then all the other properties follow). The easiest is $\det I = 1$, already checked. The rule of linearity becomes clear, if you ***separate out the factor*** a_{11} ***or*** a_{12} ***or*** $a_{1\alpha}$ ***that comes from the first row***. For 3 by 3, separate the usual 6 terms of the determinant into 3 pairs:

$$\det A = a_{11}\,(a_{22}a_{33} - a_{23}a_{32}) + a_{12}\,(a_{23}a_{31} - a_{21}a_{33}) + a_{13}\,(a_{21}a_{32} - a_{22}a_{31}). \tag{10}$$

Those three quantities in parentheses are called ***"cofactors"***. They are 2 by 2 determinants, coming from matrices in rows 2 and 3. The first row contributes the factors a_{11}, a_{12}, a_{13}. The lower rows contribute the cofactors C_{11}, C_{12}, C_{13}. Certainly the determinant $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ depends linearly on a_{11}, a_{12}, a_{13}—this is rule 3.

The cofactor of a_{11} is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. You can see it in this splitting:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

We are still choosing ***one entry from each row and column***. Since a_{11} uses up row 1 and column 1, that leaves a 2 by 2 determinant as its cofactor.

As always, we have to watch signs. The 2 by 2 determinant that goes with a_{12} looks like $a_{21}a_{33} - a_{23}a_{31}$. But in the cofactor C_{12}, *its sign is reversed.* Then $a_{12}C_{12}$ is the correct 3 by 3 determinant. The sign pattern for cofactors along the first row is plus-minus-plus-minus. ***You cross out row* 1 *and column* j *to get a submatrix* M_{1j} *of size* $n-1$.** Multiply its determinant by $(-1)^{1+j}$ to get the cofactor:

$$\text{The cofactors along row 1 are } C_{1j} = (-1)^{1+j} \det M_{1j}.$$
$$\textbf{The cofactor expansion is } \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \qquad (11)$$

In the big formula (8), the terms that multiply a_{11} combine to give $\det M_{11}$. The sign is $(-1)^{1+1}$, meaning *plus*. Equation (11) is another form of equation (8) and also equation (10), with factors from row 1 multiplying cofactors that use the other rows.

Note Whatever is possible for row 1 is possible for row i. The entries a_{ij} in that row also have cofactors C_{ij}. Those are determinants of order $n-1$, multiplied by $(-1)^{i+j}$. Since a_{ij} accounts for row i and column j, ***the submatrix M_{ij} throws out row i and column j***. The display shows a_{43} and M_{43} (with row 4 and column 3 removed). The sign $(-1)^{4+3}$ multiplies the determinant of M_{43} to give C_{43}. The sign matrix shows the $\pm$ pattern:

$$A = \begin{bmatrix} \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ & & a_{43} & \end{bmatrix} \qquad \text{signs } (-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

The determinant is the dot product of any row i of A with its cofactors using other rows:

$$\textbf{COFACTOR FORMULA} \qquad \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \qquad (12)$$

Each cofactor C_{ij} (order $n-1$, without row i and column j) includes its correct sign:

$$\textbf{Cofactor} \qquad C_{ij} = (-1)^{i+j} \det M_{ij}.$$

A determinant of order n is a combination of determinants of order $n-1$. A recursive person would keep going. Each subdeterminant breaks into determinants of order $n-2$. *We could define all determinants via equation* (12). This rule goes from order n to $n-1$

to $n-2$ and eventually to order 1. Define the 1 by 1 determinant $|a|$ to be the number a. Then the cofactor method is complete.

We preferred to construct $\det A$ from its properties (linearity, sign reversal, $\det I = 1$). The big formula (8) and the cofactor formulas (10)–(12) follow from those properties. One last formula comes from the rule that $\det A = \det A^{\mathrm{T}}$. We can expand in cofactors, *down a column* instead of across a row. Down column j the entries are a_{1j} to a_{nj}. The cofactors are C_{1j} to C_{nj}. The determinant is the dot product:

$$\textbf{Cofactors down column } j: \qquad \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \tag{13}$$

Cofactors are useful when matrices have many zeros—as in the next examples.

Example 6 The $-1, 2, -1$ matrix has only two nonzeros in its first row. So only two cofactors C_{11} and C_{12} are involved in the determinant. I will highlight C_{12}:

$$\begin{vmatrix} 2 & -1 & & \\ \mathbf{-1} & 2 & \mathbf{-1} & \\ & -1 & \mathbf{2} & \mathbf{-1} \\ & & \mathbf{-1} & \mathbf{2} \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} \mathbf{-1} & \mathbf{-1} & \\ & \mathbf{2} & \mathbf{-1} \\ & \mathbf{-1} & \mathbf{2} \end{vmatrix}. \tag{14}$$

You see 2 times C_{11} first on the right, from crossing out row 1 and column 1. This cofactor has exactly the same $-1, 2, -1$ pattern as the original A—but one size smaller.

To compute the boldface C_{12}, *use cofactors down its first column.* The only nonzero is at the top. That contributes another -1 (so we are back to minus). Its cofactor is the $-1, 2, -1$ determinant which is 2 by 2, *two sizes smaller* than the original A.

Summary ***Each determinant D_n of order n comes from D_{n-1} and D_{n-2}:***

$$D_4 = 2D_3 - D_2 \qquad \text{and generally} \qquad D_n = 2D_{n-1} - D_{n-2}. \tag{15}$$

Direct calculation gives $D_2 = 3$ and $D_3 = 4$. Equation (14) has $D_4 = 2(4) - 3 = 5$. These determinants 3, 4, 5 fit the formula $\boldsymbol{D_n = n + 1}$. That "special tridiagonal answer" also came from the product of pivots in Example 2.

The idea behind cofactors is to reduce the order one step at a time. The determinants $D_n = n + 1$ obey the recursion formula $n + 1 = 2n - (n - 1)$. As they must.

Example 7 This is the same matrix, except the first entry (upper left) is now 1:

$$B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

All pivots of this matrix turn out to be 1. So its determinant is 1. How does that come from cofactors? Expanding on row 1, the cofactors all agree with Example 6. Just change $a_{11} = 2$ to $b_{11} = 1$:

$$\det B_4 = D_3 - D_2 \qquad \text{instead of} \qquad \det A_4 = 2D_3 - D_2.$$

The determinant of B_4 is $4 - 3 = 1$. The determinant of every B_n is $n - (n - 1) = 1$. Problem 13 asks you to use cofactors of the *last* row. You still find $\det B_n = 1$.

■ REVIEW OF THE KEY IDEAS ■

1. With no row exchanges, $\det A$ = (*product of pivots*). In the upper left corner, $\det A_k$ = (*product of the first k pivots*).
2. Every term in the big formula (8) uses each row and column once. Half of the $n!$ terms have plus signs (when $\det P = +1$) and half have minus signs.
3. The cofactor C_{ij} is $(-1)^{i+j}$ times the smaller determinant that omits row i and column j (because a_{ij} uses that row and column).
4. The determinant is the dot product of any row of A with its row of cofactors. When a row of A has a lot of zeros, we only need a few cofactors.

■ WORKED EXAMPLES ■

5.2 A A *Hessenberg matrix* is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes, $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution The cofactor C_{11} for H_4 is the determinant $|H_3|$. We also need C_{12} (in boldface):

$$C_{12} = -\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{vmatrix} = -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Rows 2 and 3 stayed the same and we used linearity in row 1. The two determinants on the right are $-|H_3|$ and $+|H_2|$. Then the 4 by 4 determinant is

$$|H_4| = 2C_{11} + 1C_{12} = 2|H_3| - |H_3| + |H_2| = |H_3| + |H_2|.$$

The actual numbers are $|H_2| = 3$ and $|H_3| = 5$ (and of course $|H_1| = 2$). Since $|H_n|$ follows Fibonacci's rule $|H_{n-1}| + |H_{n-2}|$, it must be $|H_n| = F_{n+2}$.

5.2 B These questions use the $\pm$ signs (even and odd P's) in the big formula for $\det A$:

1. If A is the 10 by 10 all-ones matrix, how does the big formula give $\det A = 0$?

2. If you multiply all $n!$ permutations together into a single P, is P odd or even?

3. If you multiply each a_{ij} by the fraction i/j, why is $\det A$ unchanged?

Solution In Question **1**, with all $a_{ij} = 1$, all the products in the big formula (8) will be 1. Half of them come with a plus sign, and half with minus. So they cancel to leave $\det A = 0$. (Of course the all-ones matrix is singular.)

In Question **2**, multiplying $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ gives an odd permutation. Also for 3 by 3, the three odd permutations multiply (in any order) to give *odd*. But for $n > 3$ the product of all permutations will be *even*. There are $n!/2$ odd permutations and that is an even number as soon as it includes the factor 4.

In Question **3**, each a_{ij} is multiplied by i/j. So each product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is multiplied by all the row numbers $i = 1, 2, \ldots, n$ and divided by all the column numbers $j = 1, 2, \ldots, n$. (The columns come in some permuted order!) Then each product is unchanged and $\det A$ stays the same.

Another approach to Question **3**: We are multiplying the matrix A by the diagonal matrix $D = \mathbf{diag}(1 : n)$ when row i is multiplied by i. And we are postmultiplying by D^{-1} when column j is divided by j. The determinant of DAD^{-1} is the same as $\det A$ by the product rule.

Problem Set 5.2

Problems 1–10 use the big formula with $n!$ terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$.

1 Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2 Compute the determinants of A, B, C, D. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

3 Show that $\det A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1?
What is the rank of A?
What are the 6 terms in $\det A$?

4 Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad (B \text{ has the same zeros as } A).$$

Is $\det A$ equal to $1 + 1$ or $1 - 1$ or $-1 - 1$? What is $\det B$?

5 Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det A$ will be zero?

(b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3l}a_{4m}$ are sure to be zero?

7 How many 5 by 5 permutation matrices have $\det P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.

8 If $\det A$ is not zero, at least one of the $n!$ terms in formula (8) is not zero. Deduce from the big formula that some ordering of the rows of A leaves no zeros on the diagonal. (Don't use P from elimination; that PA can have zeros on the diagonal.)

9 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

10 How many permutations of $(1, 2, 3, 4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

Problems 11–22 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Remove row i and column j.

11 Find all cofactors and put them into cofactor matrices C, D. Find AC and $\det B$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

12 Find the cofactor matrix C and multiply A times C^{T}. Compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

13 The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are these determinants C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

14 The matrices in Problem 13 have 1's just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1's? Explain why that permutation is *even* for $n = 4, 8, 12, \ldots$ and *odd* for $n = 2, 6, 10, \ldots$. Then

$$C_n = 0 \text{ (odd } n) \qquad C_n = 1 \; (n = 4, 8, \cdots) \qquad C_n = -1 \; (n = 2, 6, \cdots).$$

15 The tridiagonal 1, 1, 1 matrix of order n has determinant E_n:

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

(b) Starting from $E_1 = 1$ and $E_2 = 0$ find $E_3, E_4, \ldots, E_8$.

(c) By noticing how these numbers eventually repeat, find E_{100}.

16 F_n is the determinant of the $1, 1, -1$ tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \quad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* $1, 2, 3, 5, 8, 13, \ldots$. The sequence usually starts $1, 1, 2, 3$ (with two 1's) so our F_n is the usual F_{n+1}.

17 The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{11} = 1$ instead of $a_{11} = 2$. Using cofactors of the *last* row of B_4 show that $|B_4| = 2|B_3| - |B_2| = 1$.

$$B_4 = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 2 \end{bmatrix}.$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is satisfied when every $|B_n| = 1$. This recursion is the same as for the A's in Example 6. The difference is in the starting values 1, 1, 1 for the determinants of sizes $n = 1, 2, 3$.

18 Go back to B_n in Problem 17. It is the same as A_n except for $b_{11} = 1$. So use linearity in the first row, where $[1 \ \ -1 \ \ 0]$ equals $[2 \ \ -1 \ \ 0]$ minus $[1 \ \ 0 \ \ 0]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} 2 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} - \begin{vmatrix} 1 & 0 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix}.$$

Linearity gives $|B_n| = |A_n| - |A_{n-1}| = \_\_\_\_$.

19 Explain why the 4 by 4 Vandermonde determinant contains x^3 but not x^4 or x^5:

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

The determinant is zero at $x =$ ____, ____, and ____. The cofactor of x^3 is $V_3 = (b-a)(c-a)(c-b)$. Then $V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$.

20 Find G_2 and G_3 and then by row operations G_4. Can you predict G_n?

$$G_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \qquad G_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

21 Compute S_1, S_2, S_3 for these 1, 3, 1 matrices. By Fibonacci guess and check S_4.

$$S_1 = |3| \qquad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \qquad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

22 Change 3 to 2 in the upper left corner of the matrices in Problem 21. Why does that subtract S_{n-1} from the determinant S_n? Show that the determinants of the new matrices become the Fibonacci numbers 2, 5, 13 (always F_{2n+1}).

Problems 23–26 are about block matrices and block determinants.

23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \qquad \text{but} \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer $\det(AD - CB)$ is also wrong.

24 With block multiplication, $A = LU$ has $A_k = L_k U_k$ in the top left corner:

$$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$

(a) Suppose the first three pivots of A are $2, 3, -1$. What are the determinants of L_1, L_2, L_3 (with diagonal 1's) and U_1, U_2, U_3 and A_1, A_2, A_3?

(b) If A_1, A_2, A_3 have determinants $5, 6, 7$ find the three pivots from equation (3).

25 Block elimination subtracts CA^{-1} times the first row $[\,A \quad B\,]$ from the second row $[C \quad D]$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these block matrices to prove correct rules if A^{-1} exists:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| \quad = \quad |AD - CB| \;\text{ provided } AC = CA.$$

26 If A is m by n and B is n by m, block multiplication gives $\det M = \det AB$:

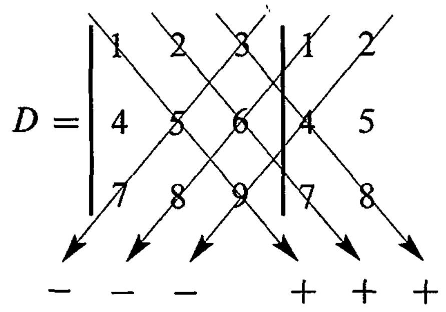
$$M = \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}.$$

If A is a single row and B is a single column what is $\det M$? If A is a column and B is a row what is $\det M$? Do a 3 by 3 example of each.

27 (A calculus question) Show that the derivative of $\det A$ with respect to a_{11} is the cofactor C_{11}. The other entries are fixed—we are only changing a_{11}.

Problems 28–33 are about the "big formula" with $n!$ terms.

28 A 3 by 3 determinant has three products "down to the right" and three "down to the left" with minus signs. Compute the six terms like $(1)(5)(9) = 45$ to find D.



Explain without determinants why this particular matrix is or is not invertible.

29 For E_4 in Problem 15, five of the $4! = 24$ terms in the big formula (8) are nonzero. Find those five terms to show that $E_4 = -1$.

30 For the 4 by 4 tridiagonal second difference matrix (entries -1, 2, -1) find the five terms in the big formula that give $\det A = 16 - 4 - 4 - 4 + 1$.

31 Find the determinant of this cyclic P by cofactors of row 1 and then the "big formula". How many exchanges reorder $4, 1, 2, 3$ into $1, 2, 3, 4$? Is $|P^2| = 1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Challenge Problems

32 Cofactors of the $1, 3, 1$ matrices in Problem 21 give a recursion $S_n = 3S_{n-1} - S_{n-2}$. Amazingly that recursion produces every second Fibonacci number. Here is the challenge.

Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$ starting with $k = 2n + 2$.

33 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain).}$$

34 This problem shows in two ways that $\det A = 0$ (the x's are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

35 If $|\det(A)| > 1$, prove that the powers A^n cannot stay bounded. But if $|\det(A)| \leq 1$, show that some entries of A^n might still grow large. Eigenvalues will give the right test for stability, determinants tell us only one number.

5.3 Cramer's Rule, Inverses, and Volumes

This section solves $Ax = b$—by algebra and not by elimination. We also invert A. In the entries of A^{-1}, you will see $\det A$ in every denominator—we divide by it. (If $\det A = 0$ then we can't divide and A^{-1} doesn't exist.) Each entry in A^{-1} and $A^{-1}b$ is a determinant divided by the determinant of A.

Cramer's Rule solves $Ax = b$. A neat idea gives the first component x_1. Replacing the first column of I by x gives a matrix with determinant x_1. When you multiply it by A, the first column becomes Ax which is b. The other columns are copied from A:

$$\textbf{Key idea}\qquad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1. \tag{1}$$

We multiplied a column at a time. *Take determinants of the three matrices*:

$$\textbf{Product rule}\qquad (\det A)(x_1) = \det B_1 \qquad \text{or} \qquad \boldsymbol{x_1 = \frac{\det B_1}{\det A}}. \tag{2}$$

This is the first component of x in Cramer's Rule! Changing a column of A gives B_1.

To find x_2, put the vector x into the *second* column of the identity matrix:

$$\textbf{Same idea}\qquad \begin{bmatrix} & & \\ a_1 & a_2 & a_3 \\ & & \end{bmatrix}\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ a_1 & b & a_3 \\ & & \end{bmatrix} = B_2. \tag{3}$$

Take determinants to find $(\det A)(x_2) = \det B_2$. This gives x_2 in Cramer's Rule:

CRAMER's RULE If $\det A$ is not zero, $Ax = b$ is solved by determinants:

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \cdots \qquad x_n = \frac{\det B_n}{\det A} \tag{4}$$

The matrix B_j has the jth column of A replaced by the vector b.

Example 1 Solving $3x_1 + 4x_2 = \mathbf{2}$ and $5x_1 + 6x_2 = \mathbf{4}$ needs three determinants:

$$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} \qquad \det B_1 = \begin{vmatrix} \mathbf{2} & 4 \\ \mathbf{4} & 6 \end{vmatrix} \qquad \det B_2 = \begin{vmatrix} 3 & \mathbf{2} \\ 5 & \mathbf{4} \end{vmatrix}$$

Those determinants are -2 and -4 and 2. All ratios divide by $\det A$:

$$\textbf{Cramer's Rule}\quad x_1 = \frac{-4}{-2} = 2 \quad x_2 = \frac{2}{-2} = -1 \quad \textbf{check} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

To solve an n by n system, Cramer's Rule evaluates $n + 1$ determinants (of A and the n different B's). When each one is the sum of $n!$ terms—applying the "big formula" with all permutations—this makes a total of $(n + 1)!$ terms. *It would be crazy to solve equations that way.* But we do finally have an explicit formula for the solution x.

Example 2 Cramer's Rule is inefficient for numbers but it is well suited to letters. For $n = 2$, find the columns of A^{-1} by solving $AA^{-1} = I$:

$$\textbf{Columns of } I \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Those share the same A. We need five determinants for $x_1,\ x_2,\ y_1,\ y_2$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} \mathbf{1} & b \\ \mathbf{0} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{1} \\ c & \mathbf{0} \end{vmatrix} \quad \begin{vmatrix} \mathbf{0} & b \\ \mathbf{1} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{0} \\ c & \mathbf{1} \end{vmatrix}$$

The last four are $d, -c, -b$, and a. (They are the cofactors!) Here is A^{-1}:

$$x_1 = \frac{d}{|A|},\ x_2 = \frac{-c}{|A|},\ y_1 = \frac{-b}{|A|},\ y_2 = \frac{a}{|A|}, \text{ and then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

I chose 2 by 2 so that the main points could come through clearly. The new idea is the appearance of the cofactors. When the right side is a column of the identity matrix I, the determinant of each matrix B_j in Cramer's Rule is a cofactor.

You can see those cofactors for $n = 3$. Solve $AA^{-1} = I$ (first column only):

$$\begin{array}{l} \textbf{Determinants} \\ \textbf{= Cofactors of } A \end{array} \quad \begin{vmatrix} \mathbf{1} & a_{12} & a_{13} \\ \mathbf{0} & a_{22} & a_{23} \\ \mathbf{0} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & \mathbf{1} & a_{13} \\ a_{21} & \mathbf{0} & a_{23} \\ a_{31} & \mathbf{0} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & \mathbf{1} \\ a_{21} & a_{22} & \mathbf{0} \\ a_{31} & a_{32} & \mathbf{0} \end{vmatrix} \tag{5}$$

That first determinant $|B_1|$ is the cofactor C_{11}. The second determinant $|B_2|$ is the cofactor C_{12}. *Notice that* the correct minus sign appears in $-(a_{21}a_{33} - a_{23}a_{31})$. This cofactor C_{12} goes into the 2, 1 entry of A^{-1}—the first column! So we transpose the cofactor matrix, and as always we divide by $\det A$.

The i, j entry of A^{-1} is the cofactor C_{ji} (not C_{ij}) divided by $\det A$:

$$\textbf{FORMULA FOR } A^{-1} \qquad (A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}. \tag{6}$$

The cofactors C_{ij} go into the *"cofactor matrix"* C. ***Its transpose leads to*** A^{-1}. To compute the i, j entry of A^{-1}, cross out row j and column i of A. Multiply the determinant by $(-1)^{i+j}$ to get the cofactor, and divide by $\det A$.

Check this rule for the 3, 1 entry of A^{-1}. This is in column 1 so we solve $Ax = (1, 0, 0)$. The third component x_3 needs the third determinant in equation (5), divided by $\det A$. That third determinant is exactly the cofactor $C_{13} = a_{21}a_{32} - a_{22}a_{31}$. So $(A^{-1})_{31} = C_{13}/\det A$ (2 by 2 determinant divided by 3 by 3).

Summary In solving $AA^{-1} = I$, the columns of I lead to the columns of A^{-1}. Then Cramer's Rule using $\boldsymbol{b}$ = columns of I gives the short formula (6) for A^{-1}.

Direct proof of the formula $A^{-1} = C^{\mathrm{T}}/\det A$ The idea is to multiply A times C^{T}:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}. \quad (7)$$

Row 1 of A times column 1 of the cofactors yields the first $\det A$ on the right:

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A \quad \text{by the cofactor rule.}$$

Similarly row 2 of A times column 2 of C^{T}(*transpose*) yields $\det A$. The entries a_{2j} are multiplying cofactors C_{2j} as they should, to give the determinant.

How to explain the zeros off the main diagonal in equation (7)? Rows of A are multiplying cofactors from *different* rows. Why is the answer zero?

Row 2 of A
Row 1 of C

$$a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = 0. \quad (8)$$

Answer: This is the cofactor rule for a new matrix, when the second row of A is copied into its first row. The new matrix A^* has two equal rows, so $\det A^* = 0$ in equation (8). Notice that A^* has the same cofactors C_{11}, C_{12}, C_{13} as A—because all rows agree after the first row. Thus the remarkable multiplication (7) is correct:

$$AC^{\mathrm{T}} = (\det A)I \qquad \text{or} \qquad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}.$$

Example 3 The "sum matrix" A has determinant 1. Then A^{-1} contains cofactors:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{C^{\mathrm{T}}}{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Cross out row 1 and column 1 of A to see the 3 by 3 cofactor $C_{11} = 1$. Now cross out row 1 and column 2 for C_{12}. The 3 by 3 submatrix is still triangular with determinant 1. But the cofactor C_{12} is -1 because of the sign $(-1)^{1+2}$. This number -1 goes into the $(2, 1)$ entry of A^{-1}—don't forget to transpose C.

The inverse of a triangular matrix is triangular. Cofactors give a reason why.

Example 4 If all cofactors are nonzero, is A sure to be invertible? *No way.*

Area of a Triangle

Everybody knows the area of a rectangle—base times height. The area of a triangle is *half* the base times the height. But here is a question that those formulas don't answer. ***If we know the corners*** (x_1, y_1) ***and*** (x_2, y_2) ***and*** (x_3, y_3) ***of a triangle, what is the area?*** Using the corners to find the base and height is not a good way.

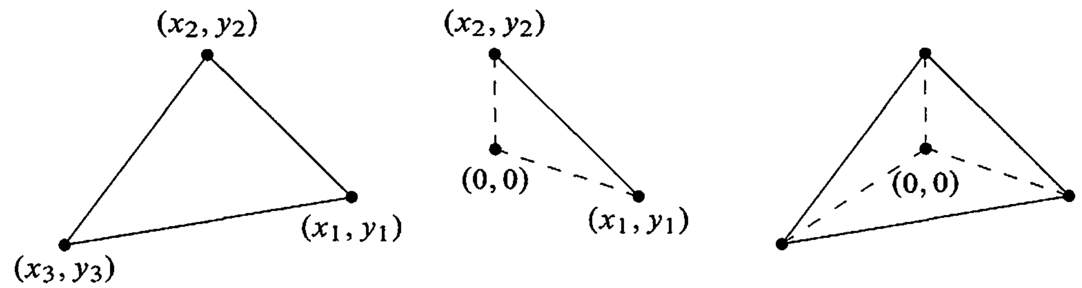


Figure 5.1: General triangle; special triangle from $(0,0)$; general from three specials.

Determinants are much better. The square roots in the base and height cancel out in the good formula. ***The area of a triangle is half of a 3 by 3 determinant.*** If one corner is at the origin, say $(x_3, y_3) = (0,0)$, the determinant is only 2 by 2.

The triangle with corners (x_1, y_1) and (x_2, y_2) and (x_3, y_3) has area $= \dfrac{\text{determinant}}{2}$:

Area of triangle $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ Area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ when $(x_3, y_3) = (0,0)$.

When you set $x_3 = y_3 = 0$ in the 3 by 3 determinant, you get the 2 by 2 determinant. These formulas have no square roots—they are reasonable to memorize. The 3 by 3 determinant breaks into a sum of three 2 by 2's, just as the third triangle in Figure 5.1 breaks into three special triangles from $(0,0)$:

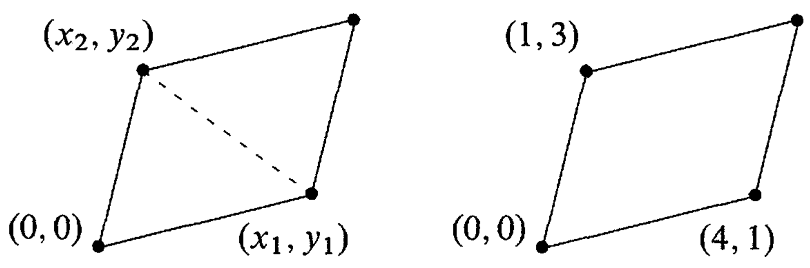
Cofactors of column 3

$$\textbf{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{array}{l} +\frac{1}{2}(x_1y_2 - x_2y_1) \\ +\frac{1}{2}(x_2y_3 - x_3y_2) \\ +\frac{1}{2}(x_3y_1 - x_1y_3). \end{array} \tag{9}$$

If $(0,0)$ is outside the triangle, two of the special areas can be negative—but the sum is still correct. The real problem is to explain the special area $\frac{1}{2}(x_1y_2 - x_2y_1)$.

Why is this the area of a triangle? We can remove the factor $\frac{1}{2}$ and change to a parallelogram (twice as big, because the parallelogram contains two equal triangles). We now prove that the parallelogram area is the determinant $x_1y_2 - x_2y_1$. This area in Figure 5.2 is 11, and therefore the triangle has area $\frac{11}{2}$.

Proof that a parallelogram starting from* $(0,0)$ *has area* $=$ *2 by 2 determinant.



Parallelogram

$$\text{Area} = \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 11$$

$$\text{Triangle: Area} = \tfrac{11}{2}$$

Figure 5.2: A triangle is half of a parallelogram. Area is half of a determinant.

There are many proofs but this one fits with the book. We show that the area has the same properties 1-2-3 as the determinant. Then area = determinant! Remember that those three rules defined the determinant and led to all its other properties.

1 When $A = I$, the parallelogram becomes the unit square. Its area is $\det I = 1$.

2 When rows are exchanged, the determinant reverses sign. The absolute value (positive area) stays the same—it is the same parallelogram.

3 If row 1 is multiplied by t, Figure 5.3a shows that the area is also multiplied by t. Suppose a new row (x_1', y_1') is added to (x_1, y_1) (keeping row 2 fixed). Figure 5.3b shows that the solid parallelogram areas add to the dotted parallelogram area (because the two triangles completed by dotted lines are the same).

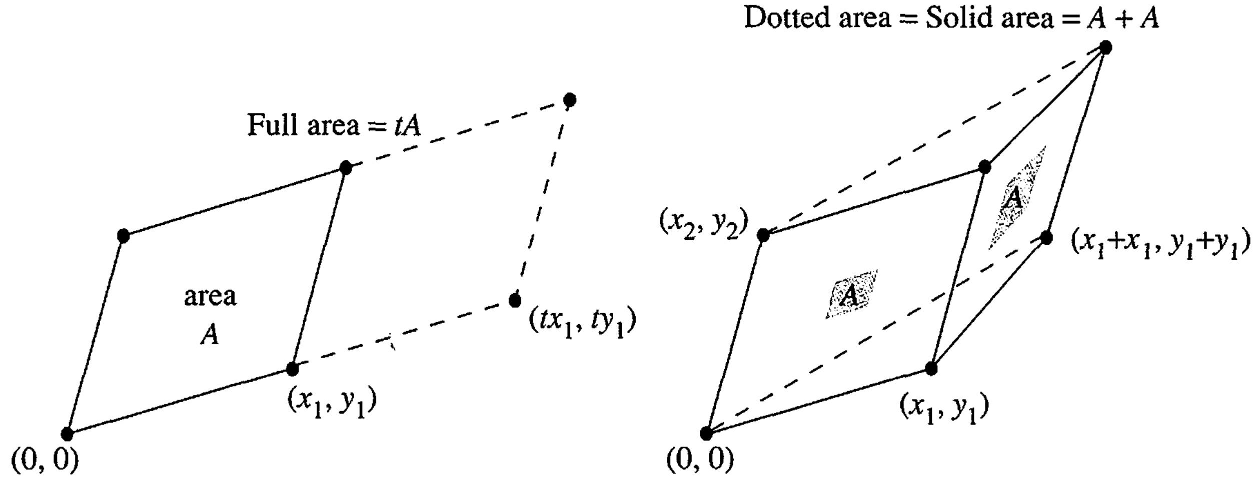


Figure 5.3: Areas obey the rule of linearity (keeping the side (x_2, y_2) constant).

That is an exotic proof, when we could use plane geometry. But the proof has a major attraction—it applies in n dimensions. The n edges going out from the origin are given by the *rows of an n by n matrix*. The box is completed by more edges, just like the parallelogram.

Figure 5.4 shows a three-dimensional box—whose edges are not at right angles. ***The volume equals the absolute value of* $\det A$.** Our proof checks again that rules 1–3 for determinants are also obeyed by volumes. When an edge is stretched by a factor t, the volume is multiplied by t. When edge 1 is added to edge $1'$, the new box has edge $1 + 1'$. Its volume is the sum of the two original volumes. This is Figure 5.3b lifted into three dimensions or n dimensions. I would draw the boxes but this paper is only two-dimensional.

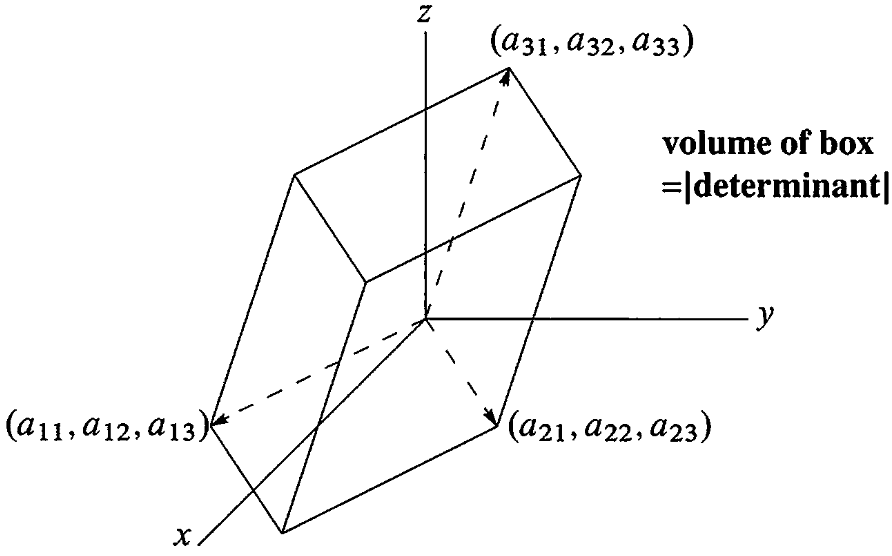


Figure 5.4: Three-dimensional box formed from the three rows of A.

The unit cube has volume $= 1$, which is $\det I$. Row exchanges or edge exchanges leave the same box and the same absolute volume. The determinant changes sign, to indicate whether the edges are a *right-handed triple* ($\det A > 0$) or a *left-handed triple* ($\det A < 0$). The box volume follows the rules for determinants, so volume of the box = absolute value of the determinant.

Example 5 Suppose a rectangular box (90° angles) has side lengths r, s, and t. Its volume is r times s times t. The diagonal matrix with entries r, s, and t produces those three sides. Then $\det A$ also equals $r\,s\,t$.

Example 6 In calculus, the box is infinitesimally small! To integrate over a circle, we might change x and y to r and θ. Those are polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. The area of a "polar box" is a determinant J times $dr\,d\theta$:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

This determinant is the r in the small area $dA = r\,dr\,d\theta$. The stretching factor J goes into double integrals just as dx/du goes into an ordinary integral $\int dx = \int (dx/du)\,du$. For triple integrals the Jacobian matrix J with nine derivatives will be 3 by 3.

The Cross Product

The *cross product* is an extra (and optional) application, special for three dimensions. Start with vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Unlike the dot product, which is a number, the cross product is a vector—also in three dimensions. It is written $\boldsymbol{u} \times \boldsymbol{v}$ and pronounced "$\boldsymbol{u}$ cross $\boldsymbol{v}$." *The components of this cross product are just* 2 *by* 2 *cofactors.* We will explain the properties that make $\boldsymbol{u} \times \boldsymbol{v}$ useful in geometry and physics.

This time we bite the bullet, and write down the formula before the properties.

DEFINITION The *cross product* of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}. \quad (10)$$

This vector is perpendicular to u and v. The cross product $\boldsymbol{v} \times \boldsymbol{u}$ is $-(\boldsymbol{u} \times \boldsymbol{v})$.

Comment The 3 by 3 determinant is the easiest way to remember $\boldsymbol{u} \times \boldsymbol{v}$. It is not especially legal, because the first row contains vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the other rows contain numbers. In the determinant, the vector $\boldsymbol{i} = (1, 0, 0)$ multiplies u_2v_3 and $-u_3v_2$. The result is $(u_2v_3 - u_3v_2, 0, 0)$, which displays the first component of the cross product.

Notice the cyclic pattern of the subscripts: 2 and 3 give component 1 of $\boldsymbol{u} \times \boldsymbol{v}$, then 3 and 1 give component 2, then 1 and 2 give component 3. This completes the definition of $\boldsymbol{u} \times \boldsymbol{v}$. Now we list the properties of the cross product:

Property 1 $\boldsymbol{v} \times \boldsymbol{u}$ reverses rows 2 and 3 in the determinant so it equals $-(\boldsymbol{u} \times \boldsymbol{v})$.

Property 2 The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ (and also to $\boldsymbol{v}$). The direct proof is to watch terms cancel. Perpendicularity is a zero dot product:

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0. \quad (11)$$

The determinant now has rows $\boldsymbol{u}, \boldsymbol{u}$ and $\boldsymbol{v}$ so it is zero.

Property 3 The cross product of any vector with itself (two equal rows) is $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$.

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, the cross product is zero. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, the dot product is zero. One involves $\sin\theta$ and the other involves $\cos\theta$:

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \, |\sin\theta| \quad \text{and} \quad |\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \, |\cos\theta|. \quad (12)$$

Example 7 Since $\boldsymbol{u} = (3, 2, 0)$ and $\boldsymbol{v} = (1, 4, 0)$ are in the xy plane, $\boldsymbol{u} \times \boldsymbol{v}$ goes up the z axis:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\boldsymbol{k}. \quad \text{The cross product is } \boldsymbol{u} \times \boldsymbol{v} = (0, 0, 10).$$

The length of $u \times v$ equals the area of the parallelogram with sides u and v. This will be important: In this example the area is 10.

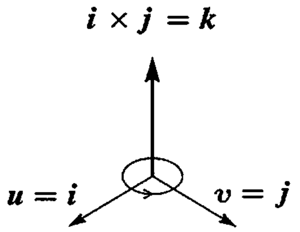
Example 8 The cross product of $\boldsymbol{u} = (1,1,1)$ and $\boldsymbol{v} = (1,1,2)$ is $(1,-1,0)$:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j}.$$

This vector $(1,-1,0)$ is perpendicular to $(1,1,1)$ and $(1,1,2)$ as predicted. Area $= \sqrt{2}$.

Example 9 The cross product of $(1,0,0)$ and $(0,1,0)$ obeys the *right hand rule*. It goes up not down:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{k}$$



Rule $\boldsymbol{u} \times \boldsymbol{v}$ points along your right thumb when the fingers curl from $\boldsymbol{u}$ to $\boldsymbol{v}$.

Thus $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$. The right hand rule also gives $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$. Note the cyclic order. In the opposite order (anti-cyclic) the thumb is reversed and the cross product goes the other way: $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$ and $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ and $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$. You see the three plus signs and three minus signs from a 3 by 3 determinant.

The definition of $\boldsymbol{u} \times \boldsymbol{v}$ can be based on vectors instead of their components:

DEFINITION The ***cross product*** is a vector with length $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\sin\theta|$. Its direction is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. It points "up" or "down" by the right hand rule.

This definition appeals to physicists, who hate to choose axes and coordinates. They see (u_1, u_2, u_3) as the position of a mass and (F_x, F_y, F_z) as a force acting on it. If $\boldsymbol{F}$ is parallel to $\boldsymbol{u}$, then $\boldsymbol{u} \times \boldsymbol{F} = \boldsymbol{0}$—there is no turning. The cross product $\boldsymbol{u} \times \boldsymbol{F}$ is the turning force or *torque*. It points along the turning axis (perpendicular to $\boldsymbol{u}$ and $\boldsymbol{F}$). Its length $\|\boldsymbol{u}\|\,\|\boldsymbol{F}\|\sin\theta$ measures the "moment" that produces turning.

Triple Product = Determinant = Volume

Since $\boldsymbol{u} \times \boldsymbol{v}$ is a vector, we can take its dot product with a third vector $\boldsymbol{w}$. That produces the ***triple product*** $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. It is called a "scalar" triple product, because it is a number. In fact it is a determinant—it gives the volume of the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box:

Triple product
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{13}$$

We can put $\boldsymbol{w}$ in the top or bottom row. The two determinants are the same because ____ row exchanges go from one to the other. Notice when this determinant is zero:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = 0 \quad \text{exactly when the vectors } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \text{ lie in the } \textit{same plane.}$$

First reason $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to that plane so its dot product with $\boldsymbol{w}$ is zero.

Second reason Three vectors in a plane are dependent. The matrix is singular (det $= 0$).

Third reason Zero volume when the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box is squashed onto a plane.

It is remarkable that $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ equals the volume of the box with sides $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. This 3 by 3 determinant carries tremendous information. Like $ad - bc$ for a 2 by 2 matrix, it separates invertible from singular. Chapter 6 will be looking for singular.

■ REVIEW OF THE KEY IDEAS ■

1. Cramer's Rule solves $A\boldsymbol{x} = \boldsymbol{b}$ by ratios like $x_1 = |B_1|/|A| = |\boldsymbol{b}\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n|/|A|$.
2. When C is the cofactor matrix for A, the inverse is $A^{-1} = C^{\mathrm{T}}/\det A$.
3. The volume of a box is $|\det A|$, when the box edges are the rows of A.
4. Area and volume are needed to change variables in double and triple integrals.
5. In $\mathbf{R}^3$, the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$.

■ WORKED EXAMPLES ■

5.3 A If A is singular, the equation $AC^{\mathrm{T}} = (\det A)I$ becomes $AC^{\mathrm{T}} =$ **zero matrix.** *Then each column of C^{T} is in the nullspace of A.* Those columns contain cofactors along rows of A. So the cofactors quickly find the nullspace of a 3 by 3 matrix—my apologies that this comes so late!

Solve $A\boldsymbol{x} = \mathbf{0}$ by $\boldsymbol{x} =$ cofactors along a row, for these singular matrices of rank 2:

Cofactors give Nullspace $\qquad A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Any nonzero column of C^{T} will give the desired solution to $A\boldsymbol{x} = \mathbf{0}$. With rank 2, A has at least one nonzero cofactor. If A has rank 1 we get $\boldsymbol{x} = \mathbf{0}$ and the idea fails.

Solution The first matrix has these cofactors along its top row (note each minus sign):

$$\begin{vmatrix} 3 & 9 \\ 2 & 8 \end{vmatrix} = 6 \qquad -\begin{vmatrix} 2 & 9 \\ 2 & 8 \end{vmatrix} = 2 \qquad \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

Then $\boldsymbol{x} = (6, 2, -2)$ solves $A\boldsymbol{x} = \mathbf{0}$. The cofactors along the second row are $(-18, -6, 6)$ which is just $-3\boldsymbol{x}$. This is also in the one-dimensional nullspace of A.

The second matrix has *zero cofactors* along its first row. The nullvector $\boldsymbol{x} = (0, 0, 0)$ is not interesting. The cofactors of row 2 give $\boldsymbol{x} = (1, -1, 0)$ which solves $A\boldsymbol{x} = \mathbf{0}$.

Every n by n matrix of rank $n - 1$ has at least one nonzero cofactor by Problem 3.3.12. But for rank $n - 2$, all cofactors are zero and we only find $\boldsymbol{x} = \mathbf{0}$.

5.3 B Use Cramer's Rule with ratios $\det B_j / \det A$ to solve $A\boldsymbol{x} = \boldsymbol{b}$. Also find the inverse matrix $A^{-1} = C^{\mathrm{T}} / \det A$. Why is the solution $\boldsymbol{x}$ for this $\boldsymbol{b}$ the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the volumes of the boxes whose edges are *columns* of A and then rows of A^{-1}.

Solution The determinants of the B_j (with right side $\boldsymbol{b}$ placed in column j) are

$$|B_1| = \begin{vmatrix} \mathbf{0} & 6 & 2 \\ \mathbf{0} & 4 & 2 \\ \mathbf{1} & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & \mathbf{0} & 2 \\ 1 & \mathbf{0} & 2 \\ 5 & \mathbf{1} & 0 \end{vmatrix} = -2 \qquad |B_3| = \begin{vmatrix} 2 & 6 & \mathbf{0} \\ 1 & 4 & \mathbf{0} \\ 5 & 9 & \mathbf{1} \end{vmatrix} = 2.$$

Those are cofactors C_{31}, C_{32}, C_{33} of row 3. Their dot product with row 3 is $\det A$:

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (5, 9, 0) \cdot (4, -2, 2) = 2.$$

The three ratios $\det B_j / \det A$ give the three components of $\boldsymbol{x} = (2, -1, 1)$. This $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b} = (0, 0, 1)$ is the third column of I. The cofactors along the other rows of A, divided by $\det A = 2$, give the other *columns* of A^{-1}:

$$A^{-1} = \frac{C^{\mathrm{T}}}{\det A} = \frac{1}{2} \begin{bmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{bmatrix}. \quad \text{Multiply to check} \quad AA^{-1} = I$$

The box from the columns of A has volume $= \det A = 2$ (the same as the box from the rows, since $|A^{\mathrm{T}}| = |A|$). The box from A^{-1} has volume $1/|A| = \frac{1}{2}$.

Problem Set 5.3

Problems 1–5 are about Cramer's Rule for $x = A^{-1}b$.

1 Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 \quad\quad &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

2 Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

(a) $\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$ (b) $\begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$

3 Cramer's Rule breaks down when $\det A = 0$. Example (a) has no solution while (b) has infinitely many. What are the ratios $x_j = \det B_j / \det A$ in these two cases?

(a) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 1 \end{aligned}$ (parallel lines) (b) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 2 \end{aligned}$ (same line)

4 *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $\boldsymbol{b} = A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ goes into the first column of A, the determinant of this matrix B_1 is

$$|\boldsymbol{b} \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = |x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1|\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1 \det A.$$

(a) What formula for x_1 comes from left side = right side?

(b) What steps lead to the middle equation?

5 If the right side $\boldsymbol{b}$ is the first column of A, solve the 3 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$. How does each determinant in Cramer's Rule lead to this solution $\boldsymbol{x}$?

Problems 6–15 are about $A^{-1} = C^{\mathrm{T}} / \det A$. Remember to transpose C.

6 Find A^{-1} from the cofactor formula $C^{\mathrm{T}} / \det A$. Use symmetry in part (b).

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

7 If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

8 Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

9 Suppose $\det A = 1$ and you know all the cofactors in C. How can you find A?

10 From the formula $AC^{\mathrm{T}} = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

11 If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example with no zero entries.

12 If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1. Hint: What is $\det A$ times $\det A^{-1}$?

13 Complete the calculation of A^{-1} by cofactors that was started in Example 5.

14 L is lower triangular and S is symmetric. Assume they are invertible:

To invert
triangular L
symmetric S

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is also lower triangular.

(b) Which three pairs of cofactors of S are equal? Then S^{-1} is also symmetric.

(c) The cofactor matrix C of an orthogonal Q will be ______. *Why*?

15 For $n = 5$ the matrix C contains ______ cofactors. Each 4 by 4 cofactor contains ______ terms and each term needs ______ multiplications. Compare with $5^3 = 125$ for the Gauss-Jordan computation of A^{-1} in Section 2.4.

Problems 16–26 are about area and volume by determinants.

16 (a) Find the area of the parallelogram with edges $\boldsymbol{v} = (3, 2)$ and $\boldsymbol{w} = (1, 4)$.

(b) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v} + \boldsymbol{w}$. Draw it.

(c) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{w} - \boldsymbol{v}$. Draw it.

17 A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$ and $(1, 3, 1)$ and $(1, 1, 3)$. Find its volume. Also find the area of each parallelogram face using $\|\boldsymbol{u} \times \boldsymbol{v}\|$.

18 (a) The corners of a triangle are $(2, 1)$ and $(3, 4)$ and $(0, 5)$. What is the area?

(b) Add a corner at $(-1, 0)$ to make a lopsided region (four sides). Find the area.

19 The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

20 The Hadamard matrix H has orthogonal rows. The box is a hypercube!

$$\text{What is} \quad |H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } \mathbf{R}^4?$$

21 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all entries of the matrix are 1 or -1, what are those lengths and the maximum determinant?

22 Show by a picture how a rectangle with area $x_1 y_2$ minus a rectangle with area $x_2 y_1$ produces the same area as our parallelogram.

23 When the edge vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are perpendicular, the volume of the box is $\|\boldsymbol{a}\|$ times $\|\boldsymbol{b}\|$ times $\|\boldsymbol{c}\|$. The matrix $A^{\mathrm{T}}A$ is _____. Find $\det A^{\mathrm{T}}A$ and $\det A$.

24 The box with edges $\boldsymbol{i}$ and $\boldsymbol{j}$ and $\boldsymbol{w} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ has height _____. What is the volume? What is the matrix with this determinant? What is $\boldsymbol{i} \times \boldsymbol{j}$ and what is its dot product with $\boldsymbol{w}$?

25 An n-dimensional cube has how many corners? How many edges? How many $(n-1)$-dimensional faces? The cube in $\mathbf{R}^n$ whose edges are the rows of $2I$ has volume _____. A hypercube computer has parallel processors at the corners with connections along the edges.

26 The triangle with corners $(0,0), (1,0), (0,1)$ has area $\frac{1}{2}$. The pyramid in $\mathbf{R}^3$ with four corners $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ has volume _____. What is the volume of a pyramid in $\mathbf{R}^4$ with five corners at $(0,0,0,0)$ and the rows of I?

Problems 27–30 are about areas dA and volumes dV in calculus.

27 Polar coordinates satisfy $x = r\cos\theta$ and $y = r\sin\theta$. Polar area is $J\,dr\,d\theta$:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

The two columns are orthogonal. Their lengths are _____. Thus $J =$ _____.

28 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho, \partial x/\partial\phi, \partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2\sin\phi$. Then dV in spherical coordinates is $\rho^2 \sin\phi\, d\rho\, d\phi d\theta$, the volume of an infinitesimal "coordinate box".

29 The matrix that connects r, θ to x, y is in Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \theta/\partial x & \partial \theta/\partial y \end{vmatrix} = \begin{vmatrix} \cos\theta & ? \\ ? & ? \end{vmatrix} = ?$$

It is surprising that $\partial r/\partial x = \partial x/\partial r$ (***Calculus***, Gilbert Strang, p. 501). Multiplying the matrices J and J^{-1} gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} = 1$.

30 The triangle with corners $(0,0)$, $(6,0)$, and $(1,4)$ has area ______. When you rotate it by $\theta = 60°$ the area is ______. The determinant of the rotation matrix is

$$J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & ? \\ ? & ? \end{vmatrix} = ?$$

Problems 31–38 are about the triple product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ in three dimensions.

31 A box has base area $\|\boldsymbol{u} \times \boldsymbol{v}\|$. Its perpendicular height is $\|\boldsymbol{w}\| \cos\theta$. Base area times height $=$ volume $= \|\boldsymbol{u} \times \boldsymbol{v}\|\,\|\boldsymbol{w}\| \cos\theta$ which is $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. Compute base area, height, and volume for $\boldsymbol{u} = (2,4,0)$, $\boldsymbol{v} = (-1,3,0)$, $\boldsymbol{w} = (1,2,2)$.

32 The volume of the same box is given more directly by a 3 by 3 determinant. Evaluate that determinant.

33 Expand the 3 by 3 determinant in equation (13) in cofactors of its row u_1, u_2, u_3. This expansion is the dot product of $\boldsymbol{u}$ with the vector ____.

34 Which of the triple products $(\boldsymbol{u} \times \boldsymbol{w}) \cdot \boldsymbol{v}$ and $(\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}$ and $(\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u}$ are the same as $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$? Which orders of the rows $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ give the correct determinant?

35 Let $P = (1,0,-1)$ and $Q = (1,1,1)$ and $R = (2,2,1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTUV$ is a tilted box and compute its volume.

36 Suppose (x,y,z) and $(1,1,0)$ and $(1,2,1)$ lie on a plane through the origin. What determinant is zero? What equation does this give for the plane?

37 Suppose (x,y,z) is a linear combination of $(2,3,1)$ and $(1,2,3)$. What determinant is zero? What equation does this give for the plane of all combinations?

38 (a) Explain from volumes why $\det 2A = 2^n \det A$ for n by n matrices.

(b) For what size matrix is the false statement $\det A + \det A = \det(A + A)$ true?

Challenge Problems

39 If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find A?

40 Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in rows 2–5 to give the determinant. Can you guess a "Jacobi formula" for $\det A$ using 2 by 2 determinants from rows 1–2 *times* 3 by 3 determinants from rows 3–5?

Test your formula on the $-1, 2, -1$ tridiagonal matrix that has determinant $= 6$.

41 The 2 by 2 matrix $AB =$(2 by 3)(3 by 2) has a "Cauchy-Binet formula" for $\det AB$:

$$\det AB = \text{sum of (2 by 2 determinants in } A\text{) (2 by 2 determinants in } B\text{)}$$

(a) Guess which 2 by 2 determinants to use from A and B.

(b) Test your formula when the rows of A are $1,2,3$ and $1,4,7$ with $B = A^{\mathrm{T}}$.

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

Linear equations $Ax = b$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $du/dt = Au$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra, based on $Ax = \lambda x$. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix. Suppose you need the hundredth power A^{100}. The starting matrix A becomes unrecognizable after a few steps, and A^{100} is very close to $[\,.6\ \ .6;\ \ .4\ \ .4\,]$:

$$\underset{A}{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix}} \quad \cdots \qquad \underset{A^{100}}{\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}}$$

A^{100} was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues (here they are 1 and $1/2$) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.

This section will explain how to compute the x's and λ's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use $\det(A - \lambda I) = 0$ to find the eigenvalues for this first example, and then derive it properly in equation (3).

Example 1 The matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda} = \mathbf{1}$ and $\boldsymbol{\lambda} = \frac{1}{2}$. For those numbers, the matrix $A - \lambda I$ becomes *singular* (zero determinant). The eigenvectors x_1 and x_2 are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)x_1 = 0$ is $Ax_1 = x_1$ and the first eigenvector is $(.6, .4)$.

$(A - \frac{1}{2}I)x_2 = 0$ is $Ax_2 = \frac{1}{2}x_2$ and the second eigenvector is $(1, -1)$:

$$x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad Ax_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = x_1 \quad (Ax = x \text{ means that } \lambda_1 = 1)$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}x_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If x_1 is multiplied again by A, we still get x_1. Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2}x_2$, and if we multiply again we get $(\frac{1}{2})^2$ times x_2.

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same x_1 and x_2. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100}$ = very small number.

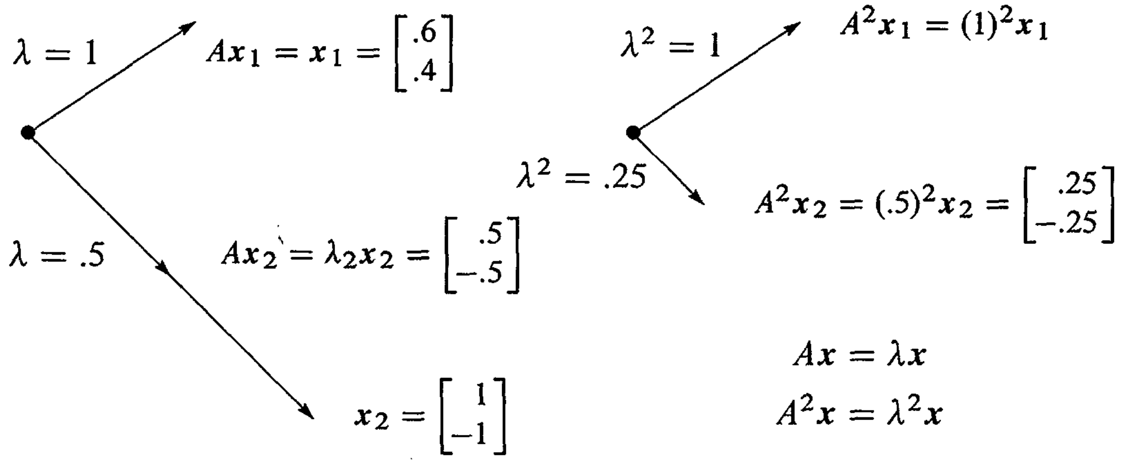


Figure 6.1: The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + (.2)x_2$:

Separate into eigenvectors
$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + (.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \tag{1}$$

Multiplying by A gives $(.7, .3)$, the first column of A^2. Do it separately for $\boldsymbol{x}_1$ and $(.2)\boldsymbol{x}_2$. Of course $A\boldsymbol{x}_1 = \boldsymbol{x}_1$. And A multiplies $\boldsymbol{x}_2$ by its eigenvalue $\frac{1}{2}$:

$$\textbf{Multiply each } \boldsymbol{x}_i \textbf{ by } \boldsymbol{\lambda}_i \qquad A\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + \frac{1}{2}(.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. We didn't need these eigenvectors to find A^2. But it is the good way to do 99 multiplications. At every step $\boldsymbol{x}_1$ is unchanged and $\boldsymbol{x}_2$ is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \boldsymbol{x}_1 + (.2)(\frac{1}{2})^{99}\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_1$ is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector $\boldsymbol{x}_2$ is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A, the closer its columns approach the steady state.

We mention that this particular A is a ***Markov matrix***. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ (as we found). Its eigenvector $\boldsymbol{x}_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 8.3 shows how Markov matrices appear in applications like Google.

For projections we can spot the steady state ($\lambda = 1$) and the nullspace ($\lambda = 0$).

Example 2 **The projection matrix** $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ **has eigenvalues** $\lambda = 1$ **and** $\lambda = 0$.

Its eigenvectors are $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (1, -1)$. For those vectors, $P\boldsymbol{x}_1 = \boldsymbol{x}_1$ (steady state) and $P\boldsymbol{x}_2 = \boldsymbol{0}$ (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special λ's and $\boldsymbol{x}$'s:

1. Each column of $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is **singular**, so $\lambda = 0$ is an eigenvalue.
3. P is **symmetric**, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\boldsymbol{x} = 0\boldsymbol{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\boldsymbol{x} = \boldsymbol{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

$$\textbf{Project each part} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \textbf{projects onto} \quad P\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Special properties of a matrix lead to special eigenvalues and eigenvectors. That is a major theme of this chapter (it is captured in a table at the very end).

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix R (a reflection and at the same time a permutation) is also special.

Example 3 **The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.**

The eigenvector $(1, 1)$ is unchanged by R. The second eigenvector is $(1, -1)$—its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because *reflection* $= 2$(*projection*) $- I$:

$$R = 2P - I \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

Here is the point. If $Px = \lambda x$ then $2Px = 2\lambda x$. The eigenvalues are doubled when the matrix is doubled. Now subtract $Ix = x$. The result is $(2P - I)x = (2\lambda - 1)x$. ***When a matrix is shifted by I, each λ is shifted by*** 1. No change in eigenvectors.

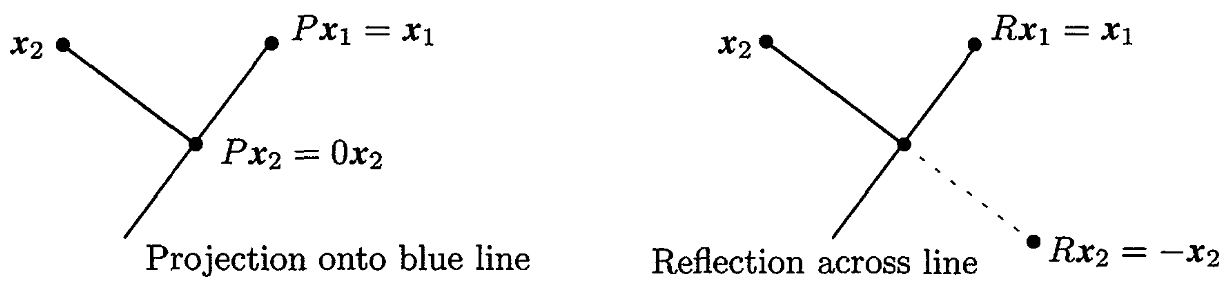


Figure 6.2: Projections P have eigenvalues 1 and 0. Reflections R have $\lambda = 1$ and -1. A typical x changes direction, but not the eigenvectors x_1 and x_2.

Key idea: The eigenvalues of R and P are related exactly as the matrices are related:

The eigenvalues of $R = 2P - I$ are $2(1) - 1 = 1$ and $2(0) - 1 = -1$.

The eigenvalues of R^2 are λ^2. In this case $R^2 = I$. Check $(1)^2 = 1$ and $(-1)^2 = 1$.

The Equation for the Eigenvalues

For projections and reflections we found λ's and x's by geometry: $Px = x, Px = \mathbf{0}$, $Rx = -x$. Now we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving $Ax = \lambda x$.

First move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. ***The eigenvectors make up the nullspace of*** $A - \lambda I$. When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)x = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)x = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $A - \lambda I$ must be zero.*** This is how to recognize an eigenvalue λ:

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

Equation for the eigenvalues $\qquad \det(A - \lambda I) = 0. \qquad (3)$

This "*characteristic polynomial*" $\det(A - \lambda I)$ involves only λ, not x. When A is n by n, equation (3) has degree n. Then A has n eigenvalues (repeats possible!) Each λ leads to x:

For each eigenvalue λ solve $(A - \lambda I)x = 0$ or $Ax = \lambda x$ to find an eigenvector x.

Example 4 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is already singular (zero determinant). Find its λ's and x's.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $Ax = 0x$ has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A - \lambda I) = 0$ is the way to find *all* λ's and x's. Always subtract λI from A:

$$\textbf{\textit{Subtract } } \lambda \textbf{\textit{ from the diagonal to find}} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}. \tag{4}$$

Take the determinant "$ad - bc$" of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the "ad" part is $\lambda^2 - 5\lambda + 4$. The "bc" part, not containing λ, is 2 times 2.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda. \tag{5}$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues} \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5\,.$$

Now find the eigenvectors. Solve $(A - \lambda I)x = 0$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)x = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)x = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)x = 0$ is $Ax = \lambda x$.

We need to emphasize: *There is nothing exceptional about* $\lambda = 0$. Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $Ax = 0x = 0$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*.

In the example, the shifted matrix $A - 5I$ is singular and 5 is the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.

2. ***Find the roots of this polynomial,*** by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.

3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

A note on the eigenvectors of 2 by 2 matrices. When $A - \lambda I$ is singular, both rows are multiples of a vector (a, b). *The eigenvector is any multiple of* $(b, -a)$. The example had $\lambda = 0$ and $\lambda = 5$:

$$\lambda = 0 : \text{rows of } A - 0I \text{ in the direction } (1,2); \text{ eigenvector in the direction } (2,-1)$$
$$\lambda = 5 : \text{rows of } A - 5I \text{ in the direction } (-4,2); \text{ eigenvector in the direction } (2,4).$$

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB's **eig**(A) divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Similarly some n by n matrices don't have n independent eigenvectors. Without n eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Good News, Bad News

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix.* For this A, the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 7$. That agrees with the sum down the main diagonal (the **trace** is $1 + 6$). These quick checks always work:

The product of the n eigenvalues equals the determinant.
The sum of the n eigenvalues equals the sum of the n diagonal entries.

The sum of the entries on the main diagonal is called the ***trace*** of A:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \textbf{\textit{trace}} = a_{11} + a_{22} + \cdots + a_{nn}. \tag{6}$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ's. But when the computation is wrong, they generally tell us so. To compute the correct λ's, go back to $\det(A - \lambda I) = 0$.

The determinant test makes the *product* of the λ's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the λ's is not the sum of the pivots—as the example showed. The individual λ's have almost nothing to do with the pivots. In this new part of linear algebra, the key equation is really *nonlinear*: λ multiplies $\boldsymbol{x}$.

Why do the eigenvalues of a triangular matrix lie on its diagonal?

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 5 *The 90° rotation* $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *has no real eigenvectors. Its eigenvalues are* $\lambda = i$ *and* $\lambda = -i$*. Sum of* λ*'s* = trace = 0. *Product = determinant* = 1.

After a rotation, *no vector* $Q\boldsymbol{x}$ *stays in the same direction as* $\boldsymbol{x}$ (except $\boldsymbol{x} = \mathbf{0}$ which is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

To see how i can help, look at Q^2 which is $-I$. If Q is rotation through 90°, then Q^2 is rotation through 180°. Its eigenvalues are -1 and -1. (Certainly $-I\boldsymbol{x} = -1\boldsymbol{x}$.) Squaring Q will square each λ, so we must have $\lambda^2 = -1$. *The eigenvalues of the* 90° *rotation matrix* Q *are* $+i$ *and* $-i$, because $i^2 = -1$.

Those λ's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are i and $-i$. We meet the imaginary number i also in the eigenvectors:

Complex eigenvectors $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Somehow these complex vectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and $-i$ also illustrate two special properties of Q:

1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.

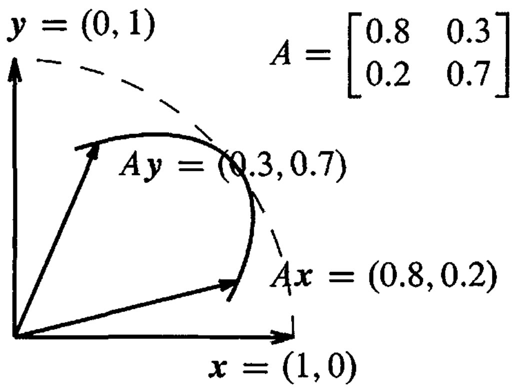
2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix ($A^{\mathrm{T}} = A$) can be compared to a real number. A skew-symmetric matrix ($A^{\mathrm{T}} = -A$) can be compared to an imaginary number. An orthogonal matrix ($A^{\mathrm{T}}A = I$) can be compared to a complex number with $|\lambda| = 1$. For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4.

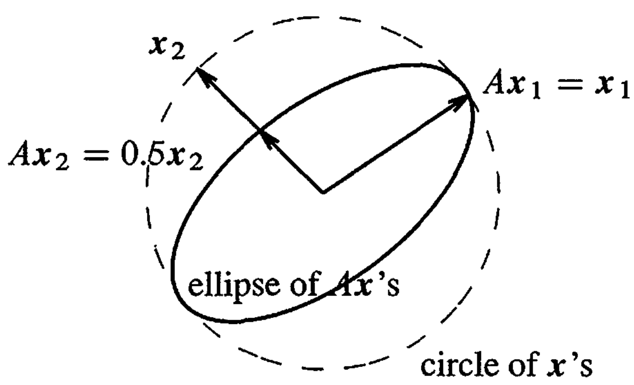
The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Chapter 10 explains the dot product of complex vectors).

Eigshow in MATLAB

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $x = (1, 0)$. *The mouse makes this vector move around the unit circle.* At the same time the screen shows Ax, in color and also moving. Possibly Ax is ahead of x. Possibly Ax is behind x. *Sometimes Ax is parallel to x.* At that parallel moment, $Ax = \lambda x$ (at x_1 and x_2 in the second figure).



These are not eigenvectors



Ax lines up with x at eigenvectors

The eigenvalue λ is the length of Ax, when the unit eigenvector x lines up. The built-in choices for A illustrate three possibilities: 0, 1, or 2 directions where Ax crosses x.

0. There are *no real eigenvectors. Ax stays behind or ahead of x.* This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q.

1. There is only *one* line of eigenvectors (unusual). The moving directions Ax and x touch but don't cross over. This happens for the last 2 by 2 matrix below.

2. There are eigenvectors in *two* independent directions. This is typical! Ax crosses x at the first eigenvector x_1, and it crosses back at the second eigenvector x_2. Then Ax and x cross again at $-x_1$ and $-x_2$.

You can mentally follow x and Ax for these five matrices. Under the matrices I will count their real eigenvectors. Can you see where Ax lines up with x?

$$A = \underset{\mathbf{2}}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} \quad \underset{\mathbf{2}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underset{\mathbf{0}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \quad \underset{\mathbf{1}}{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} \quad \underset{\mathbf{1}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

When A is singular (rank one), its column space is a line. The vector Ax goes up and down that line while x circles around. One eigenvector x is along the line. Another eigenvector appears when $Ax_2 = \mathbf{0}$. Zero is an eigenvalue of a singular matrix.

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that eigenvectors x keep the same direction when multiplied by A.
2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.
4. The sum of the λ's equals the sum down the main diagonal of A (*the trace*). The product of the λ's equals the determinant.
5. Projections P, reflections R, 90° rotations Q have special eigenvalues $1, 0, -1, i, -i$. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda-1)(\lambda-3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)x = \mathbf{0}$ which is $Ax = \lambda x$:

$$\lambda = 1: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A+4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda+4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$$

The trace of A^2 is $5+5$ which agrees with $1+9$. The determinant is $25-16=9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1+2+1=4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $x = (1,1,1)$ gives $Ax = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A. The other factors $(1-\lambda)$ and $(3-\lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue $0, 1, 3$ corresponds to an eigenvector:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad Ax_1 = \mathbf{0}x_1 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad Ax_2 = \mathbf{1}x_2 \qquad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad Ax_3 = \mathbf{3}x_3\,.$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0, 1, 3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] =$ **eig**(A) will produce unit eigenvectors in the columns of the **eigenvector matrix** S. The first one happens to have three minus signs, reversed from $(1,1,1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the ***eigenvalue matrix*** (typed as D but soon called Λ).

Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to _____.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to _____.

9 What do you do to the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^2 = P$ because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$.

(a) $P\boldsymbol{u} = \boldsymbol{u}$ comes from $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\_\_\_\_\_)$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda = 1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v} = \boldsymbol{0}$. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. Use $i^2 = -1$.

15 Every permutation matrix leaves $x = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

16 **The determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$.** Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\qquad}.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{\quad})/2$ and $\lambda = \underline{\qquad}$. Their sum is $\underline{\qquad}$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \underline{\qquad}$.

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}} B$

(c) the eigenvalues of $B^{\mathrm{T}} B$

(d) the eigenvalues of $(B^2 + I)^{-1}$.

20 Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

21 ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because $\underline{\qquad}$. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

22 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that $M^{\mathrm{T}}(1, 1, 1) = (1, 1, 1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's?

23 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that $A^2 = 0$.

24 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

25 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

26 The block B has eigenvalues 1, 2 and C has eigenvalues 3, 4 and D has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

28 Subtract I from the previous A. Find the λ's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

30 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

31 If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda = 11$. *Rank one gives* $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathrm{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

32 Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then _____ would be in the column space.

33 Suppose $\boldsymbol{u}, \boldsymbol{v}$ are orthonormal vectors in $\mathbf{R}^2$, and $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to discover the eigenvalues of A. Check that the trace of A agrees with $\lambda_1 + \lambda_2$.

34 Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Challenge Problems

35 There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P, as in Problem 15?

36 *Is there a real* 2 *by* 2 *matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct a rotation matrix as A (which angle of rotation?).

37 (a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

(b) Show that A has just one line of eigenvectors when $c = 1.6$.

(c) This is a Markov matrix when $c = .8$. Then A^n will approach what matrix A^{∞}?

6.2 Diagonalizing a Matrix

When x is an eigenvector, multiplication by A is just multiplication by a number λ: $Ax = \lambda x$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.*** This is the matrix form of our key idea. We start right off with that one essential computation.

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an ***eigenvector matrix*** S. Then $S^{-1}AS$ is the ***eigenvalue matrix*** Λ:

Eigenvector matrix S
Eigenvalue matrix Λ
$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \tag{1}$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because of the small λ's (the eigenvalues) on its diagonal.

Proof Multiply A times its eigenvectors, which are the columns of S. The first column of AS is Ax_1. That is $\lambda_1 x_1$. Each column of S is multiplied by its eigenvalue λ_i:

A times S
$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}.$$

The trick is to split this matrix AS into S times Λ:

S times Λ
$$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column x_1, as shown. The diagonalization is complete, and we can write $AS = S\Lambda$ in two good ways:

$$AS = S\Lambda \quad \text{is} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}. \tag{2}$$

The matrix S has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors $x_1, \ldots, x_n$ was to diagonalize A. Those eigenvectors in S produce $A = S\Lambda S^{-1}$. You will soon see the simplicity and importance and meaning of the nth power $A^n = S\Lambda^n S^{-1}$.

Example 1 This A is triangular so the λ's are on the diagonal: $\lambda = 1$ and $\lambda = 6$.

$$\textbf{Eigenvectors} \quad \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \quad \underset{S^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = S\Lambda S^{-1}$. Then watch $A^2 = S\Lambda S^{-1}S\Lambda S^{-1}$. When you remove $S^{-1}S = I$, this becomes $S\Lambda^2 S^{-1}$. ***Same eigenvectors in S and squared eigenvalues in*** Λ^2.

The kth power will be $A^k = S\Lambda^k S^{-1}$ which is easy to compute:

$$\textbf{Powers of } A \qquad \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 6^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}.$$

With $k = 1$ *we get* A. *With* $k = 0$ *we get* $A^0 = I$ (and $\lambda^0 = 1$). *With* $k = -1$ *we get* A^{-1}. You can see how $A^2 = [1\ \ 35;\ \ 0\ \ 36]$ fits that formula when $k = 2$.

Here are four small remarks before we use Λ again.

Remark 1 Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are independent. ***Any matrix that has no repeated eigenvalues can be diagonalized.***

Remark 2 ***We can multiply eigenvectors by any nonzero constants.*** $A\boldsymbol{x} = \lambda\boldsymbol{x}$ will remain true. In Example 1, we can divide the eigenvector $(1, 1)$ by $\sqrt{2}$ to produce a unit vector.

Remark 3 The eigenvectors in S come in the same order as the eigenvalues in Λ. To reverse the order in Λ, put $(1, 1)$ before $(1, 0)$ in S:

$$\textbf{New order } 6, 1 \qquad \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\textbf{new}}$$

To diagonalize A we *must* use an eigenvector matrix. From $S^{-1}AS = \Lambda$ we know that $AS = S\Lambda$. Suppose the first column of S is $\boldsymbol{x}$. Then the first columns of AS and $S\Lambda$ are $A\boldsymbol{x}$ and $\lambda_1\boldsymbol{x}$. For those to be equal, $\boldsymbol{x}$ must be an eigenvector.

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices cannot be diagonalized.* Here are two examples:

$$\textbf{Not diagonalizable} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. Nothing is special about $\lambda = 0$, it is the repetition of λ that counts. All eigenvectors of the first matrix are multiples of $(1, 1)$:

$$\begin{array}{l}\textbf{Only one line} \\ \textbf{of eigenvectors}\end{array} \qquad A\boldsymbol{x} = 0\boldsymbol{x} \quad \text{means} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There is no second eigenvector, so the unusual matrix A cannot be diagonalized.

Those matrices are the best examples to test any statement about eigenvectors. In many true-false questions, non-diagonalizable matrices lead to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- ***Invertibility*** is concerned with the ***eigenvalues*** ($\lambda = 0$ or $\lambda \neq 0$).
- ***Diagonalizability*** is concerned with the ***eigenvectors*** (too few or enough for S).

Each eigenvalue has at least one eigenvector! $A - \lambda I$ is singular. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ leads you to $\boldsymbol{x} = \mathbf{0}$, λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$.

Eigenvectors for n different λ's are independent. Then we can diagonalize A.

Independent $\boldsymbol{x}$ from different λ Eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof Suppose $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \mathbf{0}$. Multiply by A to find $c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \mathbf{0}$. Multiply by λ_2 to find $c_1\lambda_2\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \mathbf{0}$. Now subtract one from the other:

$$\text{Subtraction leaves} \quad (\lambda_1 - \lambda_2)c_1\boldsymbol{x}_1 = \mathbf{0}. \quad \text{Therefore } c_1 = 0.$$

Since the λ's are different and $\boldsymbol{x}_1 \neq \mathbf{0}$, we are forced to this conclusion that $c_1 = 0$. Similarly $c_2 = 0$. No other combination gives $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \mathbf{0}$, so the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must be independent.

This proof extends directly to j eigenvectors. Suppose $c_1\boldsymbol{x}_1 + \cdots + c_j\boldsymbol{x}_j = \mathbf{0}$. Multiply by A, multiply by λ_j, and subtract. This removes $\boldsymbol{x}_j$. Now multiply by A and by λ_{j-1} and subtract. This removes $\boldsymbol{x}_{j-1}$. Eventually only $\boldsymbol{x}_1$ is left:

$$(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_j)c_1\boldsymbol{x}_1 = \mathbf{0} \quad \text{which forces} \quad c_1 = 0. \tag{3}$$

Similarly every $c_i = 0$. When the λ's are all different, the eigenvectors are independent. A full set of eigenvectors can go into the columns of the eigenvector matrix S.

Example 2 **Powers of A** The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = S\Lambda S^{-1}$ with those eigenvalues in the diagonal Λ:

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = S\Lambda S^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of S. They are also the eigenvectors of A^2. Watch how A^2 has the same S, and ***the eigenvalue matrix of A^2 is Λ^2***:

$$\textbf{Same } S \textbf{ for } A^2 \qquad A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}. \tag{4}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } A \qquad A^k = S\Lambda^k S^{-1} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } k \to \infty \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The limit has the eigenvector $\boldsymbol{x}_1$ in both columns. We saw this A^∞ on the very first page of the chapter. Now we see it coming, from powers like $A^{100} = S\Lambda^{100}S^{-1}$.

Question *When does $A^k \to$ zero matrix?* **Answer** *All $|\lambda| < 1$.*

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. ***Every new Fibonacci number is the sum of the two previous F's***:

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ***comes from*** $F_{k+2} = F_{k+1} + F_k$.

These numbers turn up in a fantastic variety of applications. Plants and trees grow in a spiral pattern, and a pear tree has 8 growths for every 3 turns. For a willow those numbers can be 13 and 5. The champion is a sunflower of Daniel O'Connell, which had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12}. Our problem is more basic.

Problem: Find the Fibonacci number F_{100} The slow way is to apply the rule $F_{k+2} = F_{k+1} + F_k$ one step at a time. By adding $F_6 = 8$ to $F_7 = 13$ we reach $F_8 = 21$. Eventually we come to F_{100}. Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$. That is a *one-step* rule for vectors, while Fibonacci gave a two-step rule for scalars. We match those rules by putting two Fibonacci numbers into a vector. Then you will see the matrix A.

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}. \text{ The rule } \begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix} \text{ is } \boldsymbol{u}_{k+1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{u}_k. \tag{5}$$

Every step multiplies by $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100} = A^{100}\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \ldots, \quad \boldsymbol{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}.$$

This problem is just right for eigenvalues. Subtract λ from the diagonal of A:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{leads to} \quad \det(A - \lambda I) = \lambda^2 - \lambda - 1.$$

The equation $\lambda^2 - \lambda - 1 = 0$ is solved by the quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$:

$$\textbf{Eigenvalues} \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618.$$

These eigenvalues lead to eigenvectors $\boldsymbol{x}_1 = (\lambda_1, 1)$ and $\boldsymbol{x}_2 = (\lambda_2, 1)$. Step 2 finds the combination of those eigenvectors that gives $\boldsymbol{u}_0 = (1, 0)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right) \quad \text{or} \quad \boldsymbol{u}_0 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{6}$$

Step 3 multiplies $\boldsymbol{u}_0$ by A^{100} to find $\boldsymbol{u}_{100}$. The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ stay separate! They are multiplied by $(\lambda_1)^{100}$ and $(\lambda_2)^{100}$:

$$\textbf{100 steps from } \boldsymbol{u}_0 \qquad \boldsymbol{u}_{100} = \frac{(\lambda_1)^{100}\boldsymbol{x}_1 - (\lambda_2)^{100}\boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{7}$$

We want F_{100} = second component of $\boldsymbol{u}_{100}$. The second components of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1. The difference between $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ is $\lambda_1 - \lambda_2 = \sqrt{5}$. We have F_{100}:

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \approx 3.54 \cdot 10^{20}. \tag{8}$$

Is this a whole number? *Yes.* The fractions and square roots must disappear, because Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ stays with integers. The second term in (8) is less than $\frac{1}{2}$, so it must move the first term to the nearest whole number:

$$k\text{th Fibonacci number} = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k. \tag{9}$$

The ratio of F_6 to F_5 is $8/5 = 1.6$. The ratio F_{101}/F_{100} must be very close to the limiting ratio $\left(1+\sqrt{5}\right)/2$. The Greeks called this number the *"golden mean"*. For some reason a rectangle with sides 1.618 and 1 looks especially graceful.

Matrix Powers A^k

Fibonacci's example is a typical difference equation $\boldsymbol{u}_{k+1} = A\mathbf{u}_k$. ***Each step multiplies by A.*** The solution is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. We want to make clear how diagonalizing the matrix gives a quick way to compute A^k and find $\boldsymbol{u}_k$ in three steps.

The eigenvector matrix S produces $A = S\Lambda S^{-1}$. This is a factorization of the matrix, like $A = LU$ or $A = QR$. The new factorization is perfectly suited to computing powers, because ***every time*** $\boldsymbol{S^{-1}}$ ***multiplies S we get I***:

$$\textbf{Powers of } A \qquad A^k\boldsymbol{u}_0 = (S\Lambda S^{-1})\cdots(S\Lambda S^{-1})\boldsymbol{u}_0 = S\Lambda^k S^{-1}\boldsymbol{u}_0$$

I will split $S\Lambda^k S^{-1}\boldsymbol{u}_0$ into three steps that show how eigenvalues work:

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ of the eigenvectors. Then $\boldsymbol{c} = S^{-1}\boldsymbol{u}_0$.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by $(\lambda_i)^k$. Now we have $\Lambda^k S^{-1}\boldsymbol{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k\boldsymbol{x}_i$ to find the solution $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. This is $S\Lambda^k S^{-1}\boldsymbol{u}_0$.

$$\textbf{Solution for } \boldsymbol{u}_{k+1} = A\boldsymbol{u}_k \qquad \boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{10}$$

In matrix language A^k equals $(S\Lambda S^{-1})^k$ which is S times Λ^k times S^{-1}. In Step 1, the eigenvectors in S lead to the c's in the combination $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$:

$$\textbf{Step 1} \qquad \boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \quad \text{This says that } \boldsymbol{u}_0 = S\boldsymbol{c}. \tag{11}$$

The coefficients in Step 1 are $\boldsymbol{c} = S^{-1}\boldsymbol{u}_0$. Then Step 2 multiplies by Λ^k. The final result $\boldsymbol{u}_k = \sum c_i(\lambda_i)^k\boldsymbol{x}_i$ in Step 3 is the product of S and Λ^k and $S^{-1}\boldsymbol{u}_0$:

$$A^k\boldsymbol{u}_0 = S\Lambda^k S^{-1}\boldsymbol{u}_0 = S\Lambda^k\boldsymbol{c} = \begin{bmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{12}$$

This result is exactly $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n$. It solves $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$.

Example 3 Start from $\boldsymbol{u}_0 = (1, 0)$. Compute $A^k\boldsymbol{u}_0$ when S and Λ contain these eigenvectors and eigenvalues:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 2 \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This matrix is like Fibonacci except the rule is changed to $F_{k+2} = F_{k+1} + 2F_k$. The new numbers start $0, 1, 1, 3$. They grow faster from $\lambda = 2$.

Solution in three steps Find $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ and then $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + c_2(\lambda_2)^k\boldsymbol{x}_2$

Step 1 $\qquad \boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{so} \quad c_1 = c_2 = \frac{1}{3}$

Step 2 $\qquad$ Multiply the two parts by $(\lambda_1)^k = 2^k$ and $(\lambda_2)^k = (-1)^k$

Step 3 $\qquad$ Combine eigenvectors $c_1(\lambda_1)^k\boldsymbol{x}_1$ and $c_2(\lambda_2)^k\boldsymbol{x}_2$ into $\boldsymbol{u}_k$:

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 \qquad \boldsymbol{u}_k = \frac{1}{3}2^k\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}(-1)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{13}$$

The new number is $F_k = (2^k - (-1)^k)/3$. After $0, 1, 1, 3$ comes $F_4 = 15/3 = 5$.

Behind these numerical examples lies a fundamental idea: ***Follow the eigenvectors***. In Section 6.3 this is the crucial link from linear algebra to differential equations (powers λ^k will become $e^{\lambda t}$). Chapter 7 sees the same idea as "transforming to an eigenvector basis." The best example of all is a ***Fourier series***, built from the eigenvectors of d/dx.

Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. Eigenvectors (geometric) There are nonzero solutions to $Ax = \lambda x$.

2. Eigenvalues (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two different ways to count its multiplicity. Always GM $\leq$ AM for each λ:

1. (Geometric Multiplicity = GM) Count the **independent eigenvectors** for λ. This is the dimension of the nullspace of $A - \lambda I$.

2. (Algebraic Multiplicity = AM) Count the **repetitions of λ** among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, that eigenvalue has AM $= 3$ and GM $= 1, 2$, or 3.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM $= 2$) with only one eigenvector (GM $= 1$).

$$\begin{array}{l}\textbf{AM = 2}\\ \textbf{GM = 1}\end{array} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2. \quad \begin{array}{l}\boldsymbol{\lambda = 0, 0} \textbf{ but}\\ \textbf{1 eigenvector}\end{array}$$

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM $= 2$. But there is only one eigenvector $x = (1, 0)$. ***This shortage of eigenvectors when* GM *is below* AM *means that A is not diagonalizable.***

The vector called "repeats" in the Teaching Code **eigval** gives the algebraic multiplicity AM for each eigenvalue. When repeats $= [1 \ \ 1 \ldots \ 1]$ we know that the n eigenvalues are all different and A is diagonalizable. The sum of all components in "repeats" is always n, because every nth degree equation $\det(A - \lambda I) = 0$ has n roots (counting repetitions).

The diagonal matrix **D** in the Teaching Code **eigvec** gives the geometric multiplicity GM for each eigenvalue. This counts the independent eigenvectors. The total number of independent eigenvectors might be less than n. Then A is not diagonalizable.

We emphasize again: $\lambda = 0$ makes for easy computations, but these three matrices also have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10, determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM $= 2$. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM $= 1$. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

Eigenvalues of AB and $A+B$

The first guess about the eigenvalues of AB is not true. An eigenvalue λ of A times an eigenvalue β of B usually does *not* give an eigenvalue of AB:

False proof $$ABx = A\beta x = \beta Ax = \beta\lambda x. \tag{14}$$

It seems that β times λ is an eigenvalue. When x is an eigenvector for A and B, this proof is correct. ***The mistake is to expect that A and B automatically share the same eigenvector x.*** Usually they don't. Eigenvectors of A are not generally eigenvectors of B. A and B could have all zero eigenvalues while 1 is an eigenvalue of AB:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{then} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$. Here $\lambda + \beta = 0$ while $A + B$ has eigenvalues 1 and -1. (At least they add to zero.)

The false proof suggests what is true. Suppose x really is an eigenvector for both A and B. Then we do have $ABx = \lambda\beta x$ and $BAx = \lambda\beta x$. When all n eigenvectors are shared, we *can* multiply eigenvalues. The test $AB = BA$ for shared eigenvectors is important in quantum mechanics—time out to mention this application of linear algebra:

Commuting matrices share eigenvectors Suppose both A and B can be diagonalized. They share the same eigenvector matrix S if and only if $AB = BA$.

Heisenberg's uncertainty principle In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$ (these are infinite matrices). Then we cannot have $Px = \mathbf{0}$ at the same time as $Qx = \mathbf{0}$ (unless $x = \mathbf{0}$). If we knew the position exactly, we could not also know the momentum exactly. Problem 28 derives Heisenberg's uncertainty principle $\|Px\|\ \|Qx\| \geq \frac{1}{2}\|x\|^2$.

■ REVIEW OF THE KEY IDEAS ■

1. If A has n independent eigenvectors $x_1, \ldots, x_n$, they go into the columns of S.

   A is diagonalized by S $\qquad S^{-1}AS = \Lambda \quad \text{and} \quad A = S\Lambda S^{-1}.$

2. The powers of A are $A^k = S\Lambda^k S^{-1}$. The eigenvectors in S are unchanged.
3. The eigenvalues of A^k are $(\lambda_1)^k, \ldots, (\lambda_n)^k$ in the matrix Λ^k.
4. The solution to $u_{k+1} = Au_k$ starting from u_0 is $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0$:

   $$u_k = c_1(\lambda_1)^k x_1 + \cdots + c_n(\lambda_n)^k x_n \quad \text{provided} \quad u_0 = c_1 x_1 + \cdots + c_n x_n.$$

   That shows Steps 1, 2, 3 (c's from $S^{-1}u_0$, λ^k from Λ^k, and x's from S)

5. A is diagonalizable if every eigenvalue has enough eigenvectors (GM = AM).

■ WORKED EXAMPLES ■

6.2 A The **Lucas numbers** are like the Fibonacci numbers except they start with $L_1 = 1$ and $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$, the next Lucas numbers are $4, 7, 11, 18$. Show that the Lucas number L_{100} is $\lambda_1^{100} + \lambda_2^{100}$.

Note The key point is that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1^2 + \lambda_2^2 = 3$, when the λ's are $(1 \pm \sqrt{5})/2$. The Lucas number L_k is $\boldsymbol{\lambda_1^k + \lambda_2^k}$, since this is correct for L_1 and L_2.

Solution $\boldsymbol{u}_{k+1} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\boldsymbol{u}_k$ is the same as for Fibonacci, because $L_{k+2} = L_{k+1} + L_k$ is the same rule (with different starting values). The equation becomes a 2 by 2 system:

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} L_{k+1} \\ L_k \end{bmatrix}. \quad \text{The rule } \begin{array}{l} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{array} \quad \text{is } \boldsymbol{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}_k.$$

The eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ still come from $\lambda^2 = \lambda + 1$:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Now solve $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \boldsymbol{u}_1 = (3, 1)$. The solution is $c_1 = \lambda_1$ and $c_2 = \lambda_2$. Check:

$$\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boldsymbol{u}_1$$

$\boldsymbol{u}_{100} = A^{99}\boldsymbol{u}_1$ tells us the Lucas numbers (L_{101}, L_{100}). The second components of the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1, so the second component of $\boldsymbol{u}_{100}$ is the answer we want:

Lucas number $\qquad L_{100} = c_1\lambda_1^{99} + c_2\lambda_2^{99} = \lambda_1^{100} + \lambda_2^{100}.$

Lucas starts faster than Fibonacci, and ends up larger by a factor near $\sqrt{5}$.

6.2 B Find the inverse and the eigenvalues and the determinant of A:

$$A = 5 * \mathbf{eye}(4) - \mathbf{ones}(4) = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Describe an eigenvector matrix S that gives $S^{-1}AS = \Lambda$.

Solution What are the eigenvalues of the all-ones matrix **ones**(4)? Its rank is certainly 1, so three eigenvalues are $\lambda = 0, 0, 0$. Its trace is 4, so the other eigenvalue is $\lambda = 4$. Subtract this all-ones matrix from $5I$ to get our matrix A:

Subtract the eigenvalues 4, 0, 0, 0 from 5, 5, 5, 5. The eigenvalues of A are 1, 5, 5, 5.

The determinant of A is 125, the product of those four eigenvalues. The eigenvector for $\lambda = 1$ is $\boldsymbol{x} = (1, 1, 1, 1)$ or (c, c, c, c). The other eigenvectors are perpendicular to $\boldsymbol{x}$ (since A is symmetric). The nicest eigenvector matrix S is the symmetric orthogonal Hadamard matrix H (normalized to unit column vectors):

$$\textbf{Orthonormal eigenvectors} \quad S = H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = H^{\mathrm{T}} = H^{-1}.$$

The eigenvalues of A^{-1} are $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$. The eigenvectors are not changed so $A^{-1} = H\Lambda^{-1}H^{-1}$. The inverse matrix is surprisingly neat:

$$A^{-1} = \frac{1}{5} * (\textbf{eye}(4) + \textbf{ones}(4)) = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

A is a rank-one change from $5I$. So A^{-1} is a rank-one change $I/5 + \textbf{ones}/5$.

The determinant 125 counts the "spanning trees" in a graph with 5 nodes (all edges included). *Trees have no loops* (graphs and trees are in Section 8.2).

With 6 nodes, the matrix $6 * \textbf{eye}(5) - \textbf{ones}(5)$ has the five eigenvalues $1, 6, 6, 6, 6$.

Problem Set 6.2

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and S.

1 (a) Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = S\Lambda S^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

3 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

4 True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix S is triangular, then S^{-1} is triangular. Prove that A is also triangular.

6 Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 11–14 are about diagonalizability.

11 True or false: If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

12 True or false: If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $S\Lambda S^{-1}$.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $Ax = 5x$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

14 The matrix $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is ____. Change one entry to make A diagonalizable. Which entries could you change?

Questions 15–19 are about powers of matrices.

15 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ____.

17 Find Λ and S to diagonalize A_2 in Problem 15. What is $(A_2)^{10}u_0$ for these u_0?

$$u_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

18 Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

19 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

20 Suppose $A = S\Lambda S^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1 \lambda_2 \cdots \lambda_n$. This quick proof only works when A can be ____.

21 Show that trace ST = trace TS, by adding the diagonal entries of ST and TS:

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Choose T as ΛS^{-1}. Then $S\Lambda S^{-1}$ has the same trace as $\Lambda S^{-1} S = \Lambda$. The trace of A equals the trace of Λ = sum of the eigenvalues.

22 $AB - BA = I$ is impossible since the left side has trace = _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

23 If $A = S\Lambda S^{-1}$, diagonalize the block matrix $B = \left[\begin{smallmatrix} A & 0 \\ 0 & 2A \end{smallmatrix}\right]$. Find its eigenvalue and eigenvector (block) matrices.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix S. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same S). What is this subspace when $S = I$? What is its dimension?

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So a matrix with $A^2 = A$ can be diagonalized.

26 (Recommended) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$. If $\lambda = 0$ then $\boldsymbol{x}$ is in the nullspace. If $\lambda \neq 0$ then $\boldsymbol{x}$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}\,S^{-1}$. Why is there no real matrix square root of B?

28 **(Heisenberg's Uncertainty Principle)** $AB - BA = I$ can happen for infinite matrices with $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. Then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} \leq 2\|A\boldsymbol{x}\|\,\|B\boldsymbol{x}\|.$$

Explain that last step by using the Schwarz inequality. Then Heisenberg's inequality says that $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\|$ times $\|B\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

29 If A and B have the same λ's with the same independent eigenvectors, their factorizations into _____ are the same. So $A = B$.

30 Suppose the same S diagonalizes both A and B. They have the same eigenvectors in $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Prove that $AB = BA$.

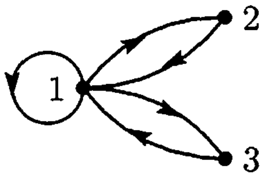
31 (a) If $A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) = \textit{zero matrix}$.

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

32 Substitute $A = S\Lambda S^{-1}$ into the product $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

33 Find the eigenvalues and eigenvectors and the kth power of A. For this "adjacency matrix" the i, j entry of A^k counts the k-step paths from i to j.

1's in A show edges between nodes $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$



34 If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$, show that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also a diagonal matrix. B has the same eigen_____ as A but different eigen_____ . These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns **a, b, c, d**—find the rank of the 4 by 4 matrix.

35 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \qquad B^{1024} = I \qquad C^{1024} = -C \qquad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

Challenge Problems

36 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = S\Lambda S^{-1}$. The eigenvectors (columns of S) are $(1, i)$ and $(i, 1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

37 The transpose of $A = S\Lambda S^{-1}$ is $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\Lambda S^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(S^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***. How do you multiply matrices to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = S\Lambda S^{-1} = \lambda_1 \boldsymbol{x}_1 \boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{x}_n \boldsymbol{y}_n^{\mathrm{T}}.$$

38 The inverse of $A = \mathbf{eye}(n) + \mathbf{ones}(n)$ is $A^{-1} = \mathbf{eye}(n) + C * \mathbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

6.3 Applications to Differential Equations

Eigenvalues and eigenvectors and $A = S\Lambda S^{-1}$ are perfect for matrix powers A^k. They are also perfect for differential equations $du/dt = Au$. This section is mostly linear algebra, but to read it you need one fact from calculus: *The derivative of* $e^{\lambda t}$ *is* $\lambda e^{\lambda t}$. The whole point of the section is this: **To convert constant-coefficient differential equations into linear algebra.**

The ordinary scalar equation $du/dt = u$ is solved by $u = e^t$. The equation $du/dt = 4u$ is solved by $u = e^{4t}$. The solutions are exponentials!

$$\textbf{One equation} \qquad \frac{du}{dt} = \lambda u \quad \text{has the solutions} \quad u(t) = Ce^{\lambda t}. \tag{1}$$

The number C turns up on both sides of $du/dt = \lambda u$. At $t = 0$ the solution $Ce^{\lambda t}$ reduces to C (because $e^0 = 1$). By choosing $C = u(0)$, ***the solution that starts from*** $u(0)$ ***at*** $t = 0$ ***is*** $u(t) = u(0)e^{\lambda t}$.

We just solved a 1 by 1 problem. Linear algebra moves to n by n. The unknown is a vector $\boldsymbol{u}$ (now boldface). It starts from the initial vector $\boldsymbol{u}(0)$, which is given. The n equations contain a square matrix A. We expect n exponentials $e^{\lambda t}\boldsymbol{x}$ in $\boldsymbol{u}(t)$.

$$\boldsymbol{n}\ \textbf{equations} \qquad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \text{starting from the vector } \boldsymbol{u}(0) \text{ at } t = 0. \tag{2}$$

These differential equations are *linear*. If $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions, so is $C\boldsymbol{u}(t) + D\boldsymbol{v}(t)$. We will need n constants like C and D to match the n components of $\boldsymbol{u}(0)$. Our first job is to find n "pure exponential solutions" $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ by using $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Notice that A is a *constant* matrix. In other linear equations, A changes as t changes. In nonlinear equations, A changes as $\boldsymbol{u}$ changes. We don't have those difficulties. $d\boldsymbol{u}/dt = A\boldsymbol{u}$ is "linear with constant coefficients". Those and only those are the differential equations that we will convert directly to linear algebra. The main point will be:

Solve linear constant coefficient equations by exponentials $e^{\lambda t}\boldsymbol{x}$, ***when*** $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$

Our pure exponential solution will be $e^{\lambda t}$ times a fixed vector $\boldsymbol{x}$. You may guess that λ is an eigenvalue of A, and $\boldsymbol{x}$ ***is the eigenvector***. Substitute $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ into the equation $d\boldsymbol{u}/dt = A\boldsymbol{u}$ to prove you are right (the factor $e^{\lambda t}$ will cancel):

$$\textbf{Use } \boldsymbol{u} = e^{\lambda t}\boldsymbol{x} \textbf{ when } A\boldsymbol{x} = \lambda\boldsymbol{x} \qquad \frac{d\boldsymbol{u}}{dt} = \lambda e^{\lambda t}\boldsymbol{x} \quad \text{agrees with} \quad A\boldsymbol{u} = Ae^{\lambda t}\boldsymbol{x} \tag{3}$$

All components of this special solution $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ share the same $e^{\lambda t}$. The solution grows when $\lambda > 0$. It decays when $\lambda < 0$. If λ is a complex number, its real part decides growth or decay. The imaginary part ω gives oscillation $e^{i\omega t}$ like a sine wave.

Example 1 Solve $du/dt = Au = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]u$ starting from $u(0) = \left[\begin{smallmatrix}4\\ 2\end{smallmatrix}\right]$.

This is a vector equation for u. It contains two scalar equations for the components y and z. They are "coupled together" because the matrix is not diagonal:

$$\frac{du}{dt} = Au \qquad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \text{ means that } \frac{dy}{dt} = z \text{ and } \frac{dz}{dt} = y.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it:

$$\frac{d}{dt}(y+z) = z + y \qquad \text{and} \qquad \frac{d}{dt}(y-z) = -(y-z).$$

The combination $y + z$ grows like e^t, because it has $\lambda = 1$. The combination $y - z$ decays like e^{-t}, because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $du/dt = Au$, looking for these special combinations. The eigenvectors and eigenvalues of A will do it for us.

This matrix A has eigenvalues 1 and -1. The eigenvectors are $(1, 1)$ and $(1, -1)$. The pure exponential solutions u_1 and u_2 take the form $e^{\lambda t}x$ with $\lambda = 1$ and -1:

$$u_1(t) = e^{\lambda_1 t}x_1 = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2(t) = e^{\lambda_2 t}x_2 = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4}$$

Notice: These u's are eigenvectors. They satisfy $Au_1 = u_1$ and $Au_2 = -u_2$, just like x_1 and x_2. The factors e^t and e^{-t} change with time. Those factors give $du_1/dt = u_1 = Au_1$ and $du_2/dt = -u_2 = Au_2$. We have two solutions to $du/dt = Au$. To find all other solutions, multiply those special solutions by any C and D and add:

$$\textbf{Complete solution} \qquad u(t) = Ce^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \tag{5}$$

With these constants C and D, we can match any starting vector $u(0)$. Set $t = 0$ and $e^0 = 1$. The problem asked for the initial value $u(0) = (4, 2)$:

$$u(0) \textbf{ gives } C, D \qquad C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + D\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{yields} \quad C = 3 \quad \text{and} \quad D = 1.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is solved.

The same three steps that solved $u_{k+1} = Au_k$ now solve $du/dt = Au$:

1. Write $u(0)$ as a combination $c_1x_1 + \cdots + c_nx_n$ of the eigenvectors of A.

2. Multiply each eigenvector x_i by $e^{\lambda_i t}$.

3. The solution is the combination of pure solutions $e^{\lambda t}x$:

$$u(t) = c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n. \tag{6}$$

Not included: If two λ's are equal, with only one eigenvector, another solution is needed. (It will be $te^{\lambda t}x$). Step 1 needs $A = S\Lambda S^{-1}$ to be diagonalizable: a basis of eigenvectors.

Example 2 Solve $du/dt = Au$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}.$$

The eigenvectors are $\boldsymbol{x}_1 = (1,0,0)$ and $\boldsymbol{x}_2 = (1,1,0)$ and $\boldsymbol{x}_3 = (1,1,1)$.

Step 1 The vector $\boldsymbol{u}(0) = (9,7,4)$ is $2\boldsymbol{x}_1 + 3\boldsymbol{x}_2 + 4\boldsymbol{x}_3$. Thus $(c_1, c_2, c_3) = (2,3,4)$.

Step 2 The pure exponential solutions are $e^t \boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.

Step 3 The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = 2e^t\boldsymbol{x}_1 + 3e^{2t}\boldsymbol{x}_2 + 4e^{3t}\boldsymbol{x}_3$.

The coefficients 2, 3, 4 came from solving the linear equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = \boldsymbol{u}(0)$:

$$\begin{bmatrix} & & \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} \quad \text{which is} \quad Sc = \boldsymbol{u}(0). \tag{7}$$

You now have the basic idea—how to solve $du/dt = Au$. The rest of this section goes further. We solve equations that contain *second* derivatives, because they arise so often in applications. We also decide whether $\boldsymbol{u}(t)$ approaches zero or blows up or just oscillates.

At the end comes the ***matrix exponential*** e^{At}. Then $e^{At}\boldsymbol{u}(0)$ solves the equation $du/dt = Au$ in the same way that $A^k\boldsymbol{u}_0$ solves the equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. In fact we ask whether $\boldsymbol{u}_k$ approaches $\boldsymbol{u}(t)$. Example 3 will show how "difference equations" help to solve differential equations. You will see real applications.

All these steps use the λ's and the $\boldsymbol{x}$'s. This section solves the constant coefficient problems that turn into linear algebra. It clarifies these simplest but most important differential equations—whose solution is completely based on $e^{\lambda t}$.

Second Order Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (*Newton's Law*!). The force includes the damping $-by'$ and the elastic restoring force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \text{becomes} \quad (m\lambda^2 + b\lambda + k)e^{\lambda t} = 0. \tag{8}$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a vector equation (first derivative only). Suppose $m = 1$. The unknown vector $\boldsymbol{u}$ has components y and y'. The equation is $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -ky - by' \end{aligned} \qquad \text{converts to} \qquad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}. \tag{9}$$

The first equation $dy/dt = y'$ is trivial (but true). The second equation connects y'' to y' and y. Together the equations connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

The equation for the λ's is the same! It is still $\lambda^2 + b\lambda + k = 0$, since $m = 1$. The roots λ_1 and λ_2 are now *eigenvalues of A*. The eigenvectors and the solution are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of $\boldsymbol{u}(t)$ has $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$—the same solution as before. It can't be anything else. In the second component of $\boldsymbol{u}(t)$ you see the velocity dy/dt. The vector problem is completely consistent with the scalar problem.

Example 3 *Motion around a circle with $y'' + y = 0$ and $y = \cos t$*

This is our master equation with mass $m = 1$ and stiffness $k = 1$ and no damping dy'. Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\lambda^2 + 1 = 0$. *The roots are $\lambda = i$ and* $\lambda = -i$. Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos t$.

As a first-order system, the initial values $y(0) = 1$, $y'(0) = 0$ go into $\boldsymbol{u}(0) = (1, 0)$:

$$\text{Use } y'' = -y \qquad \frac{d\boldsymbol{u}}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{10}$$

The eigenvalues of A are again $\lambda = i$ and $\lambda = -i$ (no surprise). A is anti-symmetric with eigenvectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The combination that matches $\boldsymbol{u}(0) = (1, 0)$ is $\frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$. Step 2 multiplies $\frac{1}{2}$ by e^{it} and e^{-it}. Step 3 combines the pure oscillations into $\boldsymbol{u}(t)$ to find $y = \cos t$ as expected:

$$\boldsymbol{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}. \qquad \text{This is } \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

All good. The vector $\boldsymbol{u} = (\cos t, -\sin t)$ goes around a circle (Figure 6.3). The radius is 1 because $\cos^2 t + \sin^2 t = 1$.

To display a circle on a screen, replace $y'' = -y$ by a ***finite difference equation***. Here are three choices using $Y(t+\Delta t) - 2Y(t) + Y(t-\Delta t)$. Divide by $(\Delta t)^2$ to approximate y''.

Forward from $n-1$ **Centered at n** **Backward from $n+1$**	$\dfrac{Y_{n+1}-2Y_n+Y_{n-1}}{(\Delta t)^2} = \begin{matrix}-Y_{n-1}\\ -Y_n \\ -Y_{n+1}\end{matrix}$	(11)

Figure 6.3 shows the exact $y(t) = \cos t$ completing a circle at $t = 2\pi$. The three difference methods *don't* complete a perfect circle in 32 steps of length $\Delta t = 2\pi/32$. Those pictures will be explained by eigenvalues:

Forward $|\lambda| > 1$ (spiral out)   Centered $|\lambda| = 1$ (best)   Backward $|\lambda| < 1$ (spiral in)

The 2-step equations (11) reduce to 1-step systems. In the continuous case $\boldsymbol{u}$ was (y, y'). Now the discrete unknown is $\boldsymbol{U}_n = (Y_n, Z_n)$ after n time steps Δt from $\boldsymbol{U}_0$:

Forward	$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\, Z_n \\ Z_{n+1} &= Z_n - \Delta t\, Y_n \end{aligned}$	becomes	$\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = A\boldsymbol{U}_n.$	(12)

Those are like $Y' = Z$ and $Z' = -Y$. Eliminating Z will bring back equation (11). From the equation for Y_{n+1}, subtract the same equation for Y_n. That produces $Y_{n+1} - Y_n$ on the left side and $Y_n - Y_{n-1}$ on the right side. Also on the right is $\Delta t(Z_n - Z_{n-1})$, which is $-(\Delta t)^2 Y_{n-1}$ from the Z equation. This is the forward choice in equation (11).

My question is simple. Do the points (Y_n, Z_n) stay on the circle $Y^2 + Z^2 = 1$? They could grow to infinity, they could decay to $(0, 0)$. The answer must be found in the eigenvalues of A. $|\lambda|^2$ is $1 + (\Delta t)^2$, the determinant of A. Figure 6.3 shows growth!

We are taking powers A^n and not e^{At}, so we test the magnitude $|\lambda|$ and not the real part of λ.

Eigenvalues of A	$\lambda = 1 \pm i\,\Delta t$	$\lvert\lambda\rvert > 1$ and (Y_n, Z_n) spirals out

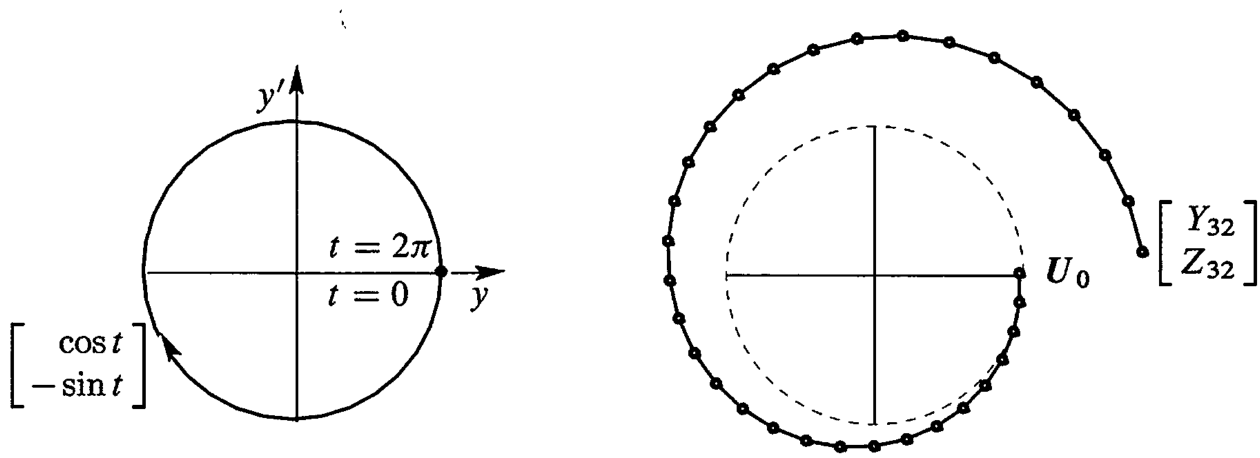


Figure 6.3: Exact $\boldsymbol{u} = (\cos t, -\sin t)$ on a circle. Forward Euler spirals out (32 steps).

The backward choice in (11) will do the opposite in Figure 6.4. Notice the difference:

$$\textbf{Backward}\quad \begin{matrix} Y_{n+1} = Y_n + \Delta t\, Z_{n+1} \\ Z_{n+1} = Z_n - \Delta t\, Y_{n+1} \end{matrix} \quad \text{is} \quad \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = U_n. \quad (13)$$

That matrix is A^{T}. It still has $\lambda = 1 \pm i\Delta t$. But now we *invert* it to reach U_{n+1}. When A^{T} has $|\lambda| > 1$, its inverse has $|\lambda| < 1$. That explains why the solution spirals in to $(0, 0)$ for backward differences.

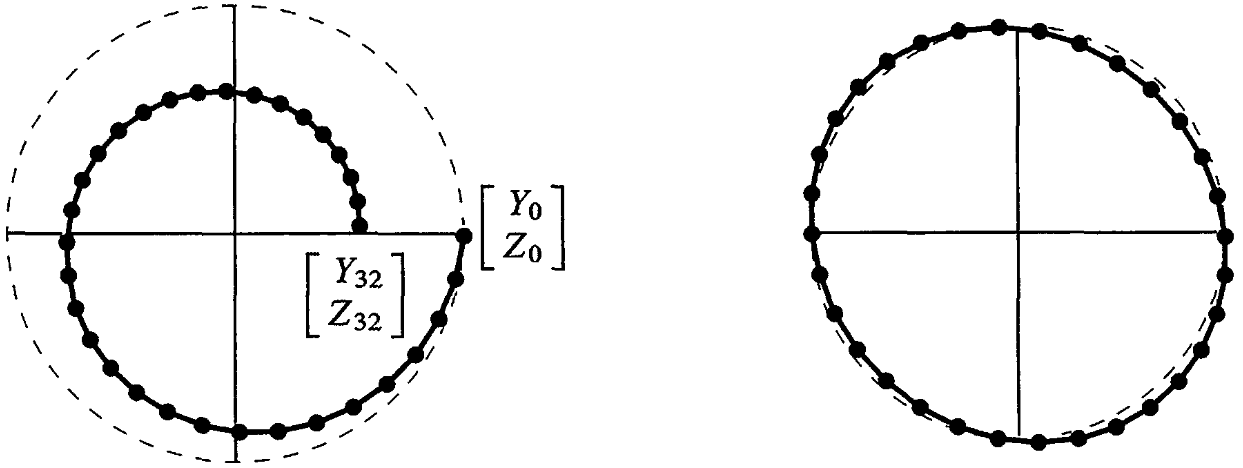


Figure 6.4: Backward differences spiral in. Leapfrog stays near the circle $Y_n^2 + Z_n^2 = 1$.

On the right side of Figure 6.4 you see 32 steps with the *centered* choice. The solution stays close to the circle (Problem 28) if $\Delta t < 2$. This is the **leapfrog method**. The second difference $Y_{n+1} - 2Y_n + Y_{n-1}$ "leaps over" the center value Y_n.

This is the way a chemist follows the motion of molecules (molecular dynamics leads to giant computations). Computational science is lively because one differential equation can be replaced by many difference equations—some unstable, some stable, some neutral. Problem 30 has a fourth (good) method that stays right on the circle.

Note Real engineering and real physics deal with systems (not just a single mass at one point). The unknown y is a vector. The coefficient of y'' is a *mass matrix* M, not a number m. The coefficient of y is a *stiffness matrix* K, not a number k. The coefficient of y' is a damping matrix which might be zero.

The equation $My'' + Ky = f$ is a major part of computational mechanics. It is controlled by the eigenvalues of $M^{-1}K$ in $Kx = \lambda Mx$.

Stability of 2 by 2 Matrices

For the solution of $du/dt = Au$, there is a fundamental question. *Does the solution approach* $u = 0$ *as* $t \to \infty$? Is the problem *stable*, by dissipating energy? The solutions in Examples 1 and 2 included e^t (unstable). Stability depends on the eigenvalues of A.

The complete solution $u(t)$ is built from pure solutions $e^{\lambda t}x$. If the eigenvalue λ is real, we know exactly when $e^{\lambda t}$ will approach zero: *The number λ must be negative.*

If the eigenvalue is a complex number $\lambda = r + is$, *the real part r must be negative.* When $e^{\lambda t}$ splits into $e^{rt}e^{ist}$, the factor e^{ist} has absolute value fixed at 1:

$$e^{ist} = \cos st + i \sin st \quad \text{has} \quad |e^{ist}|^2 = \cos^2 st + \sin^2 st = 1.$$

The factor e^{rt} controls growth ($r > 0$ is instability) or decay ($r < 0$ is stability).

The question is: ***Which matrices have negative eigenvalues?*** More accurately, when are the ***real parts*** of the λ's all negative? 2 by 2 matrices allow a clear answer.

Stability A is ***stable*** and $\boldsymbol{u}(t) \to \mathbf{0}$ when all eigenvalues have ***negative real parts***. The 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must pass two tests:

$\lambda_1 + \lambda_2 < 0$	The trace $T = a + d$ must be negative.
$\lambda_1\lambda_2 > 0$	The determinant $D = ad - bc$ must be positive.

Reason If the λ's are real and negative, their sum is negative. This is the trace T. Their product is positive. This is the determinant D. The argument also goes in the reverse direction. If $D = \lambda_1\lambda_2$ is positive, then λ_1 and λ_2 have the same sign. If $T = \lambda_1 + \lambda_2$ is negative, that sign will be negative. We can test T and D.

If the λ's are complex numbers, they must have the form $r + is$ and $r - is$. Otherwise T and D will not be real. The determinant D is automatically positive, since $(r + is)(r - is) = r^2 + s^2$. The trace T is $r + is + r - is = 2r$. So a negative trace means that the real part r is negative and the matrix is stable. Q.E.D.

Figure 6.5 shows the parabola $T^2 = 4D$ which separates real from complex eigenvalues. Solving $\lambda^2 - T\lambda + D = 0$ leads to $\sqrt{T^2 - 4D}$. This is real below the parabola and imaginary above it. The stable region is the *upper left quarter* of the figure—where the trace T is negative and the determinant D is positive.

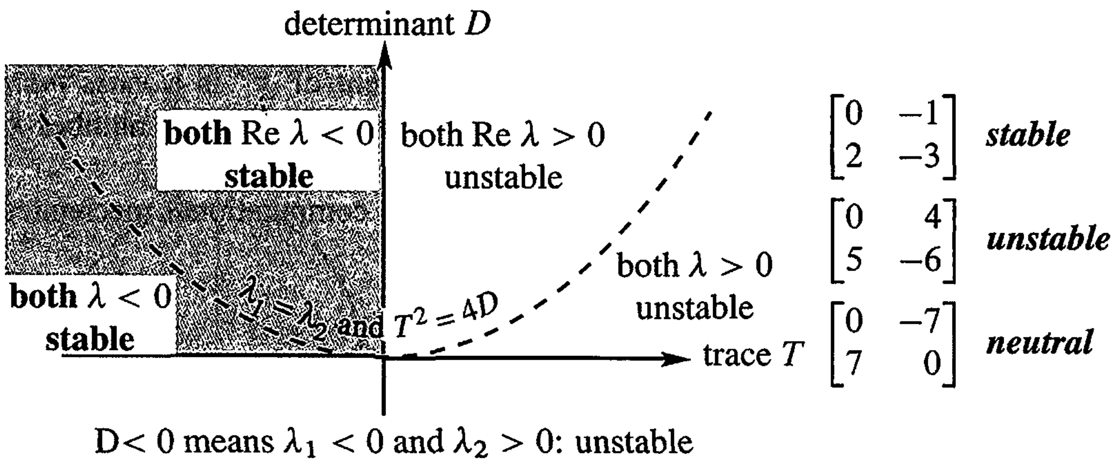


Figure 6.5: A 2 by 2 matrix is stable ($\boldsymbol{u}(t) \to \mathbf{0}$) when trace < 0 and det > 0.

The Exponential of a Matrix

We want to write the solution $u(t)$ in a new form $e^{At}u(0)$. This gives a perfect parallel with $A^k u_0$ in the previous section. First we have to say what e^{At} means, with a matrix in the exponent. To define e^{At} for matrices, we copy e^x for numbers.

The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. When you substitute any square matrix At for x, this series defines the matrix exponential e^{At}:

Matrix exponential e^{At}	$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$	(14)
Its t derivative is Ae^{At}	$A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$	
Its eigenvalues are $e^{\lambda t}$	$(I + At + \frac{1}{2}(At)^2 + \cdots)x = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots)x$	

The number that divides $(At)^n$ is "n factorial". This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after $1, 2, 6$ are $4! = 24$ and $5! = 120$. They grow quickly. The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}u(0)$ solves the differential equation with one quick formula—*even if there is a shortage of eigenvectors.*

I will use this series in Example 4, to see it work with a missing eigenvector. It will produce $te^{\lambda t}$. First let me reach $Se^{\Lambda t}S^{-1}$ in the good (diagonalizable) case.

This chapter emphasizes how to find $u(t) = e^{At}u(0)$ by diagonalization. Assume A does have n independent eigenvectors, so it is diagonalizable. Substitute $A = S\Lambda S^{-1}$ into the series for e^{At}. Whenever $S\Lambda S^{-1}S\Lambda S^{-1}$ appears, cancel $S^{-1}S$ in the middle:

Use the series $\quad e^{At} = I + S\Lambda S^{-1}t + \frac{1}{2}(S\Lambda S^{-1}t)(S\Lambda S^{-1}t) + \cdots$

Factor out S and S^{-1} $\quad = S\left[I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \cdots\right]S^{-1}$

Diagonalize e^{At} $\quad = Se^{\Lambda t}S^{-1}$. $\quad$ (15)

That equation says: e^{At} *equals* $Se^{\Lambda t}S^{-1}$. Then Λ is a diagonal matrix and so is $e^{\Lambda t}$. The numbers $e^{\lambda_i t}$ are on its diagonal. Multiply $Se^{\Lambda t}S^{-1}u(0)$ to recognize $u(t)$:

$$e^{At}u(0) = Se^{\Lambda t}S^{-1}u(0) = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{16}$$

This solution $e^{At}u(0)$ is the same answer that came in equation (6) from three steps:

1. Write $u(0) = c_1x_1 + \cdots + c_nx_n$. Here we need n independent eigenvectors.
2. Multiply each x_i by $e^{\lambda_i t}$ to follow it forward in time.
3. The best form of $e^{At}u(0)$ is $\quad u(t) = c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n$. $\quad$ (17)

Example 4 When you substitute $y = e^{\lambda t}$ into $y'' - 2y' + y = 0$, you get an equation with **repeated roots:** $\lambda^2 - 2\lambda + 1 = 0 = (\lambda - 1)^2$. A differential equations course would propose e^t and te^t as two independent solutions. Here we discover why.

Linear algebra reduces $y'' - 2y' + y = 0$ to a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' - y \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\boldsymbol{u}. \tag{18}$$

The eigenvalues of A are again $\lambda = 1, 1$ (with trace $= 2$ and $\det A = 1$). The only eigenvectors are multiples of $\boldsymbol{x} = (1, 1)$. Diagonalization is not possible, A has only one line of eigenvectors. So we compute e^{At} from its definition as a series:

$$\textbf{Short series} \qquad e^{At} = e^{It}\, e^{(A-I)t} = e^t\,[I + (A - I)t]. \tag{19}$$

The "infinite" series ends quickly because $(A - I)^2$ is the zero matrix! You can see te^t appearing in equation (19). The first component of $\boldsymbol{u}(t) = e^{At}\,\boldsymbol{u}(0)$ is our answer $y(t)$:

$$\boldsymbol{u}(t) = e^t\left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t\right]\boldsymbol{u}(0) \qquad y(t) = e^t y(0) - te^t y(0) + te^t y'(0).$$

Example 5 Use the infinite series to find e^{At} for $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Notice that $A^4 = I$:

$$A = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \quad A^3 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad A^4 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

A^5, A^6, A^7, A^8 will repeat these four matrices. The top right corner has $1, 0, -1, 0$ repeating over and over. The infinite series for e^{At} contains $t/1!$, 0, $-t^3/3!$, 0. Then $t - \frac{1}{6}t^3$ starts that top right corner, and $1 - \frac{1}{2}t^2$ starts the top left:

$$I + At + \tfrac{1}{2}(At)^2 + \tfrac{1}{6}(At)^3 + \cdots = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \cdots & t - \frac{1}{6}t^3 + \cdots \\ -t + \frac{1}{6}t^3 - \cdots & 1 - \frac{1}{2}t^2 + \cdots \end{bmatrix}.$$

On the left side is e^{At}. The top row of that matrix shows the series for $\cos t$ and $\sin t$.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{20}$$

A is a skew-symmetric matrix ($A^{\mathrm{T}} = -A$). Its exponential e^{At} is an orthogonal matrix. The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it}. Three rules:

1 *e^{At} always has the inverse e^{-At}.*

2 *The eigenvalues of e^{At} are always $e^{\lambda t}$.*

3 *When A is skew-symmetric, e^{At} is orthogonal. Inverse = transpose = e^{-At}.*

Skew-symmetric matrices have pure imaginary eigenvalues like $\lambda = i\theta$. Then e^{At} has eigenvalues $e^{i\theta t}$. Their absolute value is 1 (neutral stability, pure oscillation, energy is conserved).

Our final example has a triangular matrix A. Then the eigenvector matrix S is triangular. So are S^{-1} and e^{At}. You will see the two forms of the solution: a combination of eigenvectors and the short form $e^{At}\boldsymbol{u}(0)$.

Example 6 Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 0$.

Solution The eigenvalues 1 and 2 are on the diagonal of A (since A is triangular). The eigenvectors are $(1,0)$ and $(1,1)$. The starting $\boldsymbol{u}(0)$ is $\boldsymbol{x}_1 + \boldsymbol{x}_2$ so $c_1 = c_2 = 1$. Then $\boldsymbol{u}(t)$ is the same combination of pure exponentials (*no* $te^{\lambda t}$ *when* $\lambda = 1, 2$):

$$\textbf{Solution to } \boldsymbol{u}' = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That is the clearest form. But the matrix form produces $\boldsymbol{u}(t)$ for every $\boldsymbol{u}(0)$:

$$\boldsymbol{u}(t) = Se^{\Lambda t}S^{-1}\boldsymbol{u}(0) \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\boldsymbol{u}(0) = \begin{bmatrix} e^{t} & e^{2t} + e^{t} \\ 0 & e^{2t} \end{bmatrix}\boldsymbol{u}(0).$$

That last matrix is $\boldsymbol{e^{At}}$. It's not bad to see what a matrix exponential looks like (this is a particularly nice one). The situation is the same as for $A\boldsymbol{x} = \boldsymbol{b}$ and inverses. We don't really need A^{-1} to find $\boldsymbol{x}$, and we don't need e^{At} to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$. But as quick formulas for the answers, $A^{-1}\boldsymbol{b}$ and $e^{At}\boldsymbol{u}(0)$ are unbeatable.

■ REVIEW OF THE KEY IDEAS ■

1. The equation $\boldsymbol{u}' = A\boldsymbol{u}$ is linear with constant coefficients, starting from $\boldsymbol{u}(0)$.
2. Its solution is usually a combination of exponentials, involving each λ and $\boldsymbol{x}$:

$$\textbf{Independent eigenvectors} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.$$

3. The constants $c_1, \ldots, c_n$ are determined by $\boldsymbol{u}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n = S\boldsymbol{c}$.
4. $\boldsymbol{u}(t)$ approaches zero (stability) if every λ has negative real part.
5. The solution is always $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$, with the matrix exponential e^{At}.
6. Equations with y'' reduce to $\boldsymbol{u}' = A\boldsymbol{u}$ by combining y' and y into $\boldsymbol{u} = (y, y')$.

■ WORKED EXAMPLES ■

6.3 A Solve $y'' + 4y' + 3y = 0$ by substituting $e^{\lambda t}$ and also by linear algebra.

Solution Substituting $y = e^{\lambda t}$ yields $(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0$. That quadratic factors into $\lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0$. Therefore $\boldsymbol{\lambda_1 = -1}$ and $\boldsymbol{\lambda_2 = -3}$. The pure solutions are $y_1 = e^{-t}$ and $y_2 = e^{-3t}$. The complete solution $c_1y_1 + c_2y_2$ approaches zero.

To use linear algebra we set $\boldsymbol{u} = (y, y')$. Then the vector equation is $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -3y - 4y' \end{aligned} \quad \text{converts to} \quad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{u}.$$

This A is called a "companion matrix" and its eigenvalues are again 1 and 3 :

$$\textbf{Same quadratic} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$$

The eigenvectors of A are $(1, \lambda_1)$ and $(1, \lambda_2)$. Either way, the decay in $y(t)$ comes from e^{-t} and e^{-3t}. With constant coefficients, calculus goes back to algebra $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Note In linear algebra the serious danger is a shortage of eigenvectors. Our eigenvectors $(1, \lambda_1)$ and $(1, \lambda_2)$ are the same if $\lambda_1 = \lambda_2$. Then we can't diagonalize A. In this case we don't yet have two independent solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

In differential equations the danger is also a repeated λ. After $y = e^{\lambda t}$, a second solution has to be found. It turns out to be $y = te^{\lambda t}$. This "impure" solution (with an extra t) appears in the matrix exponential e^{At}. Example 4 showed how.

6.3 B Find the eigenvalues and eigenvectors of A and write $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ as a combination of the eigenvectors. Solve both equations $\boldsymbol{u}' = A\boldsymbol{u}$ and $\boldsymbol{u}'' = A\boldsymbol{u}$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{and} \quad \frac{d^2\boldsymbol{u}}{dt^2} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \ \text{ with } \ \frac{d\boldsymbol{u}}{dt}(0) = \boldsymbol{0}.$$

The $1, -2, 1$ diagonals make A into a *second difference matrix* (like a second derivative).

$\boldsymbol{u}' = A\boldsymbol{u}$ *is like the heat equation* $\partial u/\partial t = \partial^2 u/\partial x^2$.
Its solution $u(t)$ will decay (negative eigenvalues).
$\boldsymbol{u}'' = A\boldsymbol{u}$ *is like the wave equation* $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$.
Its solution will oscillate (imaginary eigenvalues).

Solution The eigenvalues and eigenvectors come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)[(-2-\lambda)^2 - 2] = 0.$$

One eigenvalue is $\lambda = -2$, when $-2 - \lambda$ is zero. The other factor is $\lambda^2 + 4\lambda + 2$, so the other eigenvalues (also real and negative) are $\lambda = -2 \pm \sqrt{2}$. Find the eigenvectors:

$$\boldsymbol{\lambda = -2} \qquad (A + 2I)\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{\lambda = -2 - \sqrt{2}} \qquad (A - \lambda I)\boldsymbol{x} = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = -2 + \sqrt{2}} \qquad (A - \lambda I)\boldsymbol{x} = \begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

The eigenvectors are *orthogonal* (proved in Section 6.4 for all symmetric matrices). All three λ_i are negative. This A is *negative definite* and e^{At} decays to zero (stability).

The starting $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ is $\boldsymbol{x}_3 - \boldsymbol{x}_2$. The solution is $\boldsymbol{u}(t) = e^{\lambda_3 t}\boldsymbol{x}_3 - e^{\lambda_2 t}\boldsymbol{x}_2$.

Heat equation In Figure 6.6a, the temperature at the center starts at $2\sqrt{2}$. Heat diffuses into the neighboring boxes and then to the outside boxes (frozen at 0°). The rate of heat flow between boxes is the temperature difference. From box 2, heat flows left and right at the rate $u_1 - u_2$ and $u_3 - u_2$. So the flow out is $u_1 - 2u_2 + u_3$ in the second row of $A\boldsymbol{u}$.

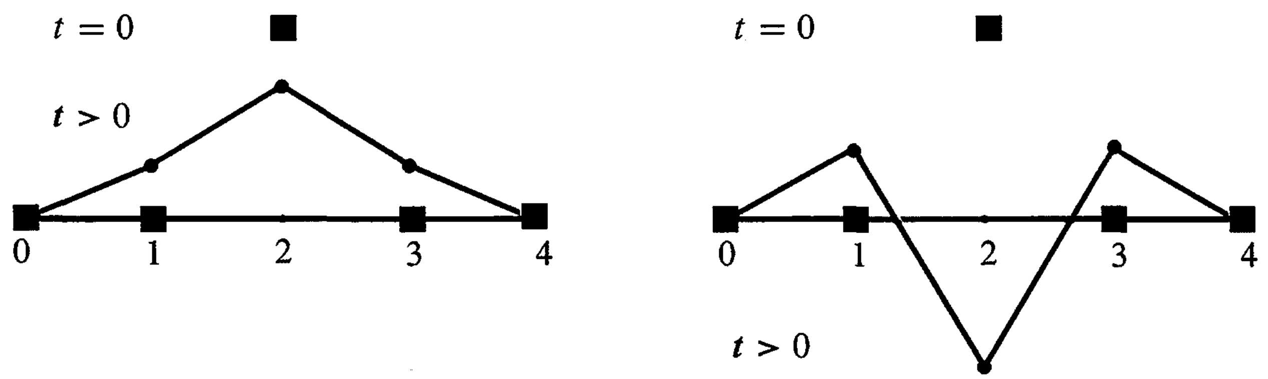


Figure 6.6: Heat diffuses away from box 2 (left). Wave travels from box 2 (right).

Wave equation $d^2\boldsymbol{u}/dt^2 = A\boldsymbol{u}$ has the same eigenvectors $\boldsymbol{x}$. But now the eigenvalues λ lead to **oscillations** $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. The frequencies come from $\omega^2 = -\lambda$:

$$\frac{d^2}{dt^2}(e^{i\omega t}\boldsymbol{x}) = A(e^{i\omega t}\boldsymbol{x}) \qquad \text{becomes} \qquad (i\omega)^2 e^{i\omega t}\boldsymbol{x} = \lambda e^{i\omega t}\boldsymbol{x} \quad \text{and} \quad \omega^2 = -\lambda.$$

There are two square roots of $-\lambda$, so we have $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. With three eigenvectors this makes *six* solutions to $\boldsymbol{u}'' = A\boldsymbol{u}$. A combination will match the six components of $\boldsymbol{u}(0)$ and $\boldsymbol{u}'(0)$. Since $\boldsymbol{u}' = \boldsymbol{0}$ in this problem, $e^{i\omega t}\boldsymbol{x}$ combines with $e^{-i\omega t}\boldsymbol{x}$ into $2\cos\omega t\,\boldsymbol{x}$.

6.3 C Solve the four equations $da/dt = 0, db/dt = a, dc/dt = 2b, dz/dt = 3c$ in that order starting from $\boldsymbol{u}(0) = (a(0), b(0), c(0), z(0))$. Solve the same equations by the matrix exponential in $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$.

Four equations **$\lambda = 0, 0, 0, 0$** **Eigenvalues on the diagonal**
$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}.$$

First find A^2, A^3, A^4 and $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3$. Why does the series stop? Why is it always true that $(e^A)(e^A) = (e^{2A})$? ***Always e^{As} times e^{At} is $e^{A(s+t)}$.***

Solution 1 Integrate $da/dt = 0$, then $db/dt = a$, then $dc/dt = 2b$ and $dz/dt = 3c$:

$$\begin{aligned} a(t) &= a(0) \\ b(t) &= ta(0) + b(0) \\ c(t) &= t^2a(0) + 2tb(0) + c(0) \\ z(t) &= t^3a(0) + 3t^2b(0) + 3tc(0) + z(0) \end{aligned}$$

The 4 by 4 matrix which is multiplying $a(0), b(0), c(0), d(0)$ to produce $a(t), b(t), c(t), d(t)$ must be the same e^{At} as below

Solution 2 *The powers of A (strictly triangular) are all zero after A^3.*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0 & 0 \\ 0 & \mathbf{6} & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{6} & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \mathbf{0}$$

The diagonals move down at each step. So the series for e^{At} stops after four terms:

Same e^{At}
$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ t^2 & 2t & 1 & \\ t^3 & 3t^2 & 3t & 1 \end{bmatrix}$$

The square of e^A is always e^{2A} for many reasons:

1. Solving with e^A from $t = 0$ to 1 and then from 1 to 2 agrees with e^{2A} from 0 to 2.

2. The squared series $(I + A + \frac{A^2}{2} + \cdots)^2$ matches $I + 2A + \frac{(2A)^2}{2} + \cdots = e^{2A}$.

3. If A can be diagonalized (this A can't!) then $(Se^{\Lambda}S^{-1})(Se^{\Lambda}S^{-1}) = Se^{2\Lambda}S^{-1}$.

But notice in Problem 23 that e^Ae^B and e^Be^A and e^{A+B} are all different.

Problem Set 6.3

1 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

2 Solve Problem 1 for $\boldsymbol{u} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \qquad \text{Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^{t}. The λ's are 4 and 1.

3 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{u}' = A\boldsymbol{u}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$

$$\text{Solve } \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{u} \text{ with } \boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{u}(\infty)?$$

4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

5 Reverse the diffusion of people in Problem 4 to $d\boldsymbol{u}/dt = -A\boldsymbol{u}$:

$$\frac{dv}{dt} = v - w \qquad \text{and} \qquad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

6 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \quad (a \text{ and } b \text{ are real})$$

Find the conditions on a and b so that all solutions of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and $d\boldsymbol{v}/dt = B\boldsymbol{v}$ approach zero as $t \to \infty$.

7 Suppose P is the projection matrix onto the 45° line $y = x$ in $\mathbf{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(0) = (3, 1)$?

8 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

9 (a) Write $(4, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ starting from $(4, 0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ to find $\boldsymbol{u}(t)$.

Questions 10–13 reduce second-order equations to first-order systems for (y, y').

10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

11 The solution to $y'' = 0$ is a straight line $y = C + Dt$. Convert to a matrix equation:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \text{ has the solution } \begin{bmatrix} y \\ y' \end{bmatrix} = e^{At}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

This matrix A has $\lambda = 0, 0$ and it cannot be diagonalized. Find A^2 and compute $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$. Multiply your e^{At} times $\big(y(0), y'(0)\big)$ to check the straight line $y(t) = y(0) + y'(0)t$.

12 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

13 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\boldsymbol{u} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

Find $\boldsymbol{u}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{u}(0) = (3, 0)$.

14 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$):

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\boldsymbol{u} \qquad \text{or} \qquad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2. \end{aligned}$$

(a) The derivative of $\|\boldsymbol{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute u_1', u_2', u_3' to get *zero*. Then $\|\boldsymbol{u}(t)\|^2$ stays equal to $\|\boldsymbol{u}(0)\|^2$.

(b) *When A is skew-symmetric, $Q = e^{At}$ is orthogonal.* Prove $Q^{\mathrm{T}} = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^{\mathrm{T}}Q = I$.

15 A particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - \boldsymbol{b}$ is $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ give $\boldsymbol{u}_n$. Find the complete solution $\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_n$:

(a) $\dfrac{du}{dt} = u - 4$ (b) $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{u} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$

16 If c is not an eigenvalue of A, substitute $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A? The "nullspace" of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ contains the usual solutions $e^{\lambda_i t}\boldsymbol{x}_i$.

17 Find a matrix A to illustrate each of the unstable regions in Figure 6.5:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

Questions 18–27 are about the matrix exponential e^{At}.

18 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{u}_0$ solves $\boldsymbol{u}' = A\boldsymbol{u}$.

19 The matrix $B = \left[\begin{smallmatrix} 0 & -4 \\ 0 & 0 \end{smallmatrix}\right]$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

20 Starting from $\boldsymbol{u}(0)$ the solution at time T is $e^{AT}\boldsymbol{u}(0)$. Go an additional time t to reach $e^{At}\ e^{AT}\boldsymbol{u}(0)$. This solution at time $t + T$ can also be written as _____. Conclusion: e^{At} times e^{AT} equals _____.

21 Write $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

22 If $A^2 = A$ show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in Problem 21 this gives $e^{At} =$ _____.

23 Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 21–22 and 19. (If $AB = BA$, all three are the same.)

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

24 Write $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check e^{At} and the derivative of e^{At} when $t = 0$.

25 Put $A = \left[\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right]$ into the infinite series to find e^{At}. First compute A^2 and A^s:

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} & \\ & \end{bmatrix} + \cdots = \begin{bmatrix} e^t & \\ 0 & \end{bmatrix}.$$

26 Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write down its inverse.

(b) Write down its eigenvalues. If $Ax = \lambda x$ then $e^{At}x =$ _____ x.

27 Find a solution $x(t), y(t)$ that gets large as $t \to \infty$. To avoid this instability a scientist exchanged the two equations:

$$\begin{aligned} dx/dt &= 0x - 4y \\ dy/dt &= -2x + 2y \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

Now the matrix $\left[\begin{smallmatrix} -2 & 2 \\ 0 & -4 \end{smallmatrix}\right]$ is stable. It has negative eigenvalues. How can this be?

Challenge Problems

28 Centering $y'' = -y$ in Example 3 will produce $Y_{n+1} - 2Y_n + Y_{n-1} = -(\Delta t)^2 Y_n$. This can be written as a one-step difference equation for $U = (Y, Z)$:

$$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\, Z_n \\ Z_{n+1} &= Z_n - \Delta t\, Y_{n+1} \end{aligned} \qquad \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

Invert the matrix on the left side to write this as $U_{n+1} = AU_n$. Show that $\det A = 1$. Choose the large time step $\Delta t = 1$ and find the eigenvalues λ_1 and $\lambda_2 = \overline{\lambda}_1$ of A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \textbf{ has } |\lambda_1| = |\lambda_2| = 1. \textbf{ Show that } A^6 \textbf{ is exactly } I.$$

After 6 steps to $t = 6$, U_6 equals U_0. The exact $y = \cos t$ returns to 1 at $t = 2\pi$.

29 That centered choice (*leapfrog method*) in Problem 28 is very successful for small time steps Δt. But find the eigenvalues of A for $\Delta t = \sqrt{2}$ and 2:

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

Both matrices have $|\lambda| = 1$. Compute A^4 in both cases and find the eigenvectors of A. That value $\Delta t = 2$ is at the border of instability. Time steps $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $\boldsymbol{U}_n = A^n \boldsymbol{U}_0$ will explode.

Note You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -1000000y$, then $\Delta t > .0002$ will give instability. Leapfrog has a very strict stability limit. $Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large.

30 Another good idea for $y'' = -y$ is the trapezoidal method (half forward/half back): ***This may be the best way to keep (Y_n, Z_n) exactly on a circle.***

$$\textbf{Trapezoidal} \quad \begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

(a) Invert the left matrix to write this equation as $\boldsymbol{U}_{n+1} = A\boldsymbol{U}_n$. *Show that A is an orthogonal matrix*: $A^{\mathrm{T}}A = I$. **These points $\boldsymbol{U}_n$ never leave the circle.** $A = (I - B)^{-1}(I + B)$ is always an orthogonal matrix if $B^{\mathrm{T}} = -B$.

(b) (Optional MATLAB) Take 32 steps from $\boldsymbol{U}_0 = (1, 0)$ to $\boldsymbol{U}_{32}$ with $\Delta t = 2\pi/32$. Is $\boldsymbol{U}_{32} = \boldsymbol{U}_0$? I think there is a small error.

31 The ***cosine of a matrix*** is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots \qquad \cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, multiply each term times $\boldsymbol{x}$ to find the eigenvalue of $\cos A$.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $C = \cos A$.

(c) The second derivative of $\cos(At)$ is $-A^2 \cos(At)$.

$$\boldsymbol{u}(t) = \mathbf{cos}(\boldsymbol{A}\boldsymbol{t})\,\boldsymbol{u}(0) \;\text{ solves }\; \frac{d^2\boldsymbol{u}}{dt^2} = -A^2\boldsymbol{u} \;\text{ starting from }\; \boldsymbol{u}'(0) = 0.$$

Construct $\boldsymbol{u}(t) = \cos(At)\,\boldsymbol{u}(0)$ by the usual three steps for that specific A:

1. Expand $\boldsymbol{u}(0) = (4, 2) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ in the eigenvectors.
2. Multiply those eigenvectors by ____ and ____ (instead of $e^{\lambda t}$).
3. Add up the solution $\boldsymbol{u}(t) = c_1$ ____ $\boldsymbol{x}_1 + c_2$ ____ $\boldsymbol{x}_2$.

6.4 Symmetric Matrices

For projection onto a plane in $\mathbf{R}^3$, the plane is full of eigenvectors (where $Px = x$). The other eigenvectors are *perpendicular* to the plane (where $Px = \mathbf{0}$). The eigenvalues $\lambda = 1, 1, 0$ are real. Three eigenvectors can be chosen perpendicular to each other. I have to write "can be chosen" because the two in the plane are not automatically perpendicular. This section makes that best possible choice for ***symmetric matrices***: ***The eigenvectors of $P = P^{\mathrm{T}}$ are perpendicular unit vectors.***

Now we open up to all symmetric matrices. It is no exaggeration to say that these are the most important matrices the world will ever see—in the theory of linear algebra and also in the applications. We come immediately to the key question about symmetry. Not only the question, but also the answer.

What is special about $Ax = \lambda x$ when A is symmetric? We are looking for special properties of the eigenvalues λ and the eigenvectors x when $A = A^{\mathrm{T}}$.

The diagonalization $A = S\Lambda S^{-1}$ will reflect the symmetry of A. We get some hint by transposing to $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\Lambda S^{\mathrm{T}}$. Those are the same since $A = A^{\mathrm{T}}$. Possibly S^{-1} in the first form equals S^{T} in the second form. Then $S^{\mathrm{T}}S = I$. That makes each eigenvector in S orthogonal to the other eigenvectors. The key facts get first place in the Table at the end of this chapter, and here they are:

1. A symmetric matrix has only ***real eigenvalues***.
2. The ***eigenvectors*** can be chosen ***orthonormal***.

Those n orthonormal eigenvectors go into the columns of S. Every symmetric matrix can be diagonalized. ***Its eigenvector matrix S becomes an orthogonal matrix Q***. Orthogonal matrices have $Q^{-1} = Q^{\mathrm{T}}$—what we suspected about S is true. To remember it we write $S = Q$, when we choose orthonormal eigenvectors.

Why do we use the word "choose"? Because the eigenvectors do not *have* to be unit vectors. Their lengths are at our disposal. We will choose unit vectors—eigenvectors of length one, which are orthonormal and not just orthogonal. Then $S\Lambda S^{-1}$ is in its special and particular form $Q\Lambda Q^{\mathrm{T}}$ for symmetric matrices:

(Spectral Theorem) Every symmetric matrix has the factorization $A = Q\Lambda Q^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in $S = Q$:

Symmetric diagonalization $\quad A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}.$

It is easy to see that $Q\Lambda Q^{\mathrm{T}}$ is symmetric. Take its transpose. You get $(Q^{\mathrm{T}})^{\mathrm{T}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}}$, which is $Q\Lambda Q^{\mathrm{T}}$ again. The harder part is to prove that every symmetric matrix has real λ's and orthonormal x's. This is the *"spectral theorem"* in mathematics and the *"principal axis* *theorem*" in geometry and physics. We have to prove it! No choice. I will approach the proof in three steps:

1. By an example, showing real λ's in Λ and orthonormal $\boldsymbol{x}$'s in Q.
2. By a proof of those facts when no eigenvalues are repeated.
3. By a proof that allows repeated eigenvalues (at the end of this section).

Example 1 Find the λ's and $\boldsymbol{x}$'s when $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$.

Solution The determinant of $A - \lambda I$ is $\lambda^2 - 5\lambda$. The eigenvalues are 0 and 5 (*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue because A is singular, and $\lambda = 5$ matches the *trace* down the diagonal of A: $0 + 5$ agrees with $1 + 4$.

Two eigenvectors are $(2, -1)$ and $(1, 2)$—orthogonal but not yet orthonormal. The eigenvector for $\lambda = 0$ is in the *nullspace* of A. The eigenvector for $\lambda = 5$ is in the *column space*. We ask ourselves, why are the nullspace and column space perpendicular? The Fundamental Theorem says that the nullspace is perpendicular to the ***row space***—not the column space. But our matrix is *symmetric*! Its row and column spaces are the same. Its eigenvectors $(2, -1)$ and $(1, 2)$ must be (and are) perpendicular.

These eigenvectors have length $\sqrt{5}$. Divide them by $\sqrt{5}$ to get unit vectors. Put those into the columns of S (which is Q). Then $Q^{-1}AQ$ is Λ and $Q^{-1} = Q^{\mathrm{T}}$:

$$Q^{-1}AQ = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda.$$

Now comes the n by n case. The λ's are real when $A = A^{\mathrm{T}}$ and $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Real Eigenvalues All the eigenvalues of a real symmetric matrix are real.

Proof Suppose that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Until we know otherwise, λ might be a complex number $a + ib$ (a and b real). *Its complex conjugate is* $\overline{\lambda} = a - ib$. Similarly the components of $\boldsymbol{x}$ may be complex numbers, and switching the signs of their imaginary parts gives $\overline{\boldsymbol{x}}$. The good thing is that $\overline{\lambda}$ times $\overline{\boldsymbol{x}}$ is always the conjugate of λ times $\boldsymbol{x}$. So we can take conjugates of $A\boldsymbol{x} = \lambda\boldsymbol{x}$, remembering that A is real:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{leads to} \quad A\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}. \qquad \text{Transpose to} \quad \overline{\boldsymbol{x}}^{\mathrm{T}}A = \overline{\boldsymbol{x}}^{\mathrm{T}}\,\overline{\lambda}. \tag{1}$$

Now take the dot product of the first equation with $\overline{\boldsymbol{x}}$ and the last equation with $\boldsymbol{x}$:

$$\overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\lambda\boldsymbol{x} \qquad \text{and also} \qquad \overline{\boldsymbol{x}}^{\mathrm{T}}A\boldsymbol{x} = \overline{\boldsymbol{x}}^{\mathrm{T}}\,\overline{\lambda}\boldsymbol{x}. \tag{2}$$

The left sides are the same so the right sides are equal. One equation has λ, the other has $\overline{\lambda}$. They multiply $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{x} = |x_1|^2 + |x_2|^2 + \cdots =$ length squared which is not zero. *Therefore λ must equal $\overline{\lambda}$*, and $a + ib$ equals $a - ib$. The imaginary part is $b = 0$. Q.E.D.

The eigenvectors come from solving the real equation $(A - \lambda I)x = \mathbf{0}$. So the x's are also real. The important fact is that they are perpendicular.

Orthogonal Eigenvectors Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof Suppose $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$. We are assuming here that $\lambda_1 \neq \lambda_2$. Take dot products of the first equation with y and the second with x:

$$\textbf{Use } A^{\mathrm{T}} = A \qquad (\lambda_1 x)^{\mathrm{T}} y = (Ax)^{\mathrm{T}} y = x^{\mathrm{T}} A^{\mathrm{T}} y = x^{\mathrm{T}} A y = x^{\mathrm{T}} \lambda_2 y. \tag{3}$$

The left side is $x^{\mathrm{T}} \lambda_1 y$, the right side is $x^{\mathrm{T}} \lambda_2 y$. Since $\lambda_1 \neq \lambda_2$, this proves that $x^{\mathrm{T}} y = 0$. The eigenvector x (for λ_1) is perpendicular to the eigenvector y (for λ_2).

Example 2 The eigenvectors of a 2 by 2 symmetric matrix have a special form:

$$\textbf{Not widely known} \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{has} \quad x_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}. \tag{4}$$

This is in the Problem Set. The point here is that x_1 is perpendicular to x_2:

$$x_1^{\mathrm{T}} x_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $x_1^{\mathrm{T}} x_2 = 0$. Eagle eyes might notice the special case $a = c$, $b = 0$ when $x_1 = x_2 = \mathbf{0}$. This case has repeated eigenvalues, as in $A = I$. It still has perpendicular eigenvectors $(1, 0)$ and $(0, 1)$.

This example shows the main goal of this section—***to diagonalize symmetric matrices A by orthogonal eigenvector matrices $S = Q$***. Look again at the result:

$$\textbf{Symmetry} \qquad A = S \Lambda S^{-1} \quad \text{becomes} \quad A = Q \Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{\mathrm{T}} Q = I.$$

This says that every 2 by 2 symmetric matrix looks like

$$A = Q \Lambda Q^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

The columns x_1 and x_2 multiply the rows $\lambda_1 x_1^{\mathrm{T}}$ and $\lambda_2 x_2^{\mathrm{T}}$ to produce A:

$$\textbf{Sum of rank-one matrices} \qquad A = \lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}. \tag{6}$$

This is the great factorization $Q \Lambda Q^{\mathrm{T}}$, written in terms of λ's and x's. When the symmetric matrix is n by n, there are n columns in Q multiplying n rows in Q^{T}. The n products $x_i x_i^{\mathrm{T}}$ are ***projection matrices***. Including the λ's, the spectral theorem $A = Q \Lambda Q^{\mathrm{T}}$ for symmetric matrices says that A is a combination of projection matrices:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n \qquad \lambda_i = \text{eigenvalue}, \quad P_i = \text{projection onto eigenspace}.$$

Complex Eigenvalues of Real Matrices

Equation (1) went from $Ax = \lambda x$ to $A\overline{x} = \overline{\lambda}\,\overline{x}$. In the end, λ and x were real. Those two equations were the same. But a *non*symmetric matrix can easily produce λ and x that are complex. In this case, $A\overline{x} = \overline{\lambda}\,\overline{x}$ is different from $Ax = \lambda x$. It gives us a new eigenvalue (which is $\overline{\lambda}$) and a new eigenvector (which is $\overline{x}$):

For real matrices, complex λ's and x's come in "conjugate pairs."

$$\textit{If} \quad Ax = \lambda x \quad \textit{then} \quad A\overline{x} = \overline{\lambda}\,\overline{x}.$$

Example 3 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Those eigenvalues are conjugate to each other. They are λ and $\overline{\lambda}$. The eigenvectors must be x and $\overline{x}$, because A is real:

$$\begin{aligned} &\textbf{This is } \lambda x \qquad Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &\textbf{This is } \overline{\lambda}\,\overline{x} \qquad A\overline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta)\begin{bmatrix} 1 \\ i \end{bmatrix}. \end{aligned} \tag{7}$$

Those eigenvectors $(1, -i)$ and $(1, i)$ are complex conjugates because A is real.

For this rotation matrix the absolute value is $|\lambda| = 1$, because $\cos^2\theta + \sin^2\theta = 1$. ***This fact $|\lambda| = 1$ holds for the eigenvalues of every orthogonal matrix.***

We apologize that a touch of complex numbers slipped in. They are unavoidable even when the matrix is real. Chapter 10 goes beyond complex numbers λ and complex vectors to complex matrices A. Then you have the whole picture.

We end with two optional discussions.

Eigenvalues versus Pivots

The eigenvalues of A are very different from the pivots. For eigenvalues, we solve $\det(A - \lambda I) = 0$. For pivots, we use elimination. The only connection so far is this:

product of pivots = determinant = product of eigenvalues.

We are assuming a full set of pivots $d_1, \ldots, d_n$. There are n real eigenvalues $\lambda_1, \ldots, \lambda_n$. The d's and λ's are not the same, but they come from the same matrix. This paragraph is about a hidden relation. ***For symmetric matrices the pivots and the eigenvalues have the same signs***:

> ***The number of positive eigenvalues of $A = A^{\mathrm{T}}$ equals the number of positive pivots.***
> Special case: A has all $\lambda_i > 0$ if and only if all pivots are positive.

That special case is an all-important fact for **positive definite matrices** in Section 6.5.

Example 4 This symmetric matrix A has one positive eigenvalue and one positive pivot:

$$\textbf{Matching signs} \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \begin{matrix}\text{has pivots 1 and } -8 \\ \text{eigenvalues 4 and } -2.\end{matrix}$$

The signs of the pivots match the signs of the eigenvalues, one plus and one minus. This could be false when the matrix is not symmetric:

$$\textbf{Opposite signs} \quad B = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix} \quad \begin{matrix}\text{has pivots 1 and 2} \\ \text{eigenvalues } -1 \text{ and } -2.\end{matrix}$$

The diagonal entries are a third set of numbers and we say nothing about them.

Here is a proof that the pivots and eigenvalues have matching signs, when $A = A^{\mathrm{T}}$.

You see it best when the pivots are divided out of the rows of U. Then A is LDL^{T}. The diagonal pivot matrix D goes between triangular matrices L and L^{T}:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \textbf{This is } A = LDL^{\mathrm{T}}\textbf{. It is symmetric.}$$

Watch the eigenvalues when L and L^{T} move toward the identity matrix: $A \to D$.

The eigenvalues of LDL^{T} are 4 and -2. The eigenvalues of IDI^{T} are 1 and -8 (the pivots!). The eigenvalues are changing, as the "3" in L moves to zero. But to change *sign*, a real eigenvalue would have to cross zero. The matrix would at that moment be singular. Our changing matrix always has pivots 1 and -8, so it is *never* singular. The signs cannot change, as the λ's move to the d's.

We repeat the proof for any $A = LDL^{\mathrm{T}}$. Move L toward I, by moving the off-diagonal entries to zero. The pivots are not changing and not zero. The eigenvalues λ of LDL^{T} change to the eigenvalues d of IDI^{T}. Since these eigenvalues cannot cross zero as they move into the pivots, their signs cannot change. Q.E.D.

This connects the two halves of applied linear algebra—pivots and eigenvalues.

All Symmetric Matrices are Diagonalizable

When no eigenvalues of A are repeated, the eigenvectors are sure to be independent. Then A can be diagonalized. But a repeated eigenvalue can produce a shortage of eigenvectors. This *sometimes* happens for nonsymmetric matrices. It *never* happens for symmetric matrices. ***There are always enough eigenvectors to diagonalize $A = A^{\mathrm{T}}$.***

Here is one idea for a proof. Change A slightly by a diagonal matrix $\mathbf{diag}(c, 2c, \ldots, nc)$. If c is very small, the new symmetric matrix will have no repeated eigenvalues. Then we know it has a full set of orthonormal eigenvectors. As $c \to 0$ we obtain n orthonormal eigenvectors of the original A—even if some eigenvalues of that A are repeated.

Every mathematician knows that this argument is incomplete. How do we guarantee that the small diagonal matrix will separate the eigenvalues? (I am sure this is true.)

A different proof comes from a useful new factorization that applies to *all matrices*, symmetric or not. This new factorization immediately produces $A = Q\Lambda Q^{\mathrm{T}}$ with a full set of real orthonormal eigenvectors when A is any symmetric matrix.

Every square matrix factors into $A = QTQ^{-1}$ where T is upper triangular and $\overline{Q}^{\mathrm{T}} = Q^{-1}$. If A has real eigenvalues then Q and T can be chosen real: $Q^{\mathrm{T}}Q = I$.

This is Schur's Theorem. We are looking for $AQ = QT$. The first column $\boldsymbol{q}_1$ of Q must be a unit eigenvector of A. Then the first columns of AQ and QT are $A\boldsymbol{q}_1$ and $t_{11}\boldsymbol{q}_1$. But the other columns of Q need not be eigenvectors when T is only triangular (not diagonal). So use any $n-1$ columns that complete $\boldsymbol{q}_1$ to a matrix Q_1 with orthonormal columns. At this point only the first columns of Q and T are set, where $A\boldsymbol{q}_1 = t_{11}\boldsymbol{q}_1$:

$$\overline{Q}_1^{\mathrm{T}} A Q_1 = \begin{bmatrix} \overline{\boldsymbol{q}}_1^{\mathrm{T}} \\ \vdots \\ \overline{\boldsymbol{q}}_n^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A\boldsymbol{q}_1 & \cdots & A\boldsymbol{q}_n \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots \\ 0 & \boxed{A_2} \\ \vdots & \\ 0 & \end{bmatrix}. \tag{8}$$

Now I will argue by "induction". *Assume Schur's factorization $A_2 = Q_2T_2Q_2^{-1}$ is possible for that matrix A_2 of size $n-1$.* Put the orthogonal (or unitary) matrix Q_2 and the triangular T_2 into the final Q and T:

$$Q = Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} t_{11} & \cdots \\ 0 & T_2 \end{bmatrix} \quad \text{and} \quad AQ = QT \quad \text{as desired.}$$

Note I had to allow $\boldsymbol{q}_1$ and Q_1 to be complex, in case A has complex eigenvalues. But if t_{11} is a real eigenvalue, then $\boldsymbol{q}_1$ and Q_1 can stay real. The induction step keeps everything real when A has real eigenvalues. Induction starts with 1 by 1, no problem.

Proof that T is the diagonal Λ when A is symmetric. Then we have $A = Q\Lambda Q^{\mathrm{T}}$.

Every symmetric A has real eigenvalues. Schur's $A = QTQ^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$ means that $T = Q^{\mathrm{T}}AQ$. This is a symmetric matrix (its transpose is $Q^{\mathrm{T}}AQ$). Now the key point: *If T is triangular and also symmetric, it must be diagonal*: $T = \Lambda$.

This proves $A = Q\Lambda Q^{\mathrm{T}}$. The matrix $A = A^{\mathrm{T}}$ has n orthonormal eigenvectors.

■ REVIEW OF THE KEY IDEAS ■

1. A symmetric matrix has *real eigenvalues* and *perpendicular eigenvectors*.
2. Diagonalization becomes $A = Q\Lambda Q^{\mathrm{T}}$ with an orthogonal matrix Q.
3. All symmetric matrices are diagonalizable, even with repeated eigenvalues.
4. The signs of the eigenvalues match the signs of the pivots, when $A = A^{\mathrm{T}}$.
5. Every square matrix can be "triangularized" by $A = QTQ^{-1}$.

■ WORKED EXAMPLES ■

6.4 A What matrix A has eigenvalues $\lambda = 1, -1$ and eigenvectors $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$? Which of these properties can be predicted in advance?

$$A = A^{\mathrm{T}} \qquad A^2 = I \qquad \det A = -1 \qquad + \text{ and } - \text{ pivot} \qquad A^{-1} = A$$

Solution All those properties can be predicted! With real eigenvalues in Λ and orthonormal eigenvectors in Q, the matrix $A = Q\Lambda Q^{\mathrm{T}}$ must be symmetric. The eigenvalues 1 and -1 tell us that $A^2 = I$ (since $\lambda^2 = 1$) and $A^{-1} = A$ (same thing) and $\det A = -1$. The two pivots are positive and negative like the eigenvalues, since A is symmetric.

The matrix must be a reflection. Vectors in the direction of $\boldsymbol{x}_1$ are unchanged by A (since $\lambda = 1$). Vectors in the perpendicular direction are reversed (since $\lambda = -1$). The reflection $A = Q\Lambda Q^{\mathrm{T}}$ is across the "θ-line". Write c for $\cos\theta$, s for $\sin\theta$:

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Notice that $\boldsymbol{x} = (1, 0)$ goes to $A\boldsymbol{x} = (\cos 2\theta, \sin 2\theta)$ on the 2θ-line. And $(\cos 2\theta, \sin 2\theta)$ goes back across the θ-line to $\boldsymbol{x} = (1, 0)$.

6.4 B Find the eigenvalues of A_3 and B_4, and check the orthogonality of their first two eigenvectors. Graph these eigenvectors to see discrete sines and cosines:

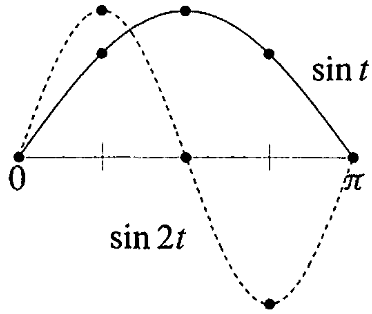
$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second difference". Section 8.1 will explain how this is like a second derivative. Then $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $B\boldsymbol{x} = \lambda\boldsymbol{x}$ are like $d^2x/dt^2 = \lambda x$. This has eigenvectors $x = \sin kt$ and $x = \cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the *Discrete Fourier Transform.* This DFT is absolutely central to all areas of digital signal processing. The favorite choice for JPEG in image processing has been B_8 of size 8.

Solution The eigenvalues of A_3 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$. (see **6.3 B**). Their sum is 6 (the trace of A_3) and their product is 4 (the determinant). The eigenvector matrix S gives the "Discrete Sine Transform" and the graph shows how the first two eigenvectors fall onto sine curves. Please draw the third eigenvector onto a third sine curve!

$$S = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

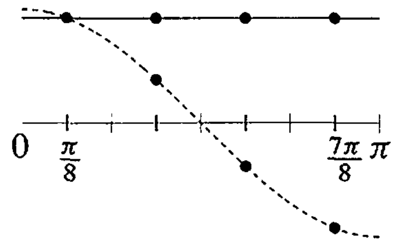
Eigenvector matrix for A_3



The eigenvalues of B_4 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$ and 0 (the same as for A_3, plus the zero eigenvalue). The trace is still 6, but the determinant is now zero. The eigenvector matrix C gives the 4-point "Discrete Cosine Transform" and the graph shows how the first two eigenvectors fall onto cosine curves. (Please plot the third eigenvector.) These eigenvectors match cosines at the *halfway points* $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Eigenvector matrix for B_4



S and C have orthogonal columns (eigenvectors of the symmetric A_3 and B_4). When we multiply a vector by S or C, that signal splits into pure frequencies—as a musical chord separates into pure notes. This is the most useful and insightful transform in all of signal processing. Here is a MATLAB code to create B_8 and its eigenvector matrix C:

```
n=8; e=ones(n-1,1); B=2*eye(n)-diag(e,-1)-diag(e,1); B(1,1)=1; B(n,n)=1;
[C, Λ] = eig(B);
plot(C(:,1:4), '-o')
```

Problem Set 6.4

1 Write A as $M + N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N \qquad (M^{\mathrm{T}} = M, N^{\mathrm{T}} = -N).$$

For any square matrix, $M = \frac{A+A^{\mathrm{T}}}{2}$ and $N =$ _____ add up to A.

2 If C is symmetric prove that $A^{\mathrm{T}}CA$ is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and $A^{\mathrm{T}}CA$?

3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. What is Λ?

5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

7 (a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know it must have a negative pivot?

(c) How do you know it can't have two negative eigenvalues?

8 If $A^3 = 0$ then the eigenvalues of A must be ____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.

9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue. (If $Ax = \lambda x$ then also $A\overline{x} = \overline{\lambda}\overline{x}$.) Prove that every real 3 by 3 matrix has at least one real eigenvalue.

10 Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$\textbf{False proof} \quad Ax = \lambda x \quad \text{gives} \quad x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x \quad \text{so} \quad \lambda = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[0 \ \ -1; \ \ 1 \ \ 0]$ with $\lambda = i$ and $x = (i, 1)$.

11 Write A and B in the form $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad (\text{keep } \|x_1\| = \|x_2\| = 1).$$

12 Every 2 by 2 symmetric matrix is $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = x_1 x_1^{\mathrm{T}} + x_2 x_2^{\mathrm{T}} = I$ from columns times rows of Q. Why is $P_1 P_2 = 0$?

13 What are the eigenvalues of $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$? Create a 4 by 4 skew-symmetric matrix ($A^{\mathrm{T}} = -A$) and verify that all its eigenvalues are imaginary.

14 (Recommended) This matrix M is skew-symmetric and also ____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|Mx\| = \|x\|$ for every x so $\|\lambda x\| = \|x\|$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

15 Show that A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\overline{A}^{\mathrm{T}} = A$ (Section 10.2). Then all λ's are real and eigenvectors are orthogonal.

16 Even if A is rectangular, the block matrix $B = \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$ is symmetric:

$$Bx = \lambda x \quad \text{is} \quad \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix} \quad \text{which is} \quad \begin{aligned} Az &= \lambda y \\ A^{\mathrm{T}} y &= \lambda z. \end{aligned}$$

(a) Show that $-\lambda$ is also an eigenvalue, with the eigenvector $(y, -z)$.

(b) Show that $A^{\mathrm{T}}Az = \lambda^2 z$, so that λ^2 is an eigenvalue of $A^{\mathrm{T}}A$.

(c) If $A = I$ (2 by 2) find all four eigenvalues and eigenvectors of B.

17 If $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in Problem 16, find all three eigenvalues and eigenvectors of B.

18 ***Another proof that eigenvectors are perpendicular when $A = A^{\mathrm{T}}$.*** Two steps:

1. Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because ____. Go carefully—why are these subspaces orthogonal?
2. If $Ay = \beta y$, apply this argument to $A - \beta I$. The eigenvalue of $A - \beta I$ moves to zero and the eigenvectors stay the same—so they are perpendicular.

19 Find the eigenvector matrix S for A and for B. Show that S doesn't collapse at $d = 1$, even though $\lambda = 1$ is repeated. Are the eigenvectors perpendicular?

$$A = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \quad \text{have} \quad \lambda = 1, d, -d.$$

20 Write a 2 by 2 *complex* matrix with $\overline{A}^{\mathrm{T}} = A$ (a "Hermitian matrix"). Find λ_1 and λ_2 for your complex matrix. Adjust equations (1) and (2) to show that *the eigenvalues of a Hermitian matrix are real.*

21 ***True*** (with reason) or ***false*** (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(c) The inverse of a symmetric matrix is symmetric.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

22 (A paradox for instructors) If $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ then A and A^{T} share the same eigenvectors (true). A and A^{T} always share the same eigenvalues. Find the flaw in this conclusion: They must have the same S and Λ. Therefore A equals A^{T}.

23 (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$?

24 What number b in $\left[\begin{smallmatrix} 2 & b \\ 1 & 0 \end{smallmatrix}\right]$ makes $A = Q\Lambda Q^{\mathrm{T}}$ possible? What number makes $A = S\Lambda S^{-1}$ impossible? What number makes A^{-1} impossible?

25 Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues?

26 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [?]$$

What is the angle between the eigenvectors?

27 (MATLAB) Take two symmetric matrices with different eigenvectors, say $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 8 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Graph the eigenvalues $\lambda_1(A+tB)$ and $\lambda_2(A+tB)$ for $-8 < t < 8$. Peter Lax says on page 113 of *Linear Algebra* that λ_1 and λ_2 appear to be on a collision course at certain values of t. "Yet at the last minute they turn aside." How close do they come?

Challenge Problems

28 For complex matrices, the symmetry $A^{\mathrm{T}} = A$ that produces real eigenvalues changes to $\overline{A}^{\mathrm{T}} = A$. From $\det(A - \lambda I) = 0$, find the eigenvalues of the 2 by 2 "Hermitian" matrix $A = [4 \quad 2+i; \quad 2-i \quad 0] = \overline{A}^{\mathrm{T}}$. To see why eigenvalues are real when $\overline{A}^{\mathrm{T}} = A$, adjust equation (1) of the text to $\overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x}$.

$$\textbf{Transpose to } \overline{x}^{\mathrm{T}}\,\overline{A}^{\mathrm{T}} = \overline{x}^{\mathrm{T}}\,\overline{\lambda}. \textbf{ With } \overline{A}^{\mathrm{T}} = A, \textbf{reach equation (2): } \lambda = \overline{\lambda}.$$

29 **Normal matrices** have $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$. For real matrices, $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ includes symmetric, skew-symmetric, and orthogonal. Those have real λ, imaginary λ, and $|\lambda| = 1$. Other normal matrices can have any complex eigenvalues λ.

Key point: ***Normal matrices have n orthonormal eigenvectors.*** Those vectors x_i probably will have complex components. In that complex case orthogonality means $\overline{x}_i^{\mathrm{T}}x_j = 0$ as Chapter 10 explains. Inner products (dot products) become $\overline{x}^{\mathrm{T}}y$.

The test for n orthonormal columns in Q becomes $\overline{Q}^{\mathrm{T}}Q = I$ instead of $Q^{\mathrm{T}}Q = I$.

A has n **orthonormal eigenvectors** $(A = Q\Lambda\overline{Q}^{\mathrm{T}})$ if and only if A is **normal**.

(a) Start from $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ with $\overline{Q}^{\mathrm{T}}Q = I$. Show that $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$: A is normal.

(b) Now start from $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$. Schur found $A = QT\overline{Q}^{\mathrm{T}}$ for every matrix A, with a triangular T. For normal matrices we must show (in 3 steps) that this T will actually be diagonal. Then $T = \Lambda$.

Step 1. Put $A = QT\overline{Q}^{\mathrm{T}}$ into $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$ to find $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$.

Step 2. Suppose $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$. Prove that $b = 0$.

Step 3. Extend Step 2 to size n. A normal triangular T must be diagonal.

30 If $\lambda_{\max}$ is the largest eigenvalue of a symmetric matrix A, no diagonal entry can be larger than $\lambda_{\max}$. What is the first entry a_{11} of $A = Q\Lambda Q^{\mathrm{T}}$? Show why $a_{11} \leq \lambda_{\max}$.

31 Suppose $A^{\mathrm{T}} = -A$ (real antisymmetric matrix). Explain these facts about A:

(a) $x^{\mathrm{T}}Ax = 0$ for every real vector x.

(b) The eigenvalues of A are pure imaginary.

(c) The determinant of A is positive or zero (not negative).

For (a), multiply out an example of $x^{\mathrm{T}}Ax$ and watch terms cancel. Or reverse $x^{\mathrm{T}}(Ax)$ to $(Ax)^{\mathrm{T}}x$. For (b), $Az = \lambda z$ leads to $\overline{z}^{\mathrm{T}}Az = \lambda\overline{z}^{\mathrm{T}}z = \lambda\|z\|^2$. Part (a) shows that $\overline{z}^{\mathrm{T}}Az = (x - iy)^{\mathrm{T}}A(x + iy)$ has zero real part. Then (b) helps with (c).

32 If A is symmetric and all its eigenvalues are $\lambda = 2$, how do you know that A must be $2I$? (Key point: Symmetry guarantees that A is diagonalizable. See "Proofs of the Spectral Theorem" on *web.mit.edu*/18.06.)

6.5 Positive Definite Matrices

This section concentrates on ***symmetric matrices that have positive eigenvalues***. If symmetry makes a matrix important, this extra property (*all* $\lambda > 0$) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called ***positive definite***.

The first problem is to recognize these matrices. You may say, just find the eigenvalues and test $\lambda > 0$. That is exactly what we want to avoid. Calculating eigenvalues is work. When the λ's are needed, we can compute them. But if we just want to know that they are positive, there are faster ways. Here are two goals of this section:

- To find *quick tests* on a symmetric matrix that guarantee *positive eigenvalues*.
- To explain important applications of positive definiteness.

The λ's are automatically real because the matrix is symmetric.

Start with 2 by 2. When does $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ ***have*** $\lambda_1 > 0$ ***and*** $\lambda_2 > 0$?

The eigenvalues of A are positive if and only if $a > 0$ ***and*** $ac - b^2 > 0$.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ is } \textit{not} \text{ positive definite because } ac - b^2 = 1 - 4 < 0$$

$$A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \text{ is positive definite because } a = 1 \text{ and } ac - b^2 = 6 - 4 > 0$$

$$A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix} \text{ is } \textit{not} \text{ positive definite (even with } \det A = +2) \text{ because } a = -1$$

Notice that we didn't compute the eigenvalues 3 and -1 of A_1. Positive trace $3 - 1 = 2$, negative determinant $(3)(-1) = -3$. And $A_3 = -A_2$ is *negative* definite. The positive eigenvalues for A_2, two negative eigenvalues for A_3.

Proof that the 2 by 2 test is passed when $\lambda_1 > 0$ *and* $\lambda_2 > 0$. Their product $\lambda_1\lambda_2$ is the determinant so $ac - b^2 > 0$. Their sum is the trace so $a + c > 0$. Then a and c are both positive (if one of them is not positive, $ac - b^2 > 0$ will fail). Problem 1 reverses the reasoning to show that the tests guarantee $\lambda_1 > 0$ and $\lambda_2 > 0$.

This test uses the 1 by 1 determinant a and the 2 by 2 determinant $ac - b^2$. When A is 3 by 3, $\det A > 0$ is the third part of the test. The next test requires *positive pivots*.

The eigenvalues of $A = A^{\mathrm{T}}$ ***are positive if and only if the pivots are positive:***

$$a > 0 \qquad \text{and} \qquad \frac{ac - b^2}{a} > 0.$$

$a > 0$ is required in both tests. So $ac > b^2$ is also required, for the determinant test and now the pivot. The point is to recognize that ratio as the *second pivot* of A:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \xrightarrow[\text{The multiplier is } b/a]{\text{The first pivot is } a} \quad \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix} \quad \begin{array}{c} \textbf{\textit{The second pivot is}} \\ c - \dfrac{b^2}{a} = \dfrac{ac - b^2}{a} \end{array}$$

This connects two big parts of linear algebra. ***Positive eigenvalues mean positive pivots and vice versa.*** We gave a proof for symmetric matrices of any size in the last section. The pivots give a quick test for $\lambda > 0$, and they are a lot faster to compute than the eigenvalues. It is very satisfying to see pivots and determinants and eigenvalues come together in this course.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix}$$

pivots 1 and -3	pivots 1 and 2	pivots -1 and -2
(indefinite)	**(positive definite)**	**(negative definite)**

Here is a different way to look at symmetric matrices with positive eigenvalues.

Energy-based Definition

From $A\boldsymbol{x} = \lambda\boldsymbol{x}$, multiply by $\boldsymbol{x}^{\mathrm{T}}$ to get $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. The right side is a positive λ times a positive number $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \|\boldsymbol{x}\|^2$. So $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ is positive for any eigenvector.

The new idea is that $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ ***is positive for all nonzero vectors*** $\boldsymbol{x}$, not just the eigenvectors. In many applications this number $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ (or $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$) is the **energy** in the system. The requirement of positive energy gives *another definition* of a positive definite matrix. I think this energy-based definition is the fundamental one.

Eigenvalues and pivots are two equivalent ways to test the new requirement $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > 0$.

Definition *A is positive definite if $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > \boldsymbol{0}$ for every nonzero vector $\boldsymbol{x}$:*

$$\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0. \tag{1}$$

The four entries a, b, b, c give the four parts of $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. From a and c come the pure squares ax^2 and cy^2. From b and b off the diagonal come the cross terms bxy and byx (the same). Adding those four parts gives $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. This energy-based definition leads to a basic fact:

If A and B are symmetric positive definite, so is $A + B$.

Reason: $\boldsymbol{x}^{\mathrm{T}}(A+B)\boldsymbol{x}$ is simply $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x}$. Those two terms are positive (for $\boldsymbol{x} \neq \boldsymbol{0}$) so $A + B$ is also positive definite. The pivots and eigenvalues are not easy to follow when matrices are added, but the energies just add.

$x^{\mathrm{T}}Ax$ also connects with our final way to recognize a positive definite matrix. Start with any matrix R, possibly rectangular. We know that $A = R^{\mathrm{T}}R$ is square and symmetric. More than that, A will be positive definite when R has independent columns:

If the columns of R are independent, then $A = R^{\mathrm{T}}R$ is positive definite.

Again eigenvalues and pivots are not easy. But the number $x^{\mathrm{T}}Ax$ is the same as $x^{\mathrm{T}}R^{\mathrm{T}}Rx$. That is exactly $(Rx)^{\mathrm{T}}(Rx)$—another important proof by parenthesis! That vector Rx is not zero when $x \neq 0$ (this is the meaning of independent columns). Then $x^{\mathrm{T}}Ax$ is the positive number $\|Rx\|^2$ and the matrix A is positive definite.

Let me collect this theory together, into five equivalent statements of positive definiteness. You will see how that key idea connects the whole subject of linear algebra: pivots, determinants, eigenvalues, and least squares (from $R^{\mathrm{T}}R$). Then come the applications.

***When a symmetric matrix has one of these five properties, it has them all*:**

1. All ***n pivots*** are positive.
2. All ***n upper left determinants*** are positive.
3. All ***n eigenvalues*** are positive.
4. $x^{\mathrm{T}}Ax$ is positive except at $x = 0$. This is the ***energy-based*** definition.
5. A equals $R^{\mathrm{T}}R$ for a matrix R with ***independent columns***.

The "upper left determinants" are 1 by 1, 2 by 2, . . ., n by n. The last one is the determinant of the complete matrix A. This remarkable theorem ties together the whole linear algebra course—at least for symmetric matrices. We believe that two examples are more helpful than a detailed proof (we nearly have a proof already).

Example 1 Test these matrices A and B for positive definiteness:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution The pivots of A are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of A are $2-\sqrt{2}$ and 2 and $2+\sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**.

We can write $x^{\mathrm{T}}Ax$ as a sum of three squares. The pivots 2, $\frac{3}{2}$, $\frac{4}{3}$ appear outside the squares. The multipliers $-\frac{1}{2}$ and $-\frac{2}{3}$ from elimination are inside the squares:

$$\begin{aligned} x^{\mathrm{T}}Ax &= 2\left(x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2\right) && \textbf{Rewrite with squares} \\ &= 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}(x_3)^2. && \textbf{This sum is positive.} \end{aligned}$$

I have two candidates to suggest for R. Either one will show that $A = R^{\mathrm{T}}R$ is positive definite. R can be a rectangular first difference matrix, 4 by 3, to produce those second differences $-1, 2, -1$ in A:

$$\boldsymbol{A = R^{\mathrm{T}}R} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The three columns of this R are independent. A is positive definite.

Another R comes from $A = LDL^{\mathrm{T}}$ (the symmetric version of $A = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $\sqrt{D}$ with L.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = \begin{matrix} (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = R^{\mathrm{T}}R. \\ \textbf{\textit{R is the Cholesky factor}} \end{matrix} \qquad (2)$$

This choice of R has square roots (not so beautiful). But it is the only R that is 3 by 3 and upper triangular. It is the "Cholesky factor" of A and it is computed by MATLAB's command $R = \mathrm{chol}(A)$. In applications, the rectangular R is how we build A and this Cholesky R is how we break it apart.

Eigenvalues give the symmetric choice $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$. This is also successful with $R^{\mathrm{T}}R = Q\Lambda Q^{\mathrm{T}} = A$. All these tests show that the $-1, 2, -1$ matrix A is positive definite.

Now turn to B, where the $(1,3)$ and $(3,1)$ entries move away from 0 to b. This b must not be too large! *The determinant test is easiest.* The 1 by 1 determinant is 2, the 2 by 2 determinant is still 3. The 3 by 3 determinant involves b:

$$\det B = 4 + 2b - 2b^2 = (1+b)(4-2b) \quad \text{must be positive.}$$

At $b = -1$ and $b = 2$ we get $\det B = 0$. *Between $b = -1$ and $b = 2$ the matrix is positive definite.* The corner entry $b = 0$ in the first matrix A was safely between.

Positive Semidefinite Matrices

Often we are at the edge of positive definiteness. The determinant is zero. The smallest eigenvalue is zero. The energy in its eigenvector is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}0\boldsymbol{x} = 0$. These matrices on the edge are called *positive semidefinite*. Here are two examples (not invertible):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ are positive semidefinite.}$$

A has eigenvalues 5 and 0. Its upper left determinants are 1 and 0. Its rank is only 1. This matrix A factors into $R^{\mathrm{T}}R$ with **dependent columns** in R:

$$\begin{matrix} \textbf{Dependent columns} \\ \textbf{Positive semidefinite} \end{matrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = R^{\mathrm{T}}R.$$

If 4 is increased by any small number, the matrix will become positive definite.

The cyclic B also has zero determinant (computed above when $b = -1$). It is singular. The eigenvector $\boldsymbol{x} = (1,1,1)$ has $B\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} = 0$. Vectors $\boldsymbol{x}$ in all other directions do give positive energy. This B can be written as $R^{\mathrm{T}}R$ in many ways, but R will always have *dependent* columns, with $(1,1,1)$ in its nullspace:

Second differences A
from first differences $R^{\mathrm{T}}R$
Cyclic A from cyclic R

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Positive semidefinite matrices have all $\lambda \geq 0$ and all $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} \geq 0$. Those weak inequalities ($\geq$ instead of $>$) include positive definite matrices and the singular matrices at the edge.

First Application: The Ellipse $ax^2 + 2bxy + cy^2 = 1$

Think of a tilted ellipse $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$. Its center is $(0,0)$, as in Figure 6.7a. Turn it to line up with the coordinate axes (X and Y axes). That is Figure 6.7b. These two pictures show the geometry behind the factorization $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$:

1. The tilted ellipse is associated with A. Its equation is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$.
2. The lined-up ellipse is associated with Λ. Its equation is $\boldsymbol{X}^{\mathrm{T}}\Lambda\boldsymbol{X} = 1$.
3. The rotation matrix that lines up the ellipse is the eigenvector matrix Q.

Example 2 Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$.

Solution Start with the positive definite matrix that matches this equation:

The equation is $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$ The matrix is $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$

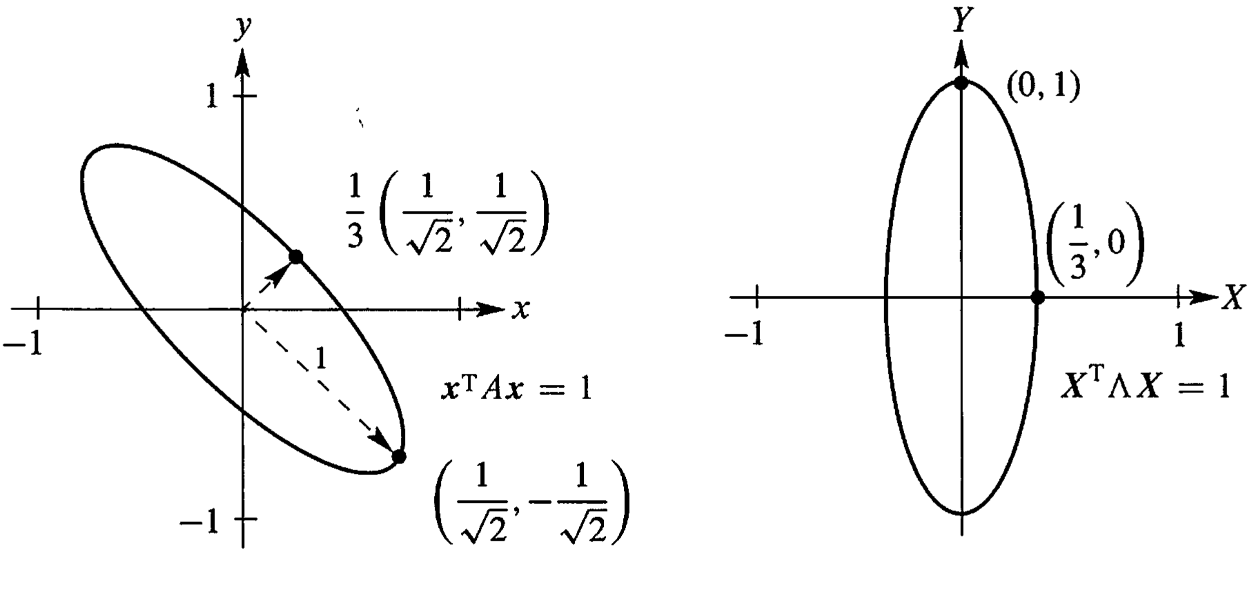


Figure 6.7: The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

The eigenvectors are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. Divide by $\sqrt{2}$ for unit vectors. Then $A = Q\Lambda Q^{\mathrm{T}}$:

$$\begin{array}{l}\textbf{Eigenvectors in } Q\\ \textbf{Eigenvalues 9 and 1}\end{array} \qquad \begin{bmatrix}5 & 4\\4 & 5\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\mathbf{9} & 0\\0 & \mathbf{1}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}.$$

Now multiply by $\begin{bmatrix}x & y\end{bmatrix}$ on the left and $\begin{bmatrix}x\\y\end{bmatrix}$ on the right to get back to $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$:

$$x^{\mathrm{T}}Ax = \textbf{sum of squares} \quad 5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2. \tag{3}$$

The coefficients are not the pivots 5 and 9/5 from D, they are the eigenvalues 9 and 1 from Λ. Inside *these* squares are the eigenvectors $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$.

The axes of the tilted ellipse point along the eigenvectors. This explains why $A = Q\Lambda Q^{\mathrm{T}}$ is called the "principal axis theorem"—it displays the axes. Not only the axis directions (from the eigenvectors) but also the axis lengths (from the eigenvalues). To see it all, use capital letters for the new coordinates that line up the ellipse:

$$\textbf{Lined up} \qquad \frac{x+y}{\sqrt{2}} = X \qquad \text{and} \qquad \frac{x-y}{\sqrt{2}} = Y \qquad \text{and} \qquad 9X^2 + Y^2 = 1.$$

The largest value of X^2 is $1/9$. The endpoint of the shorter axis has $X = 1/3$ and $Y = 0$. Notice: The *bigger* eigenvalue λ_1 gives the *shorter* axis, of half-length $1/\sqrt{\lambda_1} = 1/3$. The smaller eigenvalue $\lambda_2 = 1$ gives the greater length $1/\sqrt{\lambda_2} = 1$.

In the xy system, the axes are along the eigenvectors of A. In the XY system, the axes are along the eigenvectors of Λ—the coordinate axes. All comes from $A = Q\Lambda Q^{\mathrm{T}}$.

Suppose $A = Q\Lambda Q^{\mathrm{T}}$ is positive definite, so $\lambda_i > 0$. The graph of $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ is an ellipse:

$$\begin{bmatrix}x & y\end{bmatrix} Q\Lambda Q^{\mathrm{T}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}X & Y\end{bmatrix}\Lambda\begin{bmatrix}X\\Y\end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2 = 1.$$

The axes point along eigenvectors. The half-lengths are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$.

$A = I$ gives the circle $x^2 + y^2 = 1$. If one eigenvalue is negative (exchange 4's and 5's in A), we don't have an ellipse. The sum of squares becomes a *difference of squares*: $9X^2 - Y^2 = 1$. This indefinite matrix gives a *hyperbola*. For a negative definite matrix like $A = -I$, with both λ's negative, the graph of $-x^2 - y^2 = 1$ has no points at all.

■ REVIEW OF THE KEY IDEAS ■

1. Positive definite matrices have positive eigenvalues and positive pivots.

2. A quick test is given by the upper left determinants: $a > 0$ and $ac - b^2 > 0$.

3. The graph of x^TAx is then a "bowl" going up from $x = 0$:

$$x^TAx = ax^2 + 2bxy + cy^2 \text{ is positive except at } (x, y) = (0, 0).$$

4. $A = R^TR$ is automatically positive definite if R has independent columns.

5. The ellipse $x^TAx = 1$ has its axes along the eigenvectors of A. Lengths $1/\sqrt{\lambda}$.

■ WORKED EXAMPLES ■

6.5 A The great factorizations of a symmetric matrix are $A = LDL^T$ from pivots and multipliers, and $A = Q\Lambda Q^T$ from eigenvalues and eigenvectors. Show that $x^TAx > 0$ for all nonzero x exactly when the pivots and eigenvalues are positive. Try these n by n tests on pascal(6) and ones(6) and hilb(6) and other matrices in MATLAB's gallery.

Solution To prove $x^TAx > 0$, put parentheses into x^TLDL^Tx and $x^TQ\Lambda Q^Tx$:

$$x^TAx = (L^Tx)^TD(L^Tx) \qquad \text{and} \qquad x^TAx = (Q^Tx)^T\Lambda(Q^Tx).$$

If x is nonzero, then $y = L^Tx$ and $z = Q^Tx$ are nonzero (those matrices are invertible). So $x^TAx = y^TDy = z^T\Lambda z$ becomes a sum of squares and A is shown as positive definite:

$$\begin{array}{llllllll} \textbf{Pivots} & x^TAx & = & y^TDy & = & d_1y_1^2 + \cdots + d_ny_n^2 & > & 0 \\ \textbf{Eigenvalues} & x^TAx & = & z^T\Lambda z & = & \lambda_1z_1^2 + \cdots + \lambda_nz_n^2 & > & 0 \end{array}$$

MATLAB has a gallery of unusual matrices (type help gallery) and here are four:

pascal(6) is positive definite because all its pivots are 1 (Worked Example **2.6 A**).

ones(6) is positive *semidefinite* because its eigenvalues are $0, 0, 0, 0, 0, 6$.

H=hilb(6) is positive definite even though eig(H) shows two eigenvalues very near zero.

Hilbert matrix $x^THx = \int_0^1 (x_1 + x_2s + \cdots + x_6s^5)^2\, ds > 0$, $H_{ij} = 1/(i + j + 1)$.

rand(6)+rand(6)$'$ can be positive definite or not. Experiments gave only 2 in 20000.

$n = 20000$; $p = 0$; for $k = 1\!:\!n$, $A =$ rand(6); $p = p +$ all(eig($A + A'$) > 0); end, $p\,/\,n$

6.5 B **When is the symmetric block matrix** $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ **positive definite?**

Solution Multiply the first row of M by B^TA^{-1} and subtract from the second row, to get a block of zeros. The *Schur complement* $S = C - B^TA^{-1}B$ appears in the corner:

$$\begin{bmatrix} I & 0 \\ -B^TA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C - B^TA^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \tag{4}$$

Those two blocks A and S must be positive definite. Their pivots are the pivots of M.

6.5 C ***Second application: Test for a minimum.*** Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

Solution For $f(x)$, the test for a minimum comes from calculus: $df/dx = 0$ and $d^2 f/dx^2 > 0$. Moving to two variables x and y produces a symmetric matrix H. It contains the four second derivatives of $F(x, y)$. Positive f'' changes to positive definite H:

Second derivative matrix $$H = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$

$F(x, y)$ has a minimum if H is positive definite. Reason: H reveals the important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$!

6.5 D Find the eigenvalues of the $-1, 2, -1$ tridiagonal n by n matrix K (my favorite).

Solution The best way is to guess λ and $\boldsymbol{x}$. Then check $K\boldsymbol{x} = \lambda \boldsymbol{x}$. Guessing could not work for most matrices, but special cases are a big part of mathematics (pure and applied).

The key is hidden in a differential equation. The second difference matrix K is like a *second derivative*, and those eigenvalues are much easier to see:

Eigenvalues $\lambda_1, \lambda_2, \ldots$
Eigenfunctions $y_1, y_2, \ldots$ $$\frac{d^2 y}{dx^2} = \lambda y(x) \quad \textbf{with} \quad \begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix} \tag{5}$$

Try $y = \sin cx$. Its second derivative is $y'' = -c^2 \sin cx$. So the eigenvalue will be $\lambda = -c^2$, provided $y(x)$ satisfies the end point conditions $y(0) = 0 = y(1)$.

Certainly $\sin 0 = 0$ (this is where cosines are eliminated by $\cos 0 = 1$). At $x = 1$, we need $y(1) = \sin c = 0$. The number c must be $k\pi$, a multiple of π, and λ is $-c^2$:

Eigenvalues $\lambda = -k^2\pi^2$
Eigenfunctions $y = \sin k\pi x$ $$\frac{d^2}{dx^2} \sin k\pi x = -k^2\pi^2 \sin k\pi x. \tag{6}$$

Now we go back to the matrix K and guess its eigenvectors. They come from $\sin k\pi x$ at n points $x = h, 2h, \ldots, nh$, equally spaced between 0 and 1. The spacing Δx is $h = 1/(n+1)$, so the $(n+1)$st point comes out at $(n+1)h = 1$. Multiply that sine vector s by K:

Eigenvector of K = sine vector s $$\begin{matrix} Ks = \lambda s = (2 - 2\cos k\pi h)\, s \\ s = (\sin k\pi h, \ldots, \sin nk\pi h). \end{matrix} \tag{7}$$

I will leave that multiplication $Ks = \lambda s$ as a challenge problem. Notice what is important:

1. All eigenvalues $2 - 2\cos k\pi h$ are positive and K is positive definite.

2. The **sine matrix** S has orthogonal columns = eigenvectors $s_1, \ldots, s_n$ of K.

Discrete Sine Transform
The j, k entry is $\sin jk\pi h$

$$S = \begin{bmatrix} \sin \pi h & & \sin k\pi h & \\ \vdots & \cdots & \vdots & \cdots \\ \sin n\pi h & & \sin nk\pi h & \end{bmatrix}$$

Those eigenvectors are orthogonal just like the eigenfunctions: $\int_0^1 \sin j\pi x \sin k\pi x \, dx = 0$.

Problem Set 6.5

Problems 1–13 are about tests for positive definiteness.

1 Suppose the 2 by 2 tests $a > 0$ and $ac - b^2 > 0$ are passed. Then $c > b^2/a$ is also positive.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals _____.

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals _____.

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

2 Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compute the λ's. Find an x so that $x^{\mathrm{T}} A_1 x < 0$, so A_1 fails the test.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

3 For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \quad A = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^{T}.

4 What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

5 Write $f(x, y) = x^2 + 4xy + 3y^2$ as a *difference* of squares and find a point (x, y) where f is negative. The minimum is not at $(0, 0)$ even though f has positive coefficients.

6 The function $f(x, y) = 2xy$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix A produces this f? What are its eigenvalues?

7 Test to see if $R^{\mathrm{T}}R$ is positive definite in each case:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

8 The function $f(x, y) = 3(x+2y)^2 + 4y^2$ is positive except at $(0,0)$. What is the matrix in $f = [\,x \quad y\,]A[\,x \quad y\,]^{\mathrm{T}}$? Check that the pivots of A are 3 and 4.

9 Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

10 Which 3 by 3 symmetric matrices A and B produce these quadratics?

$$\begin{aligned} x^{\mathrm{T}}Ax &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right). \quad \text{Why is } A \text{ positive definite?} \\ x^{\mathrm{T}}Bx &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right). \quad \text{Why is } B \text{ semidefinite?} \end{aligned}$$

11 Compute the three upper left determinants of A to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

12 For what numbers c and d are A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

13 Find a matrix with $a > 0$ and $c > 0$ and $a + c > 2b$ that has a negative eigenvalue.

Problems 14–20 are about applications of the tests.

14 *If A is positive definite then A^{-1} is positive definite.* Best proof: The eigenvalues of A^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \_\_\_\_\_.$$

15 *If A and B are positive definite, their sum $A + B$ is positive definite.* Pivots and eigenvalues are not convenient for $A + B$. Better to prove $x^{\mathrm{T}}(A + B)x > 0$. Or if $A = R^{\mathrm{T}}R$ and $B = S^{\mathrm{T}}S$, show that $A + B = [\,R \ S\,]^{\mathrm{T}} \begin{bmatrix} R \\ S \end{bmatrix}$ with independent columns.

16 A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Ax > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad, \quad, \quad).$$

17 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a _____ on the main diagonal.

18 If $Ax = \lambda x$ then $x^{\mathrm{T}}Ax =$ _____. If $x^{\mathrm{T}}Ax > 0$, prove that $\lambda > 0$.

19 Reverse Problem 18 to show that *if all* $\lambda > 0$ *then* $x^{\mathrm{T}}Ax > 0$. We must do this for *every* nonzero x, not just the eigenvectors. So write x as a combination of the eigenvectors and *explain why all "cross terms" are* $x_i^{\mathrm{T}}x_j = 0$. Then $x^{\mathrm{T}}Ax$ is

$$(c_1x_1 + \cdots + c_nx_n)^{\mathrm{T}}(c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n) = c_1^2\lambda_1x_1^{\mathrm{T}}x_1 + \cdots + c_n^2\lambda_nx_n^{\mathrm{T}}x_n > 0.$$

20 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

Problems 21–24 use the eigenvalues; Problems 25–27 are based on pivots.

21 For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

22 From $A = Q\Lambda Q^{\mathrm{T}}$ compute the positive definite symmetric square root $Q\Lambda^{1/2}Q^{\mathrm{T}}$ of each matrix. Check that this square root gives $R^2 = A$:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

23 You may have seen the equation for an ellipse as $x^2/a^2 + y^2/b^2 = 1$. What are a and b when the equation is written $\lambda_1x^2 + \lambda_2y^2 = 1$? The ellipse $9x^2 + 4y^2 = 1$ has axes with half-lengths $a =$ _____ and $b =$ _____.

24 Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix A.

25 With positive pivots in D, the factorization $A = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $A = C^{\mathrm{T}}C$ which is "symmetrized LU":

$$\text{From } C = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \text{ find } A. \qquad \text{From } A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \text{ find } C = \mathbf{chol}(A).$$

26 In the Cholesky factorization $A = C^{\mathrm{T}}C$, with $C^{\mathrm{T}} = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (upper triangular) for

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$

27 The symmetric factorization $A = LDL^{\mathrm{T}}$ means that $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The left side is $ax^2 + 2bxy + cy^2$. The right side is $a\left(x + \frac{b}{a}y\right)^2 +$ ____ y^2. The second pivot completes the square! Test with $a = 2, b = 4, c = 10$.

28 Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.

29 For $F_1(x, y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x, y) = x^3 + xy - x$ find the second derivative matrices H_1 and H_2:

$$\textbf{Test for minimum}, \quad H = \begin{bmatrix} \partial^2F/\partial x^2 & \partial^2F/\partial x\partial y \\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2 \end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up ($=$ convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where first derivatives are zero).

30 The graph of $z = x^2 + y^2$ is a bowl opening upward. *The graph* of $z = x^2 - y^2$ *is a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0, 0)$?

31 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

Challenge Problems

32 A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups (as in 2.7.37)?

Invent a "subgroup" of two of these groups (not I by itself = the smallest group).

(a) Positive definite symmetric matrices A.

(b) Orthogonal matrices Q.

(c) All exponentials e^{tA} of a fixed matrix A.

(d) Matrices P with positive eigenvalues.

(e) Matrices D with determinant 1.

33 When A and B are symmetric positive definite, AB might not even be symmetric. But its eigenvalues are still positive. Start from $ABx = \lambda x$ and take dot products with Bx. Then prove $\lambda > 0$.

34 Write down the 5 by 5 sine matrix S from Worked Example **6.5 D**, containing the eigenvectors of K when $n = 5$ and $h = 1/6$. Multiply K times S to see the five positive eigenvalues.

Their sum should equal the trace 10. Their product should be det $K = 6$.

35 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \mathbf{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \mathbf{0}$ whenever $\boldsymbol{x} \neq \mathbf{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $A^{\mathrm{T}}CA$ is positive definite: *the crucial matrix in engineering.*

6.6 Similar Matrices

The key step in this chapter is to diagonalize a matrix by using its eigenvectors. When S is the eigenvector matrix, the diagonal matrix $S^{-1}AS$ is Λ—the eigenvalue matrix. But diagonalization is not possible for every A. Some matrices have too few eigenvectors—we had to leave them alone. In this new section, the eigenvector matrix S remains the best choice when we can find it, ***but now we allow any invertible matrix M.***

Starting from A we go to $M^{-1}AM$. This matrix may be diagonal—probably not. It still shares important properties of A. No matter which M we choose, ***the eigenvalues stay the same.*** The matrices A and $M^{-1}AM$ are called "similar". A typical matrix A is similar to a whole family of other matrices because there are so many choices of M.

DEFINITION Let M be any invertible matrix. Then $B = M^{-1}AM$ is ***similar*** to A.

If $B = M^{-1}AM$ then immediately $A = MBM^{-1}$. That means: If B is similar to A then A is similar to B. The matrix in this reverse direction is M^{-1}—just as good as M.

A diagonalizable matrix is similar to Λ. In that special case M is S. We have $A = S\Lambda S^{-1}$ and $\Lambda = S^{-1}AS$. They certainly have the same eigenvalues! This section is opening up to other similar matrices $B = M^{-1}AM$, by allowing all invertible M.

The combination $M^{-1}AM$ appears when we change variables in a differential equation. Start with an equation for $\boldsymbol{u}$ and set $\boldsymbol{u} = M\boldsymbol{v}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \quad \text{becomes} \quad M\frac{d\boldsymbol{v}}{dt} = AM\boldsymbol{v} \quad \text{which is} \quad \frac{d\boldsymbol{v}}{dt} = M^{-1}AM\boldsymbol{v}.$$

The original coefficient matrix was A, the new one at the right is $M^{-1}AM$. Changing $\boldsymbol{u}$ to $\boldsymbol{v}$ leads to a similar matrix. When $M = S$ the new system is diagonal—the maximum in simplicity. Other choices of M could make the new system triangular and easier to solve. Since we can always go back to $\boldsymbol{u}$, similar matrices must give the same growth or decay. More precisely, *the eigenvalues of A and B are the same.*

(No change in λ's) Similar matrices A and $M^{-1}AM$ have the **same eigenvalues**. If $\boldsymbol{x}$ is an eigenvector of A, then $M^{-1}\boldsymbol{x}$ is an eigenvector of $B = M^{-1}AM$.

The proof is quick, since $B = M^{-1}AM$ gives $A = MBM^{-1}$. Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$MBM^{-1}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \textbf{\textit{means that}} \quad B(M^{-1}\boldsymbol{x}) = \lambda(M^{-1}\boldsymbol{x}).$$

The eigenvalue of B is the same λ. The eigenvector has changed to $M^{-1}\boldsymbol{x}$.

Two matrices can have the same *repeated* λ, and fail to be similar—as we will see.

Example 1 These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0.

$$\text{The projection } A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \text{ is similar to } \Lambda = S^{-1}AS = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\text{Now choose } M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}. \text{ The similar matrix } M^{-1}AM \text{ is } \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

$$\text{Also choose } M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ The similar matrix } M^{-1}AM \text{ is } \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}.$$

All 2 by 2 matrices with those eigenvalues 1 and 0 are similar to each other. The eigenvectors change with M, the eigenvalues don't change.

The eigenvalues in that example are *not repeated*. This makes life easy. Repeated eigenvalues are harder. The next example has eigenvalues 0 and 0. The zero matrix shares those eigenvalues, but it is similar only to itself: $M^{-1}0M = 0$.

Example 2 A family of similar matrices with $\lambda = 0, 0$ (repeated eigenvalue)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is similar to } \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and all } B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix} \text{ except } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

These matrices B all have zero determinant (like A). They all have rank one (like A). One eigenvalue is zero and the trace is $cd - dc = 0$, so the other must be zero. I chose any $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $ad - bc = 1$, and $B = M^{-1}AM$.

These matrices B can't be diagonalized. In fact A is as close to diagonal as possible. It is the "***Jordan form***" for the family of matrices B. This is the outstanding member (my class says "Godfather") of the family. The Jordan form $J = A$ is as near as we can come to diagonalizing these matrices, when there is only one eigenvector. In going from A to $B = M^{-1}AM$, some things change and some don't. Here is a table to show this.

Not changed by *M*	**Changed by *M***
Eigenvalues	**Eigenvectors**
Trace and determinant	**Nullspace**
Rank	**Column space**
Number of independent eigenvectors	**Row space**
	Left nullspace
Jordan form	**Singular values**

The eigenvalues don't change for similar matrices; the eigenvectors do. The trace is the sum of the λ's (unchanged). The determinant is the product of the same λ's.[1] The nullspace consists of the eigenvectors for $\lambda = 0$ (if any), so it can change. Its dimension $n - r$ does not change! The *number* of eigenvectors stays the same for each λ, while the vectors themselves are multiplied by M^{-1}. The *singular values* depend on $A^{\mathrm{T}}A$, which definitely changes. They come in the next section.

[1]The determinant is unchanged because $\det B = (\det M^{-1})(\det A)(\det M) = \det A$.

Examples of the Jordan Form

The *Jordan form* is the serious new idea here. We lead up to it with one more example of similar matrices: *triple eigenvalue, one eigenvector.*

Example 3 This **Jordan matrix** J has $\lambda = 5, 5, 5$ on its diagonal. Its only eigenvectors are multiples of $\boldsymbol{x} = (1, 0, 0)$. Algebraic multiplicity is 3, geometric multiplicity is 1:

$$\text{If} \quad J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{then} \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalue 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 1. So every B that is similar to this "Jordan block" J has only one independent eigenvector $M^{-1}\boldsymbol{x}$.

The transpose matrix J^{T} has the same eigenvalues 5, 5, 5, and $J^{\mathrm{T}} - 5I$ has the same rank 2. ***Jordan's theorem says that J^{T} is similar to J.*** The matrix M that produces the similarity happens to be the *reverse identity*:

$$J^{\mathrm{T}} = M^{-1}JM \quad \text{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

All blank entries are zero. An eigenvector of J^{T} is $M^{-1}(1, 0, 0) = (0, 0, 1)$. There is one line of eigenvectors $(x_1, 0, 0)$ for J and another line $(0, 0, x_3)$ for J^{T}.

The key fact is that this matrix J is similar to *every* matrix A with eigenvalues 5, 5, 5 and *one line* of eigenvectors. There is an M with $M^{-1}AM = J$.

Example 4 Since J is as close to diagonal as we can get, the equation $d\boldsymbol{u}/dt = J\boldsymbol{u}$ cannot be simplified by changing variables. We must solve it as it stands:

$$\frac{d\boldsymbol{u}}{dt} = J\boldsymbol{u} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is} \quad \begin{aligned} dx/dt &= 5x + y \\ dy/dt &= 5y + z \\ dz/dt &= 5z. \end{aligned}$$

The system is triangular. We think naturally of back substitution. Solve the last equation and work upwards. Main point: *All solutions contain e^{5t}* since $\lambda = 5$:

$$\begin{array}{llll} \textbf{Last equation} & \dfrac{dz}{dt} = 5z & \text{yields} & z = z(0)e^{5t} \\[2ex] \textbf{Notice } te^{5t} & \dfrac{dy}{dt} = 5y + z & \text{yields} & y = \big(y(0) + tz(0)\big)e^{5t} \\[2ex] \textbf{Notice } t^2e^{5t} & \dfrac{dx}{dt} = 5x + y & \text{yields} & x = \big(x(0) + ty(0) + \frac{1}{2}t^2z(0)\big)e^{5t}. \end{array}$$

The two missing eigenvectors are responsible for the te^{5t} and t^2e^{5t} terms in y and x. The factors t and t^2 enter because $\lambda = 5$ is a triple eigenvalue with one eigenvector.

Note Chapter 7 will explain another approach to similar matrices. Instead of changing variables by $\boldsymbol{u} = M\boldsymbol{v}$, we "***change the basis***". In this approach, similar matrices will represent the same transformation of n-dimensional space. When we choose a basis for $\mathbf{R}^n$, we get a matrix. The standard basis vectors ($M = I$) lead to $I^{-1}AI$ which is A. Other bases lead to similar matrices $B = M^{-1}AM$.

The Jordan Form

For every A, we want to choose M so that $M^{-1}AM$ is as ***nearly diagonal as possible***. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = S$. The matrix $S^{-1}AS$ is diagonal, period. This matrix Λ is the Jordan form of A—when A can be diagonalized. In the general case, eigenvectors are missing and Λ can't be reached.

Suppose A has s independent eigenvectors. Then it is similar to a matrix with s blocks. Each block is like J in Example 3. *The eigenvalue is on the diagonal with* 1's *just above it.* This block accounts for one eigenvector of A. When there are n eigenvectors and n blocks, they are all 1 by 1. In that case J is Λ.

(Jordan form) If A has s independent eigenvectors, it is similar to a matrix J that has s Jordan blocks on its diagonal: Some matrix M puts A into Jordan form:

Jordan form
$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J. \tag{1}$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

Jordan block
$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}. \tag{2}$$

A is similar to B if they share the same Jordan form J—not otherwise.

The Jordan form J has an off-diagonal 1 for each missing eigenvector (and the 1's are next to the eigenvalues). This is the big theorem about matrix similarity. In every family of similar matrices, we are picking one outstanding member called J. It is nearly diagonal (or if possible completely diagonal). For that J, we can solve $d\boldsymbol{u}/dt = J\boldsymbol{u}$ as in Example 4. We can take powers J^k as in Problems 9–10. Every other matrix in the family has the form $A = MJM^{-1}$. The connection through M solves $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

The point you must see is that $MJM^{-1}MJM^{-1} = MJ^2M^{-1}$. That cancellation of $M^{-1}M$ in the middle has been used through this chapter (when M was S). We found A^{100} from $S\Lambda^{100}S^{-1}$—by diagonalizing the matrix. Now we can't quite diagonalize A. So we use $MJ^{100}M^{-1}$ instead.

Jordan's Theorem is proved in my textbook *Linear Algebra and Its Applications*. Please refer to that book (or more advanced books) for the proof. The reasoning is rather intricate and in actual computations the Jordan form is not at all popular—its calculation is not stable. A slight change in A will separate the repeated eigenvalues and remove the off-diagonal 1's—switching to a diagonal Λ.

Proved or not, you have caught the central idea of similarity—to make A as simple as possible while preserving its essential properties.

■ REVIEW OF THE KEY IDEAS ■

1. B is similar to A if $B = M^{-1}AM$, for some invertible matrix M.
2. Similar matrices have the same eigenvalues. Eigenvectors are multiplied by M^{-1}.
3. If A has n independent eigenvectors then A is similar to Λ (take $M = S$).
4. Every matrix is similar to a Jordan matrix J (which has Λ as its diagonal part). J has a block for each eigenvector, and 1's for missing eigenvectors.

■ WORKED EXAMPLES ■

6.6 A The 4 by 4 triangular Pascal matrix A and its inverse (alternating diagonals) are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Check that A and A^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $A^{-1} = D^{-1}AD$. This A is similar to A^{-1}, which is unusual.

These similar matrices must have the same Jordan form J. This J has only one block because the Pascal matrix has only one line of eigenvectors.

Solution The triangular matrices A and A^{-1} both have $\lambda = 1, 1, 1, 1$ on their main diagonals. Choose D with alternating 1 and -1 on its diagonal. *D equals D^{-1}*:

$$D^{-1}AD = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = A^{-1}.$$

Check: Changing signs in rows 1 and 3 of A, and columns 1 and 3, produces the four negative entries in A^{-1}. We are multiplying row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals in A^{-1}. Then AD has *columns with alternating signs.*

6.6 B ***The best way to compute eigenvalues of a large matrix is not from solving*** $\det(A - \lambda I) = 0$. That high degree polynomial is a numerical disaster.

Instead we compute similar matrices $A_1, A_2, \ldots$ that approach a triangular matrix. Then the eigenvalues of A (unchanged) are almost sitting on the main diagonal.

One way is to factor $A = QR$ by "Gram-Schmidt". Reverse the order to $A_1 = RQ$. This matrix is similar to A because $RQ = Q^{-1}(QR)Q$. An example with $c = \cos\theta$ and $s = \sin\theta$ shows how a small off-diagonal s can be *cubed* in A_1:

$$A = \begin{bmatrix} c & s \\ s & 0 \end{bmatrix} \quad \text{factors into} \quad \begin{bmatrix} c & s \\ s & -c \end{bmatrix}\begin{bmatrix} 1 & cs \\ 0 & s^2 \end{bmatrix} = QR.$$

$$A_1 = RQ = \begin{bmatrix} c + cs^2 & s^3 \\ s^3 & -cs^2 \end{bmatrix} \quad \text{has } s^3 \text{ below the diagonal}$$

Another step can factor $A_1 = Q_1 R_1$ and reverse to $A_2 = R_1 Q_1$. This ***QR method*** is in Section 9.3 with a further improvement for A_1. Add cs^2 to its diagonal (to get zero in the corner) and then subtract back from A_2:

Shift and factor $A_1 + cs^2 I = Q_1 R_1$ **Reverse and shift back** $A_2 = R_1 Q_1 - cs^2 I$

Shifted QR is an amazing success—just about the best way to compute eigenvalues.

Problem Set 6.6

1 If $C = F^{-1}AF$ and also $C = G^{-1}BG$, what matrix M gives $B = M^{-1}AM$? Conclusion: If C is similar to A and also to B then _____.

2 If $A = \text{diag}(1, 3)$ and $B = \text{diag}(3, 1)$ show that A and B are similar (find an M).

3 Show that A and B are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

4 If a 2 by 2 matrix A has eigenvalues 0 and 1, why is it similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Deduce from Problem 1 that all 2 by 2 matrices with those eigenvalues are similar.

5 Which of these six matrices are similar? Check their eigenvalues.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

6 There are sixteen 2 by 2 matrices whose entries are 0's and 1's. Similar matrices go into the same family. How many families? How many matrices (total 16) in each family?

7 (a) If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

(b) The nullspaces of A and $M^{-1}AM$ have the same (vectors)(basis)(dimension).

8 Suppose $Ax = \lambda x$ and $Bx = \lambda x$ with the same λ's and x's. With n independent eigenvectors we have $A = B$: *Why*? Find $A \neq B$ when both have eigenvalues $0, 0$ but only one line of eigenvectors $(x_1, 0)$.

9 By direct multiplication find A^2 and A^3 and A^5 when

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Guess the form of A^k. Set $k = 0$ to find A^0 and $k = -1$ to find A^{-1}.

Questions 10–14 are about the Jordan form.

10 By direct multiplication, find J^2 and J^3 when

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Guess the form of J^k. Set $k = 0$ to find J^0. Set $k = -1$ to find J^{-1}.

11 Solve $du/dt = Ju$ for J in Problem 10, starting from $u(0) = (5, 2)$. Remember $te^{\lambda t}$.

12 These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: *J is not similar to K*.

13 Based on Problem 12, what are the five Jordan forms when $\lambda = 0, 0, 0, 0$?

14 Prove that *A^{T} is always similar to A* (we know the λ's are the same):

1. For one Jordan block J_i: Find M_i so that $M_i^{-1}J_iM_i = J_i^{\mathrm{T}}$ (see Example 3).
2. For any J with blocks J_i: Build M_0 from blocks so that $M_0^{-1}JM_0 = J^{\mathrm{T}}$.
3. For any $A = MJM^{-1}$: Show that A^{T} is similar to J^{T} and so to J and to A.

15 Prove that $\det(A - \lambda I) = \det(M^{-1}AM - \lambda I)$. (You could write $I = M^{-1}M$ and factor out $\det M^{-1}$ and $\det M$.) Since these *characteristic polynomials* of A and $M^{-1}AM$ are the same, the eigenvalues are the same (with the same multiplicities).

16 Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

17 True or false, with a good reason:

(a) A symmetric matrix can't be similar to a nonsymmetric matrix.

(b) An invertible matrix can't be similar to a singular matrix.

(c) A can't be similar to $-A$ unless $A = 0$.

(d) A can't be similar to $A + I$.

18 If B is invertible, prove that AB is similar to BA. *They have the same eigenvalues.*

19 If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But with blocks

$$M^{-1}FM = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

(a) What sizes are the four blocks (the same four sizes in each matrix)?

(b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has the 6 eigenvalues of AB plus 4 zeros; G has the 4 eigenvalues of BA plus 6 zeros. ***AB has the same eigenvalues as BA plus _____ zeros.***

20 Why are these statements all true?

(a) If A is similar to B then A^2 is similar to B^2.

(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same**. In this case $M =$ _____.

21 If J is the 5 by 5 Jordan block with $\lambda = 0$, find J^2 and count its eigenvectors and find its Jordan form (there will be two blocks).

Challenge Problems

22 If an n by n matrix A has all eigenvalues $\lambda = 0$, prove that $A^n =$ zero matrix. (Maybe prove first that $J^n =$ zero matrix, by direct multiplication. Or use the Cayley-Hamilton Theorem?)

23 For the shifted QR method in the Worked Example **6.6 B**, show that A_2 is similar to A_1. No change in eigenvalues, and the A's quickly approach a diagonal matrix.

24 If A is similar to A^{-1}, must all the eigenvalues equal 1 or -1?

6.7 Singular Value Decomposition (SVD)

The Singular Value Decomposition is a highlight of linear algebra. A is any m by n matrix, square or rectangular. Its rank is r. We will diagonalize this A, but not by $S^{-1}AS$. The eigenvectors in S have three big problems: They are usually not orthogonal, there are not always enough eigenvectors, and $Ax = \lambda x$ requires A to be square. The ***singular vectors*** of A solve all those problems in a perfect way.

The price we pay is to have two sets of singular vectors, $\boldsymbol{u}$'s and $\boldsymbol{v}$'s. The $\boldsymbol{u}$'s are eigenvectors of AA^{T} and the $\boldsymbol{v}$'s are eigenvectors of $A^{\mathrm{T}}A$. Since those matrices are both symmetric, their eigenvectors can be chosen orthonormal. In equation (13) below, the simple fact that A times $A^{\mathrm{T}}A$ is the same as AA^{T} times A will lead to a remarkable property of these $\boldsymbol{u}$'s and $\boldsymbol{v}$'s:

$$\textbf{“}\boldsymbol{A}\textbf{ is diagonalized”} \qquad A\boldsymbol{v}_1 = \sigma_1\boldsymbol{u}_1 \quad A\boldsymbol{v}_2 = \sigma_2\boldsymbol{u}_2 \quad \ldots \quad A\boldsymbol{v}_r = \sigma_r\boldsymbol{u}_r \qquad (1)$$

The singular vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are in the *row space* of A. The outputs $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ are in the *column space* of A. The *singular values* $\sigma_1, \ldots, \sigma_r$ are all positive numbers. When the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s go into the columns of V and U, orthogonality gives $V^{\mathrm{T}}V = I$ and $U^{\mathrm{T}}U = I$. The σ's go into a diagonal matrix Σ.

Just as $A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i$ led to the diagonalization $AS = S\Lambda$, the equations $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$ tell us column by column that $\boldsymbol{AV = U\Sigma}$:

$$\begin{matrix}(m \text{ by } n)(n \text{ by } r)\\ \text{equals}\\ (m \text{ by } r)(r \text{ by } r)\end{matrix} \qquad A\begin{bmatrix} & & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_r \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_r \\ & & \end{bmatrix}\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}. \qquad (2)$$

This is the heart of the SVD, but there is more. Those $\boldsymbol{v}$'s and $\boldsymbol{u}$'s account for the row space and column space of A. We need $n - r$ more $\boldsymbol{v}$'s and $m - r$ more $\boldsymbol{u}$'s, from the nullspace $\boldsymbol{N}(A)$ and the left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$. They can be orthonormal bases for those two nullspaces (and then automatically orthogonal to the first r $\boldsymbol{v}$'s and $\boldsymbol{u}$'s). Include all the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s in V and U, so these matrices become *square*. ***We still have*** $\boldsymbol{AV = U\Sigma}$.

$$\begin{matrix}(m \text{ by } n)(n \text{ by } n)\\ \text{equals}\\ (m \text{ by } m)(m \text{ by } n)\end{matrix} \qquad A\begin{bmatrix} & & & & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_r & \cdots & \boldsymbol{v}_n \\ & & & & \end{bmatrix} = \begin{bmatrix} & & & & \\ \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_r & \cdots & \boldsymbol{u}_m \\ & & & & \end{bmatrix}\begin{bmatrix}\sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \end{bmatrix} \qquad (3)$$

The new Σ is m by n. It is just the old r by r matrix (call that Σ_r) with $m - r$ new zero rows and $n - r$ new zero columns. The real change is in the shapes of U and V and Σ. Still $V^{\mathrm{T}}V = I$ and $U^{\mathrm{T}}U = I$, with sizes n and m.

V is now a square orthogonal matrix, with inverse $V^{-1} = V^{\mathrm{T}}$. So $AV = U\Sigma$ can become $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$. This is the ***Singular Value Decomposition***:

$$\textbf{SVD} \qquad A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \boldsymbol{u}_r\sigma_r\boldsymbol{v}_r^{\mathrm{T}}. \qquad (4)$$

I would write the earlier "reduced SVD" from equation (2) as $A = U_r \Sigma_r V_r^{\mathrm{T}}$. That is equally true, without the extra zeros in Σ. This reduced SVD gives the same splitting of A into a sum of r matrices, each of rank one.

We will see that $\sigma_i^2 = \lambda_i$ is an eigenvalue of $A^{\mathrm{T}}A$ and also AA^{T}. When we put the singular values in descending order, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$, the splitting in equation (4) gives the r rank-one pieces of A ***in order of importance***.

Example 1 When is $U\Sigma V^{\mathrm{T}}$ (singular values) the same as $S\Lambda S^{-1}$ (eigenvalues)?

Solution We need orthonormal eigenvectors in $S = U$. We need nonnegative eigenvalues in $\Lambda = \Sigma$. So A must be a ***positive semidefinite (or definite) symmetric matrix*** $\boldsymbol{Q\Lambda Q^{\mathrm{T}}}$.

Example 2 If $A = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ with unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, what is the SVD of A?

Solution The reduced SVD in (2) is exactly $\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$, with rank $r = 1$. It has $\boldsymbol{u}_1 = \boldsymbol{x}$ and $\boldsymbol{v}_1 = \boldsymbol{y}$ and $\sigma_1 = 1$. For the full SVD, complete $\boldsymbol{u}_1 = \boldsymbol{x}$ to an orthonormal basis of $\boldsymbol{u}$'s, and complete $\boldsymbol{v}_1 = \boldsymbol{y}$ to an orthonormal basis of $\boldsymbol{v}$'s. No new σ's.

I will describe an application before proving that $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$. This key equation gave the diagonalizations (2) and (3) and (4) of the SVD: $A = U\Sigma V^{\mathrm{T}}$.

Image Compression

Unusually, I am going to stop the theory and describe applications. This is the century of data, and often that data is stored in a matrix. *A digital image is really a matrix of pixel values.* Each little picture element or "pixel" has a gray scale number between black and white (it has three numbers for a color picture). The picture might have $512 = 2^9$ pixels in each row and $256 = 2^8$ pixels down each column. We have a 256 by 512 pixel matrix with 2^{17} entries! To store one picture, the computer has no problem. But a CT or MR scan produces an image at every cross section—a ton of data. If the pictures are frames in a movie, 30 frames a second means 108,000 images per hour. Compression is especially needed for high definition digital TV, or the equipment could not keep up in real time.

What is compression? We want to replace those 2^{17} matrix entries by a smaller number, *without losing picture quality*. A simple way would be to use larger pixels—replace groups of four pixels by their average value. This is 4 : 1 compression. But if we carry it further, like 16 : 1, our image becomes "blocky". We want to replace the mn entries by a smaller number, in a way that the human visual system won't notice.

Compression is a billion dollar problem and everyone has ideas. Later in this book I will describe Fourier transforms (used in **jpeg**) and wavelets (now in **JPEG2000**). Here we try an SVD approach: *Replace the* 256 *by* 512 *pixel matrix by* ***a matrix of rank one****: a column times a row.* If this is successful, the storage requirement becomes 256 + 512 (add instead of multiply). The compression ratio (256)(512)/(256 + 512) is better than 170 to 1. This is more than we hope for. We may actually use five matrices of rank one (so a matrix approximation of rank 5). The compression is still 34 : 1 and the crucial question is the picture quality.

Where does the SVD come in? *The best rank one approximation to A is the matrix $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$*. It uses the largest singular value σ_1. The best rank 5 approximation includes also $\sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}} + \cdots + \sigma_5 \boldsymbol{u}_5 \boldsymbol{v}_5^{\mathrm{T}}$. **The SVD puts the pieces of A in descending order.**

A library compresses a different matrix. The rows correspond to key words. Columns correspond to titles in the library. The entry in this *word-title matrix* is $a_{ij} = 1$ if word i is in title j (otherwise $a_{ij} = 0$). We normalize the columns so long titles don't get an advantage. We might use a table of contents or an abstract. (Other books might share the title "*Introduction to Linear Algebra*".) Instead of $a_{ij} = 1$, the entries of A can include the *frequency* of the search words. See Section 8.6 for the SVD in statistics.

Once the indexing matrix is created, the search is a linear algebra problem. This giant matrix has to be compressed. The SVD approach gives an optimal low rank approximation, better for library matrices than for natural images. There is an ever-present tradeoff in the cost to compute the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s. We still need a better way (with sparse matrices).

The Bases and the SVD

Start with a 2 by 2 matrix. Let its rank be $r = 2$, so A is invertible. We want $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to be perpendicular unit vectors. ***We also want $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ to be perpendicular.*** (This is the tricky part. It is what makes the bases special.) Then the unit vectors $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\|A\boldsymbol{v}_1\|$ and $\boldsymbol{u}_2 = A\boldsymbol{v}_2/\|A\boldsymbol{v}_2\|$ will be orthonormal. Here is a specific example:

$$\textbf{Unsymmetric matrix} \qquad A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}. \tag{5}$$

No orthogonal matrix Q will make $Q^{-1}AQ$ diagonal. We need $U^{-1}AV$. The two bases will be different—one basis cannot do it. The output is $A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1$ when the input is $\boldsymbol{v}_1$. The "singular values" σ_1 and σ_2 are the lengths $\|A\boldsymbol{v}_1\|$ and $\|A\boldsymbol{v}_2\|$.

$$\begin{matrix} \boldsymbol{AV = U\Sigma} \\ \boldsymbol{A = U\Sigma V^{\mathrm{T}}} \end{matrix} \qquad A \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \boldsymbol{u}_1 & \sigma_2 \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}. \tag{6}$$

There is a neat way to remove U and see V by itself. *Multiply A^{T} times A.*

$$A^{\mathrm{T}}A = (U\Sigma V^{\mathrm{T}})^{\mathrm{T}}(U\Sigma V^{\mathrm{T}}) = V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}. \tag{7}$$

$U^{\mathrm{T}}U$ disappears because it equals I. (We require $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u}_1 = 1 = \boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_2$ and $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u}_2 = 0$.) Multiplying those diagonal Σ^{T} and Σ gives σ_1^2 and σ_2^2. That leaves an ordinary diagonalization of the crucial symmetric matrix $A^{\mathrm{T}}A$, whose eigenvalues are σ_1^2 and σ_2^2:

$$\begin{matrix} \textbf{Eigenvalues } \sigma_1^2, \sigma_2^2 \\ \textbf{Eigenvectors } \boldsymbol{v}_1, \boldsymbol{v}_2 \end{matrix} \qquad A^{\mathrm{T}}A = V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^{\mathrm{T}}. \tag{8}$$

This is exactly like $A = Q\Lambda Q^{\mathrm{T}}$. But the symmetric matrix is not A itself. Now the symmetric matrix is $A^{\mathrm{T}}A$! *And the columns of V are the eigenvectors of $A^{\mathrm{T}}A$.* Last is U:

Compute the eigenvectors v and eigenvalues σ^2 of $A^{\mathrm{T}}A$. Then each $u = Av/\sigma$.

For large matrices LAPACK finds a special way to avoid multiplying $A^{\mathrm{T}}A$ in **svd** (A).

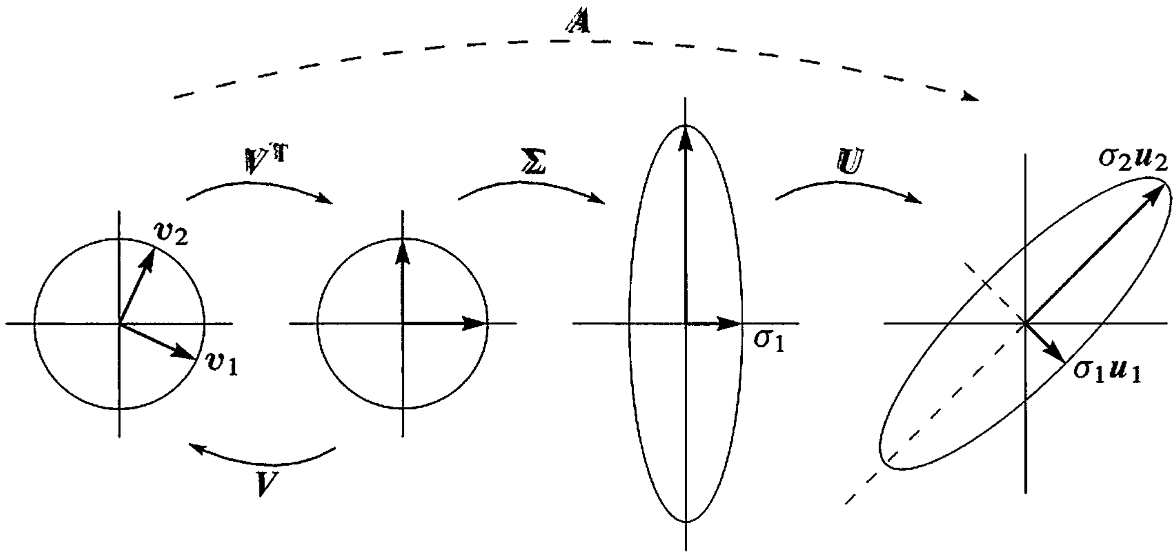


Figure 6.8: U and V are rotations and reflections. Σ stretches circle to ellipse.

Example 3 Find the singular value decomposition of that matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution Compute $A^{\mathrm{T}}A$ and its eigenvectors. Then make them unit vectors:

$$A^{\mathrm{T}}A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{has unit eigenvectors} \quad v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

The eigenvalues of $A^{\mathrm{T}}A$ are 8 and 2. The v's are perpendicular, because eigenvectors of every symmetric matrix are perpendicular—and $A^{\mathrm{T}}A$ is automatically symmetric.

Now the u's are quick to find, because Av_1 is going to be in the direction of u_1:

$$Av_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}. \quad \text{The unit vector is} \quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly Av_1 is the same as $2\sqrt{2}\, u_1$. The first singular value is $\sigma_1 = 2\sqrt{2}$. Then $\sigma_1^2 = 8$.

$$Av_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}. \quad \text{The unit vector is} \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now Av_2 is $\sqrt{2}\, u_2$ and $\sigma_2 = \sqrt{2}$. Thus σ_2^2 agrees with the other eigenvalue 2 of $A^{\mathrm{T}}A$.

$$A = U\Sigma V^{\mathrm{T}} \quad \text{is} \quad \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \tag{9}$$

This matrix, and every invertible 2 by 2 matrix, ***transforms the unit circle to an ellipse.*** You can see that in the figure, which was created by Cliff Long and Tom Hern.

One final point about that example. We found the $\boldsymbol{u}$'s from the $\boldsymbol{v}$'s. Could we find the $\boldsymbol{u}$'s directly? *Yes*, by multiplying AA^{T} instead of $A^{\mathrm{T}}A$:

$$\text{Use } V^{\mathrm{T}}V = I \qquad AA^{\mathrm{T}} = (U\Sigma V^{\mathrm{T}})(V\Sigma^{\mathrm{T}}U^{\mathrm{T}}) = U\Sigma\Sigma^{\mathrm{T}}U^{\mathrm{T}}. \tag{10}$$

Multiplying $\Sigma\Sigma^{\mathrm{T}}$ gives σ_1^2 and σ_2^2 as before. *The $\boldsymbol{u}$'s are eigenvectors of AA^{T}*:

$$\textbf{Diagonal in this example} \qquad AA^{\mathrm{T}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvectors $(1,0)$ and $(0,1)$ agree with $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ found earlier. Why take the first eigenvector to be $(1,0)$ instead of $(-1,0)$ or $(0,1)$? Because we have to follow $A\boldsymbol{v}_1$ (I missed that in my video lecture ...). Notice that AA^{T} has the same eigenvalues (8 and 2) as $A^{\mathrm{T}}A$. The singular values are $\sqrt{8}$ and $\sqrt{2}$.

Example 4 Find the SVD of the singular matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The rank is $r = 1$.

Solution The row space has only one basis vector $\boldsymbol{v}_1 = (1,1)/\sqrt{2}$. The column space has only one basis vector $\boldsymbol{u}_1 = (2,1)/\sqrt{5}$. Then $A\boldsymbol{v}_1 = (4,2)/\sqrt{2}$ must equal $\sigma_1\boldsymbol{u}_1$. It does, with $\sigma_1 = \sqrt{10}$.

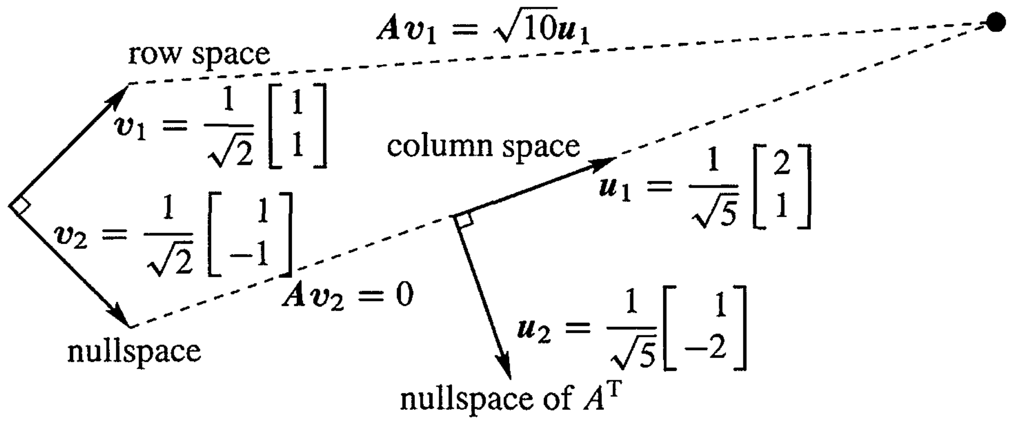


Figure 6.9: The SVD chooses orthonormal bases for 4 subspaces so that $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$.

The SVD could stop after the row space and column space (it usually doesn't). It is customary for U and V to be square. The matrices need a second column. ***The vector $\boldsymbol{v}_2$ is in the nullspace.*** It is perpendicular to $\boldsymbol{v}_1$ in the row space. Multiply by A to get $A\boldsymbol{v}_2 = \mathbf{0}$. We could say that the second singular value is $\sigma_2 = 0$, but singular values are like pivots—only the r nonzeros are counted.

$$\begin{array}{l} A = U\Sigma V^{\mathrm{T}} \\ \textbf{Full size} \end{array} \qquad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{11}$$

The matrices U and V contain orthonormal bases for all four subspaces:

first	r	columns of V :	*row space* of A
last	$n-r$	columns of V :	*nullspace* of A
first	r	columns of U :	*column space* of A
last	$m-r$	columns of U :	*nullspace* of A^{T}

The first columns $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$ are eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. We now explain why Av_i falls in the direction of u_i. The last v's and u's (in the nullspaces) are easier. As long as those are orthonormal, the SVD will be correct.

Proof of the SVD: Start from $A^{\mathrm{T}}Av_i = \sigma_i^2 v_i$, which gives the v's and σ's. Multiplying by v_i^{T} leads to $\|Av_i\|^2$. To prove that $Av_i = \sigma_i u_i$, the key step is to multiply by A:

$$v_i^{\mathrm{T}}A^{\mathrm{T}}Av_i = \sigma_i^2 v_i^{\mathrm{T}}v_i \quad \textbf{gives} \quad \|Av_i\|^2 = \sigma_i^2 \quad \textbf{so that} \quad \|Av_i\| = \sigma_i \tag{12}$$

$$AA^{\mathrm{T}}Av_i = \sigma_i^2 Av_i \quad \textbf{gives} \quad u_i = Av_i/\sigma_i \quad \textbf{as a unit eigenvector of } AA^{\mathrm{T}}. \tag{13}$$

Equation (12) used the small trick of placing parentheses in $(v_i^{\mathrm{T}}A^{\mathrm{T}})(Av_i) = \|Av_i\|^2$. Equation (13) placed the all-important parentheses in $(AA^{\mathrm{T}})(Av_i)$. This shows that Av_i is an eigenvector of AA^{T}. Divide by its length σ_i to get the unit vector $u_i = Av_i/\sigma_i$. These u's are orthogonal because $(Av_i)^{\mathrm{T}}(Av_j) = v_i^{\mathrm{T}}(A^{\mathrm{T}}Av_j) = v_i^{\mathrm{T}}(\sigma_j^2 v_j) = 0$.

I will give my opinion directly. The SVD is the climax of this linear algebra course. I think of it as the final step in the Fundamental Theorem. First come the *dimensions* of the four subspaces. Then their *orthogonality*. Then *the orthonormal bases diagonalize A*. It is all in the formula $A = U\Sigma V^{\mathrm{T}}$. You have made it to the top.

Eigshow (Part 2)

Section 6.1 described the MATLAB demo called **eigshow**. The first option is *eig*, when x moves in a circle and Ax follows on an ellipse. The second option is *svd*, when two vectors x and y stay perpendicular as they travel around a circle. Then Ax and Ay move too (not usually perpendicular). The four vectors are in the Java demo on **web.mit.edu/18.06**.

The SVD is seen graphically when Ax is perpendicular to Ay. Their directions at that moment give an orthonormal basis u_1, u_2. Their lengths give the singular values σ_1, σ_2. The vectors x and y at that same moment are the orthonormal basis v_1, v_2.

Searching the Web

I will end with an application of the SVD to web search engines. When you google a word, you get a list of web sites in order of importance. You could try "four subspaces".

The HITS algorithm that we describe is one way to produce that ranked list. It begins with about 200 sites found from an index of key words, and after that we look only at links between pages. Search engines are link-based more than content-based.

Start with the 200 sites and all sites that link to them and all sites they link to. That is our list, to be put in order. Importance can be measured by links out and links in.

1. The site is an ***authority***: *links come in from many sites.* Especially from hubs.

2. The site is a ***hub***: *links go out to many sites in the list.* Especially to authorities.

We want numbers $x_1, \ldots, x_N$ to rank the authorities and $y_1, \ldots, y_N$ to rank the hubs. Start with a simple count: x_i^0 and y_i^0 count the links into and out of site i.

Here is the point: *A good authority has links from important sites* (like hubs). Links from universities count more heavily than links from friends. *A good hub is linked to important sites* (like authorities). A link to **amazon.com** unfortunately means more than a link to **wellesleycambridge.com**. The rankings x^0 and y^0 from counting links are updated to x^1 and y^1 by taking account of *good* links (measuring their quality by x^0 and y^0):

$$\textbf{Authority values} \quad x_i^1 = \sum_{j \text{ links to } i} y_j^0 \qquad \textbf{Hub values} \quad y_i^1 = \sum_{i \text{ links to } j} x_j^0 \tag{14}$$

In matrix language those are $x^1 = A^\mathrm{T} y^0$ and $y^1 = Ax^0$. The matrix A contains 1's and 0's, with $a_{ij} = 1$ when i links to j. In the language of graphs, A is an "adjacency matrix" for the World Wide Web (an enormous matrix). The new x^1 and y^1 give better rankings, but not the best. Take another step like (14), to reach x^2 and y^2:

$$\boldsymbol{A^\mathrm{T}A} \textbf{ and } \boldsymbol{AA^\mathrm{T}} \textbf{ appear} \quad x^2 = A^\mathrm{T}y^1 = A^\mathrm{T}Ax^0 \quad \text{and} \quad y^2 = A^\mathrm{T}x^1 = AA^\mathrm{T}y^0. \tag{15}$$

In two steps we are multiplying by $A^\mathrm{T}A$ and AA^T. Twenty steps will multiply by $(A^\mathrm{T}A)^{10}$ and $(AA^\mathrm{T})^{10}$. When we take powers, the largest eigenvalue σ_1^2 begins to dominate. And the vectors x and y line up with the leading eigenvectors v_1 and u_1 of $A^\mathrm{T}A$ and AA^T. We are computing the top terms in the SVD, by the *power method* that is discussed in Section 9.3. It is wonderful that linear algebra helps to understand the Web.

Google actually creates rankings by a random walk that follows web links. The more often this random walk goes to a site, the higher the ranking. The frequency of visits gives the leading eigenvector ($\lambda = 1$) of the normalized adjacency matrix for the Web. ***That Markov matrix has 2.7 billion rows and columns, from 2.7 billion web sites.***

This is the largest eigenvalue problem ever solved. The excellent book by Langville and Meyer, *Google's PageRank and Beyond*, explains in detail the science of search engines. See `mathworks.com/company/newsletter/clevescorner/oct02_cleve.shtml`

But many of the important techniques are well-kept secrets of Google. Probably Google starts with last month's eigenvector as a first approximation, and runs the random walk very fast. To get a high ranking, you want a lot of links from important sites. The HITS algorithm is described in the 1999 *Scientific American* (June 16). But I don't think the SVD is mentioned there...

■ REVIEW OF THE KEY IDEAS ■

1. The SVD factors A into $U\Sigma V^\mathrm{T}$, with r singular values $\sigma_1 \geq \ldots \geq \sigma_r > 0$.

2. The numbers $\sigma_1^2, \ldots, \sigma_r^2$ are the nonzero eigenvalues of AA^{T} and $A^{\mathrm{T}}A$.

3. The orthonormal columns of U and V are eigenvectors of AA^{T} and $A^{\mathrm{T}}A$.

4. Those columns hold orthonormal bases for the four fundamental subspaces of A.

5. Those bases diagonalize the matrix: $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ for $i \le r$. This is $AV = U\Sigma$.

■ WORKED EXAMPLES ■

6.7 A Identify by name these decompositions $A = \boldsymbol{c}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{c}_r\boldsymbol{b}_r$ of an m by n matrix. Each term is a rank one matrix (column $\boldsymbol{c}$ times row $\boldsymbol{b}$). The rank of A is r.

1. *Orthogonal* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *orthogonal* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.
2. *Orthogonal* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *triangular* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.
3. *Triangular* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *triangular* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.

$A = CB$ is (m by r)(r by n). Triangular vectors $\boldsymbol{c}_i$ and $\boldsymbol{b}_i$ have zeros up to component i. The matrix C with columns $\boldsymbol{c}_i$ is lower triangular, the matrix B with rows $\boldsymbol{b}_i$ is upper triangular. Where do the rank and the pivots and singular values come into this picture?

Solution These three splittings $A = CB$ are basic to linear algebra, pure or applied:

1. **Singular Value Decomposition** $A = U\Sigma V^{\mathrm{T}}$ (*orthogonal* U, *orthogonal* ΣV^{T})

2. **Gram-Schmidt Orthogonalization** $A = QR$ (*orthogonal* Q, *triangular* R)

3. **Gaussian Elimination** $A = LU$ (*triangular* L, *triangular* U)

You might prefer to separate out the σ_i and pivots d_i and heights h_i:

1. $A = U\Sigma V^{\mathrm{T}}$ with unit vectors in U and V. The singular values are in Σ.

2. $A = QHR$ with unit vectors in Q and diagonal 1's in R. The heights h_i are in H.

3. $A = LDU$ with diagonal 1's in L and U. The pivots are in D.

Each h_i tells the height of column i above the base from earlier columns. The volume of the full n-dimensional box ($r = m = n$) comes from $A = U\Sigma V^{\mathrm{T}} = LDU = QHR$:

$$|\,\mathbf{det}\,A\,| = |\,\textit{\textbf{product of}}\ \sigma\text{'s}\,| = |\,\textit{\textbf{product of}}\ d\text{'s}\,| = |\,\textit{\textbf{product of}}\ h\text{'s}\,|.$$

6.7.B For $A = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ of rank one (2 by 2), compare $A = U\Sigma V^{\mathrm{T}}$ with $A = S\Lambda S^{-1}$.

Comment This started as an exam problem in 2007. It led further and became interesting. Now there is an essay called "The Four Fundamental Subspaces: 4 Lines" on **web.mit.edu/18.06**. The Jordan form enters when $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$ and $\lambda = 0$ is repeated.

6.7.C Show that $\sigma_1 \geq |\lambda|_{\max}$. The largest singular value dominates all eigenvalues.
Show that $\sigma_1 \geq |a_{ij}|_{\max}$. The largest singular value dominates all entries of A.

Solution Start from $A = U\Sigma V^{\mathrm{T}}$. Remember that multiplying by an orthogonal matrix *does not change length*: $\|Qx\| = \|x\|$ because $\|Qx\|^2 = x^{\mathrm{T}}Q^{\mathrm{T}}Qx = x^{\mathrm{T}}x = \|x\|^2$. This applies to $Q = U$ and $Q = V^{\mathrm{T}}$. In between is the diagonal matrix Σ.

$$\|Ax\| = \|U\Sigma V^{\mathrm{T}}x\| = \|\Sigma V^{\mathrm{T}}x\| \leq \sigma_1\|V^{\mathrm{T}}x\| = \sigma_1\|x\|. \tag{16}$$

An eigenvector has $\|Ax\| = |\lambda|\|x\|$. So (16) says that $|\lambda|\|x\| \leq \sigma_1\|x\|$. Then $|\lambda| \leq \sigma_1$.

Apply also to the unit vector $x = (1, 0, \ldots, 0)$. Now Ax is the first column of A. Then by inequality (16), this column has length $\leq \sigma_1$. Every entry must have magnitude $\leq \sigma_1$.

Example 5 Estimate the singular values σ_1 and σ_2 of A and A^{-1}:

$$\textbf{Eigenvalues} = \mathbf{1} \qquad A = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -C & 1 \end{bmatrix}. \tag{17}$$

Solution The length of the first column is $\sqrt{1 + C^2} \leq \sigma_1$, from the reasoning above. This confirms that $\sigma_1 \geq 1$ and $\sigma_1 \geq C$. Then σ_1 dominates the eigenvalues 1, 1 and the entry C. If C is very large then σ_1 is much bigger than the eigenvalues.

This matrix A has determinant $= 1$. $A^{\mathrm{T}}A$ also has determinant $= 1$ and then $\sigma_1\sigma_2 = 1$. For this matrix, $\sigma_1 \geq 1$ and $\sigma_1 \geq C$ lead to $\sigma_2 \leq 1$ and $\sigma_2 \leq 1/C$.

Conclusion: If $C = 1000$ then $\sigma_1 \geq 1000$ and $\sigma_2 \leq 1/1000$. ***A is ill-conditioned***, slightly sick. Inverting A is easy by algebra, but solving $Ax = b$ by elimination could be dangerous. *A is close to a singular matrix even though both eigenvalues are $\lambda = 1$.* By slightly changing the 1, 2 entry from zero to $1/C = 1/1000$, the matrix becomes singular.

Section 9.2 will explain how the ratio $\sigma_{\max}/\sigma_{\min}$ governs the roundoff error in elimination. MATLAB warns you if this "*condition number*" is large. Here $\sigma_1/\sigma_2 \geq C^2$.

Problem Set 6.7

Problems 1–3 compute the SVD of a square singular matrix A.

1 Find the eigenvalues and unit eigenvectors v_1, v_2 of $A^{\mathrm{T}}A$. Then find $u_1 = Av_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \text{ and } A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \text{ and } AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that u_1 is a unit eigenvector of AA^{T}. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$

2 Write down orthonormal bases for the four fundamental subspaces of this A.

3 (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2 ?

(b) For every rank-one matrix, why is σ_1^2 = sum of all a_{ij}^2?

Problems 4–7 ask for the SVD of matrices of rank 2.

4 Find the eigenvalues and unit eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. Keep each $Av = \sigma u$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^{\mathrm{T}}$.

5 Use the **svd** part of the MATLAB demo **eigshow** to find those v's graphically.

6 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ has the same shape as A.

7 What is the closest rank-one approximation to that 2 by 3 matrix?

8 A square invertible matrix has $A^{-1} = V\Sigma^{-1}U^{\mathrm{T}}$. This says that the *singular values of* A^{-1} *are* $1/\sigma(A)$. Show that $\sigma_{\max}(A^{-1})\,\sigma_{\max}(A) \geq 1$.

9 Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are orthonormal bases for $\mathbf{R}^n$. Construct the matrix A that transforms each v_j into u_j to give $Av_1 = u_1, \ldots, Av_n = u_n$.

10 Construct the matrix with rank one that has $Av = 12u$ for $v = \frac{1}{2}(1, 1, 1, 1)$ and $u = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 =$ _____.

11 Suppose A has orthogonal columns $w_1, w_2, \ldots, w_n$ of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. What are U, Σ, and V in the SVD?

12 Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are the matrices U, Σ, V^{T} in its SVD?

13 If $A = QR$ with an orthogonal matrix Q, the SVD of A is almost the same as the SVD of R. Which of the three matrices U, Σ, V is changed because of Q?

14 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by ***as small a matrix as possible*** to produce a singular matrix A_0. Hint: U and V do not change:

$$\text{From} \qquad A = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} \qquad \text{find the nearest } A_0.$$

15 Why doesn't the SVD for $A + I$ just use $\Sigma + I$?

Challenge Problems

16 (Search engine) Run a **random walk** $x(2), \ldots, x(n)$ starting from web site $x(1) = 1$. Count the visits to each site. At each step the code chooses the next website $x(k)$ with probabilities given by column $x(k-1)$ of A. At the end, $\boldsymbol{p}$ gives the fraction of time at each site from a histogram: count visits. ***The rankings are based on p.***

Please compare $\boldsymbol{p}$ to the steady state eigenvector of the Markov matrix A:

$$A = [0\ \ .1\ \ .2\ \ .7;\ \ .05\ \ 0\ \ .15\ \ .8;\ \ .15\ \ .25\ \ 0\ \ .6;\ \ .1\ \ .3\ \ .6\ \ 0]'$$

$n = 100;\ \ x = \text{zeros}(1, n);\ \ x(1) = 1;$
for $k = 2 : n\ \ x(k) = \text{min(find(rand} < \text{cumsum}(A(:, x(k-1)))));$ end
$\boldsymbol{p} = \text{hist}(x, 1:4)/n$

17 The $1, -1$ first difference matrix A has $A^{\mathrm{T}}A$ = second difference matrix. The singular vectors of A are *sine vectors* $\boldsymbol{v}$ and *cosine vectors* $\boldsymbol{u}$. Then $A\boldsymbol{v} = \sigma\boldsymbol{u}$ is the discrete form of $d/dx(\sin cx) = c(\cos cx)$. This is the best SVD I have seen.

$$\textbf{SVD of } A \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\textbf{\textit{Orthogonal sine matrix}} \qquad V = \frac{1}{\sqrt{2}} \begin{bmatrix} \sin \pi/4 & \sin 2\pi/4 & \sin 3\pi/4 \\ \sin 2\pi/4 & \sin 4\pi/4 & \sin 6\pi/4 \\ \sin 3\pi/4 & \sin 6\pi/4 & \sin 9\pi/4 \end{bmatrix}$$

(a) Put numbers in V: The unit eigenvectors of $A^{\mathrm{T}}A$ are singular vectors of A. Show that the columns of V have $A^{\mathrm{T}}A\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\lambda = 2 - \sqrt{2},\ 2,\ 2 + \sqrt{2}$.

(b) Multiply AV and verify that its columns are orthogonal. They are $\sigma_1\boldsymbol{u}_1$ and $\sigma_2\boldsymbol{u}_2$ and $\sigma_3\boldsymbol{u}_3$. The first columns of the cosine matrix U are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

(c) Since A is 4 by 3, we need a fourth orthogonal vector $\boldsymbol{u}_4$. It comes from the nullspace of A^{T}. What is $\boldsymbol{u}_4$?

The cosine vectors in U are eigenvectors of AA^{T}. The fourth cosine is $(1, 1, 1, 1)/2$.

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \pi/8 & \cos 2\pi/8 & \cos 3\pi/8 \\ \cos 3\pi/8 & \cos 6\pi/8 & \cos 9\pi/8 \\ \cos 5\pi/8 & \cos 10\pi/8 & \cos 15\pi/8 \\ \cos 7\pi/8 & \cos 14\pi/8 & \cos 21\pi/8 \end{bmatrix}$$

Those angles $\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$ fit 4 points with spacing $\pi/4$ between 0 and π. The sine transform has three points $\pi/4, 2\pi/4, 3\pi/4$. The full cosine transform includes $\boldsymbol{u}_4$ from the "zero frequency" or *direct current* eigenvector $(1, 1, 1, 1)$.

The 8 by 8 cosine transform in 2D is the workhorse of **jpeg compression**. Linear algebra (circulant, Toeplitz, orthogonal matrices) is at the heart of signal processing.

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors x_i.

Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $x_i^{\mathrm{T}} x_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Positive Definite: $x^{\mathrm{T}} A x > 0$	all $\lambda > 0$	orthogonal since $A^{\mathrm{T}} = A$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $x > 0$
Similar: $B = M^{-1} A M$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1} x(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$x = (1, i)$ and $(1, -i)$
Reflection: $I - 2uu^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	u; whole plane $u^{\perp}$
Rank One: uv^{T}	$\lambda = v^{\mathrm{T}} u;\ 0, .., 0$	u; whole plane $v^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi i k/n}$	$x_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$x_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^{\mathrm{T}} A = AA^{\mathrm{T}}$
Jordan: $J = M^{-1} A M$	diagonal of J	each block gives $x = (0, .., 1, .., 0)$
Rectangular: $A = U\Sigma V^{\mathrm{T}}$	rank(A) = rank(Σ)	eigenvectors of $A^{\mathrm{T}} A$, AA^{T} in V, U

Chapter 7

Linear Transformations

7.1 The Idea of a Linear Transformation

When a matrix A multiplies a vector $\boldsymbol{v}$, it "transforms" $\boldsymbol{v}$ into another vector $A\boldsymbol{v}$. ***In goes*** $\boldsymbol{v}$, ***out comes*** $T(\boldsymbol{v}) = A\boldsymbol{v}$. A transformation T follows the same idea as a function. In goes a number x, out comes $f(x)$. For one vector $\boldsymbol{v}$ or one number x, we multiply by the matrix or we evaluate the function. The deeper goal is to see all $\boldsymbol{v}$'s at once. We are transforming the whole space $\mathbf{V}$ when we multiply every $\boldsymbol{v}$ by A.

Start again with a matrix A. It transforms $\boldsymbol{v}$ to $A\boldsymbol{v}$. It transforms $\boldsymbol{w}$ to $A\boldsymbol{w}$. Then we *know* what happens to $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$. There is no doubt about $A\boldsymbol{u}$, it has to equal $A\boldsymbol{v} + A\boldsymbol{w}$. Matrix multiplication $T(\boldsymbol{v}) = A\boldsymbol{v}$ gives a ***linear transformation***:

A ***transformation*** T assigns an output $T(\boldsymbol{v})$ to each input vector $\boldsymbol{v}$ in $\mathbf{V}$. The transformation is ***linear*** if it meets these requirements for all $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\text{(a)}\ \ T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w}) \qquad \text{(b)}\ \ T(c\boldsymbol{v}) = cT(\boldsymbol{v}) \quad \text{for all } c.$$

If the input is $\boldsymbol{v} = \mathbf{0}$, the output must be $T(\boldsymbol{v}) = \mathbf{0}$. We combine (a) and (b) into one:

$$\textbf{Linear transformation} \qquad T(c\boldsymbol{v} + d\boldsymbol{w}) \quad \textbf{\textit{must equal}} \quad cT(\boldsymbol{v}) + dT(\boldsymbol{w}).$$

Again I can test matrix multiplication for linearity: $A(c\boldsymbol{v} + d\boldsymbol{w}) = cA\boldsymbol{v} + dA\boldsymbol{w}$ is *true*.

A linear transformation is highly restricted. Suppose T adds $\boldsymbol{u}_0$ to every vector. Then $T(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{u}_0$ and $T(\boldsymbol{w}) = \boldsymbol{w} + \boldsymbol{u}_0$. This isn't good, or at least *it isn't linear*. Applying T to $\boldsymbol{v} + \boldsymbol{w}$ produces $\boldsymbol{v} + \boldsymbol{w} + \boldsymbol{u}_0$. That is not the same as $T(\boldsymbol{v}) + T(\boldsymbol{w})$:

$$\textbf{Shift is not linear} \qquad \boldsymbol{v}+\boldsymbol{w}+\boldsymbol{u}_0 \quad \text{is not} \quad T(\boldsymbol{v}) + T(\boldsymbol{w}) = \boldsymbol{v}+\boldsymbol{u}_0+\boldsymbol{w}+\boldsymbol{u}_0.$$

The exception is when $\boldsymbol{u}_0 = \mathbf{0}$. The transformation reduces to $T(\boldsymbol{v}) = \boldsymbol{v}$. This is the ***identity transformation*** (nothing moves, as in multiplication by the identity matrix). That is certainly linear. In this case the input space $\mathbf{V}$ is the same as the output space $\mathbf{W}$.

The linear-plus-shift transformation $T(v) = Av + u_0$ is called *"affine"*. Straight lines stay straight although T is not linear. Computer graphics works with affine transformations in Section 8.6, because we must be able to move images.

Example 1 Choose a fixed vector $a = (1, 3, 4)$, and let $T(v)$ be the dot product $a \cdot v$:

The input is $\quad v = (v_1, v_2, v_3)$. $\qquad$ The output is $\quad T(v) = a \cdot v = v_1 + 3v_2 + 4v_3$.

This is linear. The inputs v come from three-dimensional space, so $\mathbf{V} = \mathbf{R}^3$. The outputs are just numbers, so the output space is $\mathbf{W} = \mathbf{R}^1$. We are multiplying by the row matrix $A = [1 \;\; 3 \;\; 4]$. Then $T(v) = Av$.

You will get good at recognizing which transformations are linear. If the output involves squares or products or lengths, v_1^2 or v_1v_2 or $\|v\|$, then T is not linear.

Example 2 The length $T(v) = \|v\|$ is not linear. Requirement (a) for linearity would be $\|v + w\| = \|v\| + \|w\|$. Requirement (b) would be $\|cv\| = c\|v\|$. Both are false!

Not (a): The sides of a triangle satisfy an *inequality* $\|v + w\| \leq \|v\| + \|w\|$.

Not (b): The length $\|-v\|$ is not $-\|v\|$. For negative c, we fail.

Example 3 (Important) T is the transformation that *rotates every vector by* 30°. The "*domain*" is the xy plane (all input vectors v). The "*range*" is also the xy plane (all rotated vectors $T(v)$). We described T without a matrix: rotate by 30°.

Is rotation linear? *Yes it is.* We can rotate two vectors and add the results. The sum of rotations $T(v) + T(w)$ is the same as the rotation $T(v + w)$ of the sum. The whole plane is turning together, in this linear transformation.

Lines to Lines, Triangles to Triangles

Figure 7.1 shows the line from v to w in the input space. It also shows the line from $T(v)$ to $T(w)$ in the output space. Linearity tells us: Every point on the input line goes onto the output line. And more than that: ***Equally spaced points go to equally spaced points.*** The middle point $u = \frac{1}{2}v + \frac{1}{2}w$ goes to the middle point $T(u) = \frac{1}{2}T(v) + \frac{1}{2}T(w)$.

The second figure moves up a dimension. Now we have three corners v_1, v_2, v_3. Those inputs have three outputs $T(v_1)$, $T(v_2)$, $T(v_3)$. *The input triangle goes onto the output triangle.* Equally spaced points stay equally spaced (along the edges, and then between the edges). The middle point $u = \frac{1}{3}(v_1 + v_2 + v_3)$ goes to the middle point $T(u) = \frac{1}{3}(T(v_1) + T(v_2) + T(v_3))$.

The rule of linearity extends to combinations of three vectors or n vectors:

Linearity
$$u = c_1v_1 + c_2v_2 + \cdots + c_nv_n \quad \textbf{transforms to}$$
$$T(u) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) \tag{1}$$

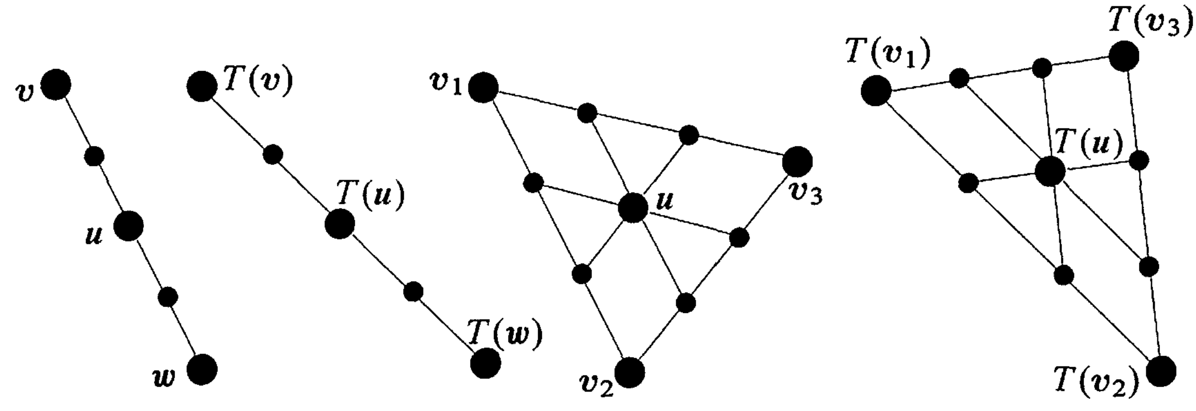


Figure 7.1: Lines to lines, equal spacing to equal spacing, $\boldsymbol{u} = \mathbf{0}$ to $T(\boldsymbol{u}) = \mathbf{0}$.

Note Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued. The column space consisted of all outputs $A\boldsymbol{v}$. The nullspace consisted of all inputs for which $A\boldsymbol{v} = \mathbf{0}$. Translate those into "range" and "kernel":

Range of T = set of all outputs $T(\boldsymbol{v})$: range corresponds to column space

Kernel of T = set of all inputs for which $T(\boldsymbol{v}) = \mathbf{0}$: kernel corresponds to nullspace.

The range is in the output space $\mathbf{W}$. The kernel is in the input space $\mathbf{V}$. When T is multiplication by a matrix, $T(\boldsymbol{v}) = A\boldsymbol{v}$, you can translate to column space and nullspace.

Examples of Transformations (mostly linear)

Example 4 Project every 3-dimensional vector straight down onto the xy plane. Then $T(x, y, z) = (x, y, 0)$. The range is that plane, which contains every $T(\boldsymbol{v})$. The kernel is the z axis (which projects down to zero). This projection is linear.

Example 5 Project every 3-dimensional vector onto the horizontal plane $z = 1$. The vector $\boldsymbol{v} = (x, y, z)$ is transformed to $T(\boldsymbol{v}) = (x, y, 1)$. This transformation is not linear. Why not? It doesn't even transform $\boldsymbol{v} = \mathbf{0}$ into $T(\boldsymbol{v}) = \mathbf{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix A. This $T(\boldsymbol{v}) = A\boldsymbol{v}$ is linear.

$$T(\boldsymbol{v} + \boldsymbol{w}) = A(\boldsymbol{v} + \boldsymbol{w}) \quad \textbf{does equal} \quad A\boldsymbol{v} + A\boldsymbol{w} = T(\boldsymbol{v}) + T(\boldsymbol{w}).$$

Example 6 Suppose A is an *invertible matrix*. The kernel of T is the zero vector; the range $\mathbf{W}$ equals the domain $\mathbf{V}$. Another linear transformation is multiplication by A^{-1}. This is the ***inverse transformation*** T^{-1}, which brings every vector $T(\boldsymbol{v})$ back to $\boldsymbol{v}$:

$$T^{-1}(T(\boldsymbol{v})) = \boldsymbol{v} \quad \text{matches the matrix multiplication} \quad A^{-1}(A\boldsymbol{v}) = \boldsymbol{v}.$$

We are reaching an unavoidable question. ***Are all linear transformations from*** $\mathbf{V} = \mathbf{R}^n$ ***to*** $\mathbf{W} = \mathbf{R}^m$ ***produced by matrices?*** When a linear T is described as a "rotation" or "projection" or "...", is there always a matrix hiding behind T?

The answer is *yes*. This is an approach to linear algebra that doesn't start with matrices. The next section shows that we still end up with matrices.

Linear Transformations of the Plane

It is more interesting to *see* a transformation than to define it. When a 2 by 2 matrix A multiplies all vectors in $\mathbf{R}^2$, we can watch how it acts. Start with a "house" that has eleven endpoints. Those eleven vectors v are transformed into eleven vectors Av. Straight lines between v's become straight lines between the transformed vectors Av. (The transformation from house to house is linear!) Applying A to a standard house produces a new house—possibly stretched or rotated or otherwise unlivable.

This part of the book is visual, not theoretical. We will show four houses and the matrices that produce them. The columns of H are the eleven corners of the first house. (H is 2 by 12, so **plot2d** will connect the 11th corner to the first.) The 11 points in the house matrix H are multiplied by A to produce the corners AH of the other houses.

House matrix $$H = \begin{bmatrix} -6 & -6 & -7 & 0 & 7 & 6 & 6 & -3 & -3 & 0 & 0 & -6 \\ -7 & 2 & 1 & 8 & 1 & 2 & -7 & -7 & -2 & -2 & -7 & -7 \end{bmatrix}.$$

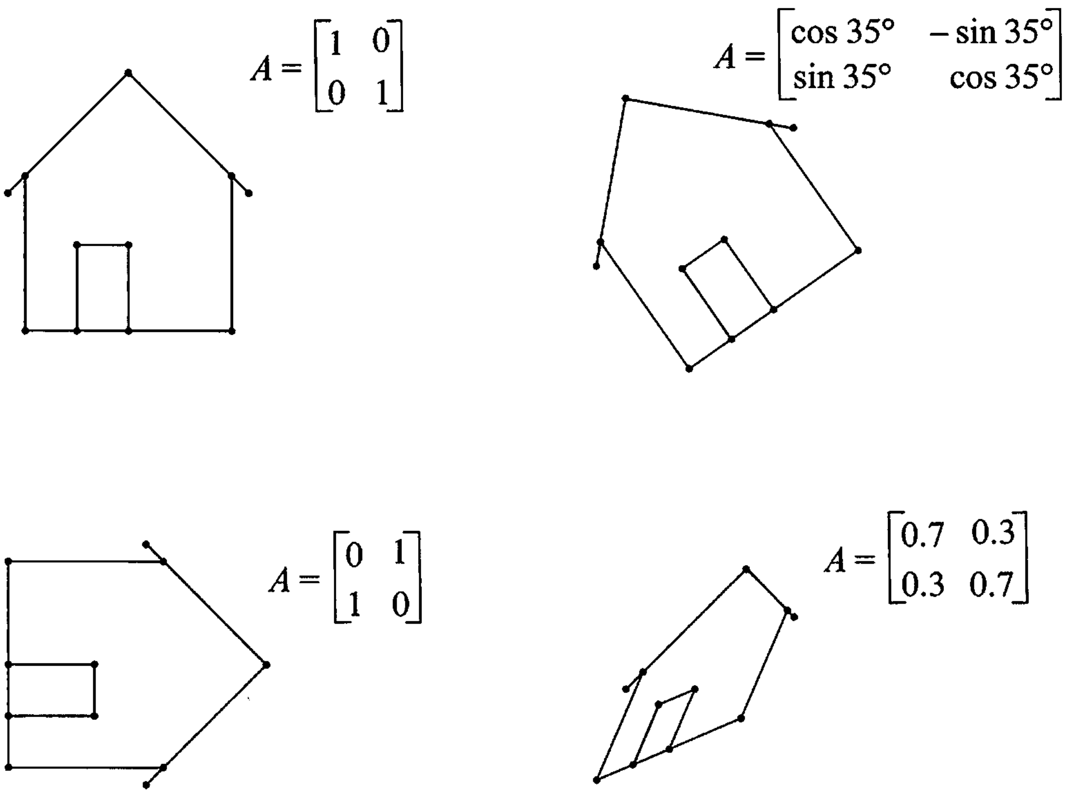


Figure 7.2: Linear transformations of a house drawn by **plot2d**$(A * H)$.

■ REVIEW OF THE KEY IDEAS ■

1. A transformation T takes each v in the input space to $T(v)$ in the output space.
2. T is **linear** if $T(v + w) = T(v) + T(w)$ and $T(cv) = cT(v)$: lines to lines.

3. Combinations to combinations: $T\,(c_1 v_1 + \cdots + c_n v_n) = c_1\,T(v_1) + \cdots + c_n\,T(v_n)$.

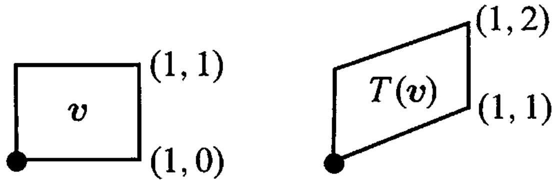
4. The transformation $T(v) = Av + v_0$ is linear only if $v_0 = \mathbf{0}$. Then $T(v) = Av$.

■ WORKED EXAMPLES ■

7.1 A The elimination matrix $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$ gives a ***shearing transformation*** from (x, y) to $T(x, y) = (x, x + y)$. Draw the xy plane and show what happens to $(1, 0)$ and $(1, 1)$. What happens to points on the vertical lines $x = 0$ and $x = a$? If the inputs fill the unit square $0 \le x \le 1,\ 0 \le y \le 1$, draw the outputs (the transformed square).

Solution The points $(1, 0)$ and $(2, 0)$ on the x axis transform by T to $(1, 1)$ and $(2, 2)$. The horizontal x axis transforms to the 45° line (going through $(0, 0)$ of course). The points on the y axis are *not moved* because $T(0, y) = (0, y)$. The y axis is the line of eigenvectors of T with $\lambda = 1$. Points with $x = a$ move up by a.

Vertical lines slide up
This is the shearing
Squares to parallelograms



7.1 B A **nonlinear transformation** T is invertible if every $\boldsymbol{b}$ in the output space comes from exactly one $\boldsymbol{x}$ in the input space: $T(\boldsymbol{x}) = \boldsymbol{b}$ always has exactly one solution. Which of these transformations (on real numbers x) is invertible and what is T^{-1}? ***None are linear, not even T_3.*** When you solve $T(\boldsymbol{x}) = \boldsymbol{b}$, you are inverting T:

$$T_1(x) = x^2 \quad T_2(x) = x^3 \quad T_3(x) = x + 9 \quad T_4(x) = e^x \quad T_5(x) = \frac{1}{x} \text{ for nonzero } x\text{'s}$$

Solution T_1 is not invertible: $x^2 = 1$ has *two* solutions and $x^2 = -1$ has *no* solution. T_4 is not invertible because $e^x = -1$ has no solution. (If the output space changes to *positive* b's then the inverse of $e^x = b$ is $x = \ln b$.)

Notice $T_5^2 = $ identity. But $T_3^2(x) = x + 18$. What are $T_2^2(x)$ and $T_4^2(x)$?

T_2, T_3, T_5 are invertible. The solutions to $x^3 = b$ and $x + 9 = b$ and $\frac{1}{x} = b$ are unique:

$$x = T_2^{-1}(b) = \boldsymbol{b}^{1/3} \qquad x = T_3^{-1}(b) = \boldsymbol{b - 9} \qquad x = T_5^{-1}(b) = \boldsymbol{1/b}$$

Problem Set 7.1

1 A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ by choosing $\boldsymbol{w} =$ _____ (and finish the proof). Prove it also from $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ by choosing $c =$ _____.

2 Requirement (b) gives $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ and also $T(d\boldsymbol{w}) = dT(\boldsymbol{w})$. Then by addition, requirement (a) gives $T(\quad) = (\quad)$. What is $T(c\boldsymbol{v} + d\boldsymbol{w} + e\boldsymbol{u})$?

3 Which of these transformations are not linear? The input is $\boldsymbol{v} = (v_1, v_2)$:

(a) $T(\boldsymbol{v}) = (v_2, v_1)$ (b) $T(\boldsymbol{v}) = (v_1, v_1)$ (c) $T(\boldsymbol{v}) = (0, v_1)$

(d) $T(\boldsymbol{v}) = (0, 1)$ (e) $T(\boldsymbol{v}) = v_1 - v_2$ (f) $T(\boldsymbol{v}) = v_1 v_2$.

4 If S and T are linear transformations, is $S(T(\boldsymbol{v}))$ linear or quadratic?

(a) (Special case) If $S(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{v}) = \boldsymbol{v}$, then $S(T(\boldsymbol{v})) = \boldsymbol{v}$ or $\boldsymbol{v}^2$?

(b) (General case) $S(\boldsymbol{w}_1+\boldsymbol{w}_2) = S(\boldsymbol{w}_1)+S(\boldsymbol{w}_2)$ and $T(\boldsymbol{v}_1+\boldsymbol{v}_2) = T(\boldsymbol{v}_1)+T(\boldsymbol{v}_2)$ combine into

$$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(\_\_\_\_) = \_\_\_\_ + \_\_\_\_.$$

5 Suppose $T(\boldsymbol{v}) = \boldsymbol{v}$ except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ but not $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$.

6 Which of these transformations satisfy $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ and which satisfy $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$?

(a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ (b) $T(\boldsymbol{v}) = v_1+v_2+v_3$ (c) $T(\boldsymbol{v}) = (v_1, 2v_2, 3v_3)$

(d) $T(\boldsymbol{v}) =$ largest component of $\boldsymbol{v}$.

7 For these transformations of $\mathbf{V} = \mathbf{R}^2$ to $\mathbf{W} = \mathbf{R}^2$, find $T(T(\boldsymbol{v}))$. Is this transformation T^2 linear?

(a) $T(\boldsymbol{v}) = -\boldsymbol{v}$ (b) $T(\boldsymbol{v}) = \boldsymbol{v} + (1, 1)$

(c) $T(\boldsymbol{v}) = 90°$ rotation $= (-v_2, v_1)$

(d) $T(\boldsymbol{v}) =$ projection $= \left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)$.

8 Find the range and kernel (like the column space and nullspace) of T:

(a) $T(v_1, v_2) = (v_1 - v_2, 0)$ (b) $T(v_1, v_2, v_3) = (v_1, v_2)$

(c) $T(v_1, v_2) = (0, 0)$ (d) $T(v_1, v_2) = (v_1, v_1)$.

9 The "cyclic" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(\boldsymbol{v}))$? What is $T^3(\boldsymbol{v})$? What is $T^{100}(\boldsymbol{v})$? Apply T a hundred times to $\boldsymbol{v}$.

10 A linear transformation from $\mathbf{V}$ to $\mathbf{W}$ has an *inverse* from $\mathbf{W}$ to $\mathbf{V}$ when the range is all of $\mathbf{W}$ and the kernel contains only $\boldsymbol{v} = \mathbf{0}$. Then $T(\boldsymbol{v}) = \boldsymbol{w}$ has one solution $\boldsymbol{v}$ for each $\boldsymbol{w}$ in $\mathbf{W}$. Why are these T's not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\quad \mathbf{W} = \mathbf{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\quad \mathbf{W} = \mathbf{R}^3$

(c) $T(v_1, v_2) = v_1$ $\quad \mathbf{W} = \mathbf{R}^1$

11 If $T(\boldsymbol{v}) = A\boldsymbol{v}$ and A is m by n, then T is "multiplication by A."

(a) What are the input and output spaces $\mathbf{V}$ and $\mathbf{W}$?

(b) Why is range of T = column space of A?

(c) Why is kernel of T = nullspace of A?

12 Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(\boldsymbol{v})$:

(a) $\boldsymbol{v} = (2, 2)$ (b) $\boldsymbol{v} = (3, 1)$ (c) $\boldsymbol{v} = (-1, 1)$ (d) $\boldsymbol{v} = (a, b)$.

Problems 13-19 may be harder. The input space V contains all 2 by 2 matrices M.

13 M is any 2 by 2 matrix and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

14 Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Show that the range of T is the whole matrix space $\mathbf{V}$ and the kernel is the zero matrix:

(1) If $AM = 0$ prove that M must be the zero matrix.

(2) Find a solution to $AM = B$ for any 2 by 2 matrix B.

15 Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Show that the identity matrix I is not in the range of T. Find a nonzero matrix M such that $T(M) = AM$ is zero.

16 Suppose T transposes every matrix M. Try to find a matrix A which gives $AM = M^{\mathrm{T}}$ for every M. Show that no matrix A will do it. *To professors:* Is this a linear transformation that doesn't come from a matrix?

17 The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?

(a) T^2 = identity transformation.

(b) The kernel of T is the zero matrix.

(c) Every matrix is in the range of T.

(d) $T(M) = -M$ is impossible.

18 Suppose $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find a matrix with $T(M) \neq 0$. Describe all matrices with $T(M) = 0$ (the kernel) and all output matrices $T(M)$ (the range).

19 If A and B are invertible and $T(M) = AMB$, find $T^{-1}(M)$ in the form $(\quad)M(\quad)$.

Questions 20–26 are about house transformations. The output is $T(H) = AH$.

20 How can you tell from the picture of T (house) that A is

(a) a diagonal matrix?

(b) a rank-one matrix?

(c) a lower triangular matrix?

21 Draw a picture of T (house) for these matrices:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

22 What are the conditions on $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to ensure that T (house) will

(a) sit straight up?

(b) expand the house by 3 in all directions?

(c) rotate the house with no change in its shape?

23 Describe T (house) when $T(v) = -v + (1, 0)$. This T is "affine".

24 Change the house matrix H to add a chimney.

25 The standard house is drawn by **plot2d**(H). Circles from o and lines from –:

$$x = H(1,:)'; y = H(2,:)';$$
$$\text{axis}([-1010-1010]), \text{axis}(\text{'square'})$$
$$\text{plot}(x, y, 'o', x, y, '-');$$

Test **plot2d**(A′∗ H) and **plot2d**(A′∗ A ∗ H) with the matrices in Figure 7.1.

26 Without a computer sketch the houses $A * H$ for these matrices A:

$$\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

27 This code creates a vector theta of 50 angles. It draws the unit circle and then T (circle) = ellipse. $T(v) = Av$ **takes circles to ellipses.**

```
A = [2 1;1 2]     % You can change A
theta = [0:2 * pi/50:2 * pi];
circle = [cos(theta); sin(theta)];
ellipse = A * circle;
axis([−4 4 −4 4]); axis('square')
plot(circle(1,:), circle(2,:), ellipse(1,:), ellipse(2,:))
```

28 Add two eyes and a smile to the circle in Problem 27. (If one eye is dark and the other is light, you can tell when the face is reflected across the y axis.) Multiply by matrices A to get new faces.

Challenge Problems

29 What conditions on $\det A = ad - bc$ ensure that the output house AH will

(a) be squashed onto a line?

(b) keep its endpoints in clockwise order (not reflected)?

(c) have the same area as the original house?

30 From $A = U\Sigma V^{\mathrm{T}}$ (Singular Value Decomposition) A takes **circles to ellipses.** $AV = U\Sigma$ says that the radius vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the circle go to the semi-axes $\sigma_1 \boldsymbol{u}_1$ and $\sigma_2 \boldsymbol{u}_2$ of the ellipse. Draw the circle and the ellipse for $\theta = 30°$:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

31 Why does every linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ take squares to parallelograms? Rectangles also go to parallelograms (squashed if T is not invertible).

7.2 The Matrix of a Linear Transformation

The next pages assign a matrix to every linear transformation T. For ordinary column vectors, the input v is in $\mathbf{V} = \mathbf{R}^n$ and the output $T(v)$ is in $\mathbf{W} = \mathbf{R}^m$. The matrix A for this transformation T will be m by n. Our choice of bases in $\mathbf{V}$ and $\mathbf{W}$ will decide A.

The standard basis vectors for $\mathbf{R}^n$ and $\mathbf{R}^m$ are the columns of I. That choice leads to a standard matrix, and $T(v) = Av$ in the normal way. But these spaces also have other bases, so the same T is represented by other matrices. A main theme of linear algebra is to choose the bases that give the best matrix for T.

When $\mathbf{V}$ and $\mathbf{W}$ are not $\mathbf{R}^n$ and $\mathbf{R}^m$, they still have bases. Each choice of basis leads to a matrix for T. When the input basis is different from the output basis, the matrix for $T(v) = v$ will not be the identity I. It will be the "change of basis matrix".

Key idea of this section

> ***Suppose we know*** $T(v_1), \ldots, T(v_n)$ ***for the basis vectors*** $v_1, \ldots, v_n$.
> ***Then linearity produces*** $T(v)$ ***for every other input vector*** v.

Reason Every v is a unique combination $c_1v_1 + \cdots + c_nv_n$ of the basis vectors v_i. Since T is a linear transformation (here is the moment for linearity), $T(v)$ must be **the same combination** $c_1T(v_1) + \cdots + c_nT(v_n)$ **of the known outputs** $T(v_i)$.

Our first example gives the outputs $T(v)$ for the standard basis vectors $(1, 0)$ and $(0, 1)$.

Example 1 Suppose T transforms $v_1 = (1, 0)$ to $T(v_1) = (2, 3, 4)$. Suppose the second basis vector $v_2 = (0, 1)$ goes to $T(v_2) = (5, 5, 5)$. If T is linear from $\mathbf{R}^2$ to $\mathbf{R}^3$ then its "standard matrix" is 3 by 2. Those outputs $T(v_1)$ and $T(v_2)$ go into its columns:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}. \qquad \begin{array}{l} T(v_1 + v_2) = T(v_1) + T(v_2) \\ \text{combines the columns} \end{array} \qquad \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Example 2 The derivatives of the functions $1, x, x^2, x^3$ are $0, 1, 2x, 3x^2$. Those are four facts about the transformation T that "***takes the derivative***". The inputs and the outputs are functions! Now add the crucial fact that the "derivative transformation" T is linear:

$$T(v) = \frac{dv}{dx} \quad \text{obeys the linearity rule} \quad \frac{d}{dx}(cv + dw) = c\frac{dv}{dx} + d\frac{dw}{dx}. \qquad (1)$$

It is exactly this linearity that you use to find all other derivatives. From the derivative of each separate power $1, x, x^2, x^3$ (those are the basis vectors v_1, v_2, v_3, v_4) you find the derivative of any polynomial like $4 + x + x^2 + x^3$:

$$\frac{d}{dx}(4 + x + x^2 + x^3) = 1 + 2x + 3x^2 \qquad \text{(because of linearity!)}$$

This example applies T (the derivative d/dx) to the input $v = 4v_1 + v_2 + v_3 + v_4$. Here the input space $\mathbf{V}$ contains all combinations of $1, x, x^2, x^3$. I call them vectors, you might call them functions. Those four vectors are a basis for the space $\mathbf{V}$ of cubic polynomials (degree ≤ 3). Four derivatives tell us all derivatives in $\mathbf{V}$.

For the nullspace of A, we solve $Av = \mathbf{0}$. For the kernel of the derivative T, we solve $dv/dx = \mathbf{0}$. The solution is $v =$ constant. The nullspace of T is one-dimensional, containing all constant functions (like the first basis function $v_1 = 1$).

To find the range (or column space), look at all outputs from $T(v) = dv/dx$. The inputs are cubic polynomials $a + bx + cx^2 + dx^3$, so the outputs are *quadratic polynomials* (degree ≤ 2). For the output space $\mathbf{W}$ we have a choice. If $\mathbf{W} =$ cubics, then the range of T (the quadratics) is a subspace. If $\mathbf{W} =$ quadratics, then the range is all of $\mathbf{W}$.

That second choice emphasizes the difference between the domain or input space ($\mathbf{V} =$ cubics) and the image or output space ($\mathbf{W} =$ quadratics). $\mathbf{V}$ has dimension $n = 4$ and $\mathbf{W}$ has dimension $m = 3$. The "derivative matrix" below will be 3 by 4.

The range of T is a three-dimensional subspace. The matrix will have rank $r = 3$. The kernel is one-dimensional. The sum $3 + 1 = 4$ is the dimension of the input space. This was $r + (n - r) = n$ in the Fundamental Theorem of Linear Algebra. Always ***(dimension of range) + (dimension of kernel) = dimension of input space.***

Example 3 The ***integral*** is the inverse of the derivative. That is the Fundamental Theorem of Calculus. We see it now in linear algebra. The transformation T^{-1} that "takes the integral from 0 to x" is linear! Apply T^{-1} to $1, x, x^2$, which are w_1, w_2, w_3:

$$\textbf{Integration is } T^{-1} \qquad \int_0^x 1\,dx = x, \quad \int_0^x x\,dx = \tfrac{1}{2}x^2, \quad \int_0^x x^2\,dx = \tfrac{1}{3}x^3.$$

By linearity, the integral of $w = B + Cx + Dx^2$ is $T^{-1}(w) = Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$. The integral of a quadratic is a cubic. The input space of T^{-1} is the quadratics, the output space is the cubics. ***Integration takes* $\mathbf{W}$ *back to* $\mathbf{V}$.** Its matrix will be 4 by 3.

Range of T^{-1} The outputs $Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$ are cubics with no constant term.

Kernel of T^{-1} The output is zero only if $B = C = D = 0$. The nullspace is $\mathbf{Z} = \{\mathbf{0}\}$.

Fundamental Theorem $3 + 0$ is the dimension of the input space $\mathbf{W}$ for T^{-1}.

Matrices for the Derivative and Integral

We will show how the matrices A and A^{-1} copy the derivative T and the integral T^{-1}. This is an excellent example from calculus. (I write A^{-1} but I don't quite mean it.) Then comes the general rule—how to represent any linear transformation T by a matrix A.

The derivative transforms the space $\mathbf{V}$ of cubics to the space $\mathbf{W}$ of quadratics. The basis for $\mathbf{V}$ is $1, x, x^2, x^3$. The basis for $\mathbf{W}$ is $1, x, x^2$. *The derivative matrix is* 3 *by* 4*:*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \textbf{matrix form of derivative } T. \tag{2}$$

Why is A the correct matrix? Because ***multiplying by A agrees with transforming by T.*** The derivative of $v = a+bx+cx^2+dx^3$ is $T(v) = b+2cx+3dx^2$. The same numbers b and $2c$ and $3d$ appear when we multiply by the matrix A:

$$\textbf{Take the derivative} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}. \tag{3}$$

Look also at T^{-1}. The integration matrix is 4 by 3. Watch how the following matrix starts with $\boldsymbol{w} = B + Cx + Dx^2$ and produces its integral $0 + Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$:

$$\textbf{Take the integral} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ \frac{1}{2}C \\ \frac{1}{3}D \end{bmatrix}. \tag{4}$$

I want to call that matrix A^{-1}, and I will. But you realize that rectangular matrices don't have inverses. At least they don't have two-sided inverses. This rectangular A has a *one-sided inverse*. The integral is a one-sided inverse of the derivative!

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{but} \quad A^{-1}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If you integrate a function and then differentiate, you get back to the start. So $AA^{-1} = I$. But if you differentiate before integrating, the constant term is lost. ***The integral of the derivative of* 1 *is zero:***

$$T^{-1}T(1) = \text{integral of zero function} = 0.$$

This matches $A^{-1}A$, whose first column is all zero. The derivative T has a kernel (the constant functions). Its matrix A has a nullspace. Main point again: Av copies $T(v)$.

Construction of the Matrix

Now we construct a matrix for any linear transformation. Suppose T transforms the space $\mathbf{V}$ (n-dimensional) to the space $\mathbf{W}$ (m-dimensional). We choose a basis $v_1, \ldots, v_n$ for $\mathbf{V}$ and we choose a basis $w_1, \ldots, w_m$ for $\mathbf{W}$. The matrix A will be m by n. To find the first column of A, apply T to the first basis vector v_1. The output $T(v_1)$ is in $\mathbf{W}$.

$$T(v_1) \quad \textit{is a combination} \quad a_{11}w_1 + \cdots + a_{m1}w_m \quad \textit{of the output basis for } \mathbf{W}.$$

These numbers $a_{11}, \ldots, a_{m1}$ *go into the first column of* A. Transforming v_1 to $T(v_1)$ matches multiplying $(1, 0, \ldots, 0)$ by A. It yields that first column of the matrix.

When T is the derivative and the first basis vector is 1, its derivative is $T(\boldsymbol{v}_1) = \mathbf{0}$. So for the derivative matrix, the first column of A was all zero.

For the integral, the first basis function is again 1. Its integral is the second basis function x. So the first column of A^{-1} was $(0, 1, 0, 0)$. Here is the construction of A.

Key rule: The jth column of A is found by applying T to the jth basis vector $\boldsymbol{v}_j$

$$T(\boldsymbol{v}_j) = \text{combination of basis vectors of } \mathbf{W} = a_{1j}\boldsymbol{w}_1 + \cdots + a_{mj}\boldsymbol{w}_m. \tag{5}$$

These numbers $a_{1j}, \ldots, a_{mj}$ go into column j of A. *The matrix is constructed to get the basis vectors right.* ***Then linearity gets all other vectors right.*** Every $\boldsymbol{v}$ is a combination $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$, and $T(\boldsymbol{v})$ is a combination of the $\boldsymbol{w}$'s. When A multiplies the coefficient vector $\boldsymbol{c} = (c_1, \ldots, c_n)$ in the $\boldsymbol{v}$ combination, $A\boldsymbol{c}$ produces the coefficients in the $T(\boldsymbol{v})$ combination. This is because matrix multiplication (combining columns) is linear like T.

The matrix A tells us what T does. Every linear transformation from $\mathbf{V}$ to $\mathbf{W}$ can be converted to a matrix. This matrix depends on the bases.

Example 4 ***If the bases change, T is the same but the matrix A is different.***

Suppose we reorder the basis to $x, x^2, x^3, 1$ for the cubics in $\mathbf{V}$. Keep the original basis $1, x, x^2$ for the quadratics in $\mathbf{W}$. The derivative of the first basis vector $\boldsymbol{v}_1 = x$ is the first basis vector $\boldsymbol{w}_1 = 1$. So the first column of A looks different:

$$A_{\text{new}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{array}{l} \text{matrix for the derivative } T \\ \text{when the bases change to} \\ x, x^2, x^3, 1 \text{ and } 1, x, x^2. \end{array}$$

When we reorder the basis of $\mathbf{V}$, we reorder the columns of A. The input basis vector $\boldsymbol{v}_j$ is responsible for column j. The output basis vector $\boldsymbol{w}_i$ is responsible for row i. Soon the changes in the bases will be more than permutations.

Products AB Match Transformations TS

The examples of derivative and integral made three points. First, linear transformations T are everywhere—in calculus and differential equations and linear algebra. Second, spaces other than $\mathbf{R}^n$ are important—we had functions in $\mathbf{V}$ and $\mathbf{W}$. Third, T still boils down to a matrix A. Now we make sure that we can find this matrix.

The next examples have $\mathbf{V} = \mathbf{W}$. We choose the same basis for both spaces. Then we can compare the matrices A^2 and AB with the transformations T^2 and TS.

Example 5 T rotates every vector by the angle θ. Here $\mathbf{V} = \mathbf{W} = \mathbf{R}^2$. *Find A.*

Solution The standard basis is $\boldsymbol{v}_1 = (1, 0)$ and $\boldsymbol{v}_2 = (0, 1)$. To find A, apply T to those basis vectors. In Figure 7.3a, they are rotated by θ. *The first vector* $(1, 0)$ ***swings around to*** $(\cos\theta, \sin\theta)$. This equals $\cos\theta$ times $(1, 0)$ plus $\sin\theta$ times $(0, 1)$. Therefore those numbers $\cos\theta$ and $\sin\theta$ go into the *first column* of A:

$$\begin{bmatrix}\cos\theta & \\ \sin\theta & \end{bmatrix} \text{ shows column 1} \quad A = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \text{ shows both columns.}$$

For the second column, transform the second vector $(0, 1)$. The figure shows it rotated to $(-\sin\theta, \cos\theta)$. *Those numbers go into the second column.* Multiplying A times $(0, 1)$ produces that column. A agrees with T on the basis, and on all $\boldsymbol{v}$.

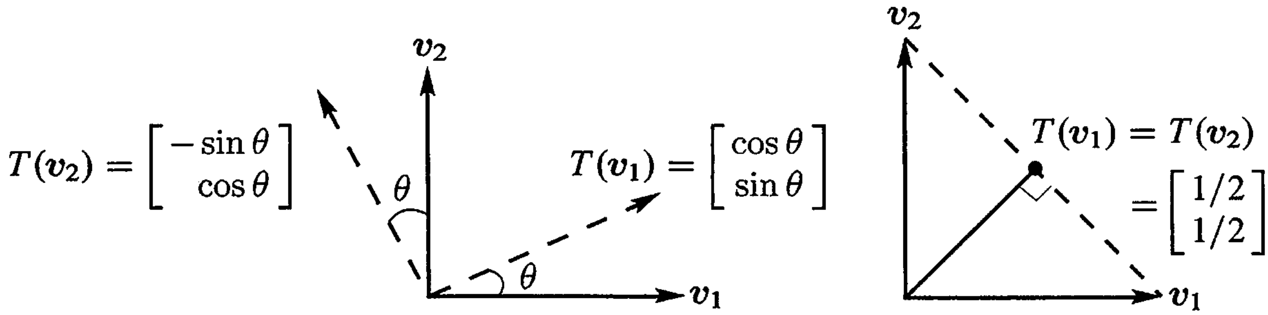


Figure 7.3: Two transformations: Rotation by θ and projection onto the 45° line.

Example 6 (*Projection*) Suppose T projects every plane vector onto the 45° line. Find its matrix for two different choices of the basis. We will find two matrices.

Solution Start with a specially chosen basis, not drawn in Figure 7.3. The basis vector $\boldsymbol{v}_1$ is along the 45° line. *It projects to itself*: $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$. So the first column of A contains 1 and 0. The second basis vector $\boldsymbol{v}_2$ is along the perpendicular line (135°). *This basis vector projects to zero.* So the second column of A contains 0 and 0:

$$\textbf{Projection} \quad A = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} \quad \text{when } \mathbf{V} \text{ and } \mathbf{W} \text{ have the 45° and 135° basis.}$$

Now take the standard basis $(1, 0)$ and $(0, 1)$. Figure 7.3b shows how $(1, 0)$ projects to $(\frac{1}{2}, \frac{1}{2})$. That gives the first column of A. The other basis vector $(0, 1)$ also projects to $(\frac{1}{2}, \frac{1}{2})$. So the standard matrix for this projection is A:

$$\textbf{Same projection} \quad A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} \quad \text{for the standard basis.}$$

Both A's are projection matrices. If you square A it doesn't change. Projecting twice is the same as projecting once: $T^2 = T$ so $A^2 = A$. Notice what is hidden in that statement: ***The matrix for T^2 is A^2.***

We have come to something important—the real reason for the way matrices are multiplied. *At last we discover why*! Two transformations S and T are represented by two matrices B and A. When we apply T to the output from S, we get the "composition" TS. When we apply A after B, we get the matrix product AB. ***Matrix multiplication gives the correct matrix AB to represent TS.***

The transformation S is from a space $\mathbf{U}$ to $\mathbf{V}$. Its matrix B uses a basis $u_1, \ldots, u_p$ for $\mathbf{U}$ and a basis $v_1, \ldots, v_n$ for $\mathbf{V}$. The matrix is n by p. The transformation T is from $\mathbf{V}$ to $\mathbf{W}$ as before. *Its matrix A must use the same basis $v_1, \ldots, v_n$ for* $\mathbf{V}$—this is the output space for S and the input space for T. ***Then the matrix AB matches TS:***

Multiplication The linear transformation TS starts with any vector u in $\mathbf{U}$, goes to $S(u)$ in $\mathbf{V}$ and then to $T(S(u))$ in $\mathbf{W}$. The matrix AB starts with any x in $\mathbf{R}^p$, goes to Bx in $\mathbf{R}^n$ and then to ABx in $\mathbf{R}^m$. The matrix AB correctly represents TS:

$$TS: \quad \mathbf{U} \to \mathbf{V} \to \mathbf{W} \qquad AB: \quad (m \text{ by } n)(n \text{ by } p) = (m \text{ by } p).$$

The input is $u = x_1 u_1 + \cdots + x_p u_p$. The output $T(S(u))$ matches the output ABx. ***Product of transformations matches product of matrices.***

The most important cases are when the spaces $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are the same and their bases are the same. With $m = n = p$ we have square matrices.

Example 7 S rotates the plane by θ and T also rotates by θ. Then TS rotates by 2θ. This transformation T^2 corresponds to the rotation matrix A^2 through 2θ:

$$T = S \qquad A = B \qquad T^2 = \text{rotation by } 2\theta \qquad A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}. \tag{6}$$

By matching (transformation)^2 with (matrix)^2, we pick up the formulas for $\cos 2\theta$ and $\sin 2\theta$. Multiply A times A:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{7}$$

Comparing (6) with (7) produces $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. Trigonometry (the double angle rule) comes from linear algebra.

Example 8 S rotates by θ and T rotates by $-\theta$. Then $TS = I$ matches $AB = I$.

In this case $T(S(u))$ is u. We rotate forward and back. For the matrices to match, ABx must be x. The two matrices are inverses. Check this by putting $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ into the backward rotation matrix:

$$AB = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = I.$$

Earlier T took the derivative and S took the integral. The transformation TS is the identity but not ST. Therefore AB is the identity matrix but not BA:

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = I \quad \text{but} \quad BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Identity Transformation and the Change of Basis Matrix

We now find the matrix for the special and boring transformation $T(v) = v$. This ***identity transformation*** does nothing to v. The matrix for $T = I$ also does nothing, *provided* the output basis is the same as the input basis. The output $T(v_1)$ is v_1. When the bases are the same, this is w_1. So the first column of A is $(1, 0, \ldots, 0)$.

When each output $T(v_j) = v_j$ is the same as w_j, the matrix is just I.

This seems reasonable: The identity transformation is represented by the identity matrix. But suppose the bases are *different*. Then $T(v_1) = v_1$ is a combination of the w's. That combination $m_{11}w_1 + \cdots + m_{n1}w_n$ tells the first column of the matrix (call it M).

Identity transformation ***When the outputs $T(v_j) = v_j$ are combinations $\sum_{i=1}^{n} m_{ij} w_i$, the "change of basis matrix" is M.***

The basis is changing but the vectors themselves are not changing: $T(v) = v$. When the inputs have one basis and the outputs have another basis, the matrix is not I.

Example 9 The input basis is $v_1 = (3, 7)$ and $v_2 = (2, 5)$. The output basis is $w_1 = (1, 0)$ and $w_2 = (0, 1)$. Then the matrix M is easy to compute:

Change of basis The matrix for $T(v) = v$ is $M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$.

Reason The first input is the basis vector $v_1 = (3, 7)$. The output is also $(3, 7)$ which we express as $3w_1 + 7w_2$. Then the first column of M contains 3 and 7.

This seems too simple to be important. It becomes trickier when the change of basis goes the other way. We get the inverse of the previous matrix M:

Example 10 The input basis is now $v_1 = (1, 0)$ and $v_2 = (0, 1)$. The outputs are just $T(v) = v$. But the output *basis* is now $w_1 = (3, 7)$ and $w_2 = (2, 5)$.

Reverse the bases
Invert the matrix The matrix for $T(v) = v$ is $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

Reason The first input is $v_1 = (1, 0)$. The output is also v_1 but we express it as $5w_1 - 7w_2$. Check that $5(3, 7) - 7(2, 5)$ does produce $(1, 0)$. We are combining the columns of the previous M to get the columns of I. The matrix to do that is M^{-1}.

$$\begin{matrix}\textbf{Change basis}\\ \textbf{Change back}\end{matrix} \quad \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \text{ is } MM^{-1} = I.$$

A mathematician would say that the matrix MM^{-1} corresponds to the product of two identity transformations. We start and end with the same basis $(1,0)$ and $(0,1)$. Matrix multiplication must give I. So the two change of basis matrices are inverses.

One thing is sure. Multiplying A times $(1,0,\ldots,0)$ gives column 1 of the matrix. The novelty of this section is that $(1,0,\ldots,0)$ stands for the first vector v_1, *written in the basis of* v*'s*. Then column 1 of the matrix is that same vector v_1, *written in the standard basis.*

Wavelet Transform = Change to Wavelet Basis

Wavelets are little waves. They have different lengths and they are localized at different places. The first basis vector is not actually a wavelet, it is the very useful flat vector of all ones. This example shows "Haar wavelets":

$$\textbf{Haar basis} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad w_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{8}$$

Those vectors are *orthogonal*, which is good. You see how w_3 is localized in the first half and w_4 is localized in the second half. The **wavelet transform** finds the coefficients c_1, c_2, c_3, c_4 when the input signal $v = (v_1, v_2, v_3, v_4)$ is expressed in the wavelet basis:

$$\textbf{Transform } v \textbf{ to } c \qquad v = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 = Wc \tag{9}$$

The coefficients c_3 and c_4 tell us about details in the first half and last half of v. The coefficient c_1 is the average.

Why do want to change the basis? I think of v_1, v_2, v_3, v_4 as the intensities of a signal. In audio they are volumes of sound. In images they are pixel values on a scale of black to white. An electrocardiogram is a medical signal. Of course $n = 4$ is very short, and $n = 10{,}000$ is more realistic. We may need to *compress* that long signal, by keeping only the largest 5% of the coefficients. This is 20 : 1 compression and (to give only two of its applications) it makes High Definition TV and video conferencing possible.

If we keep only 5% of the *standard* basis coefficients, we lose 95% of the signal. In image processing, 95% of the image disappears. In audio, 95% of the tape goes blank. But if we choose a better basis of w's, 5% of the basis vectors can combine to come very close to the original signal. In image processing and audio coding, you can't see or hear the difference. We don't need the other 95%!

One good basis vector is the flat $(1,1,1,1)$. That part alone can represent the constant background of our image. A short wave like $(0,0,1,-1)$ or in higher dimensions $(0,0,0,0,0,0,1,-1)$ represents a detail at the end of the signal.

The three steps are the transform and compression and inverse transform.

input v	$\rightarrow$	coefficients c	$\rightarrow$	compressed $\widehat{c}$	$\rightarrow$	compressed $\widehat{v}$
	[lossless]		***[lossy]***		***[reconstruct]***	

In linear algebra, where everything is perfect, we omit the compression step. The output $\widehat{v}$ is exactly the same as the input v. The transform gives $c = W^{-1}v$ and the reconstruction brings back $v = Wc$. In true signal processing, where nothing is perfect but everything is fast, the transform (lossless) and the compression (which only loses unnecessary information) are absolutely the keys to success. The output is $\widehat{v} = W\widehat{c}$.

I will show those steps for a typical vector like $v = (6, 4, 5, 1)$. Its wavelet coefficients are $c = (4, 1, 1, 2)$. The reconstruction $4\boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3 + 2\boldsymbol{w}_4$ is $v = Wc$:

$$\begin{bmatrix} \mathbf{6} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{1} \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{4} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{2} \end{bmatrix}. \qquad (10)$$

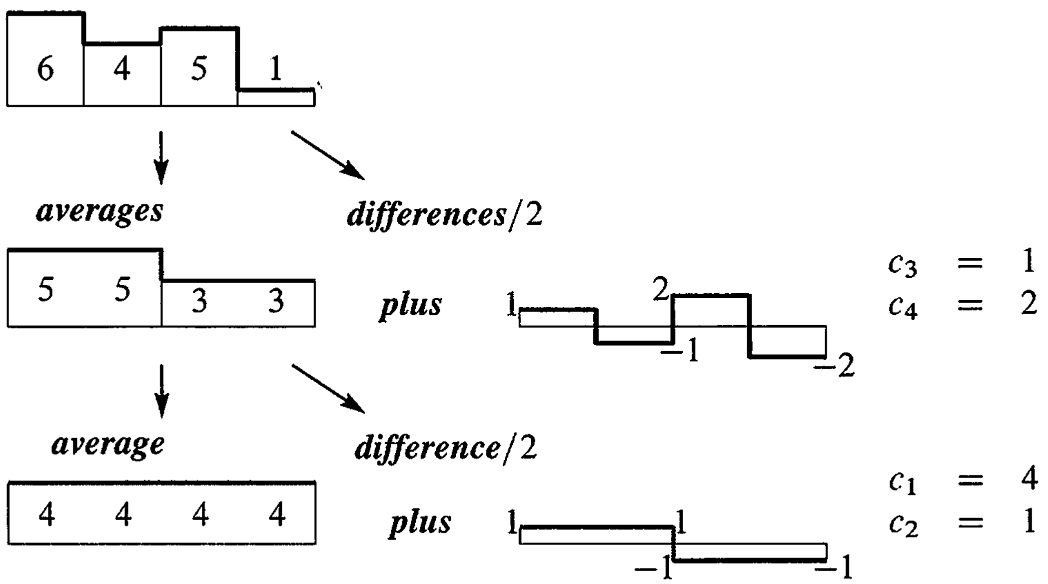
Those coefficients c are $W^{-1}v$. Inverting this basis matrix W is easy because the $\boldsymbol{w}$'s in its columns are orthogonal. But they are not unit vectors, so rescale:

$$W^{-1} = \begin{bmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The $\frac{1}{4}$'s in the first row of $c = W^{-1}v$ mean that $c_1 = 4$ is the average of $6, 4, 5, 1$.

Example 11 **(Same wavelet basis by recursion)** I can't resist showing you a faster way to find the c's. The special point of the wavelet basis is that you can pick off the details in c_3 and c_4, before the coarse details in c_2 and the overall average in c_1. A picture will explain this "multiscale" method, which is in Chapter 1 of my textbook with Nguyen on *Wavelets and Filter Banks* (Wellesley-Cambridge Press).

Split $v = (6, 4, 5, 1)$ ***into averages and waves at small scale and then large scale:***



Fourier Transform (DFT) = Change to Fourier Basis

The first thing an electrical engineer does with a signal is to take its Fourier transform. For finite vectors we are speaking about the ***Discrete Fourier Transform.*** The DFT involves complex numbers (powers of $e^{2\pi i/n}$). But if we choose $n = 4$, the matrices are small and the only complex numbers are i and $i^3 = -i$. A true electrical engineer would write j instead of i for $\sqrt{-1}$.

Fourier basis w_1 to w_n in the columns of F

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

The first column is the useful flat basis vector $(1, 1, 1, 1)$. It represents the average signal or the direct current (the DC term). It is a wave at zero frequency. The third column is $(1, -1, 1, -1)$, which alternates at the highest frequency. *The Fourier transform decomposes the signal into waves at equally spaced frequencies.*

The Fourier matrix F is absolutely the most important complex matrix in mathematics and science and engineering. Section 10.3 of this book explains the ***Fast Fourier Transform***: it can be seen as a factorization of F into matrices with many zeros. The FFT has revolutionized entire industries, by speeding up the Fourier transform. The beautiful thing is that F^{-1} looks like F, with i changed to $-i$:

Fourier transform v to c
$v = c_1 w_1 + \cdots + c_n w_n = Fc$
Fourier coefficients $c = F^{-1}v$

$$F^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (-i) & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \frac{1}{4}\overline{F}.$$

The MATLAB command $c = \text{fft}(v)$ produces the Fourier coefficients $c_1, \ldots, c_n$ of the vector v. It multiplies v by F^{-1} (fast).

■ REVIEW OF THE KEY IDEAS ■

1. If we know $T(v_1), \ldots, T(v_n)$ for a basis, linearity will determine all other $T(v)$.

2. $\left\{\begin{array}{l} \text{Linear transformation } T \\ \text{Input basis } v_1, \ldots, v_n \\ \text{Output basis } w_1, \ldots, w_m \end{array}\right\} \rightarrow \begin{array}{c} \text{Matrix } A \text{ } (m \text{ by } n) \\ \text{represents } T \\ \text{in these bases} \end{array}$

3. The derivative and integral matrices are one-sided inverses: $d(\text{constant})/dx = 0$:

   (Derivative) (Integral) $= I$ is the Fundamental Theorem of Calculus.

4. If A and B represent T and S, and the output basis for S is the input basis for T, then the matrix AB represents the transformation $T(S(u))$.

5. The change of basis matrix M represents $T(\boldsymbol{v}) = \boldsymbol{v}$. Its columns are the coefficients of the output basis expressed in the input basis: $\boldsymbol{w}_j = m_{1j}\boldsymbol{v}_1 + \cdots + m_{nj}\boldsymbol{v}_n$.

■ WORKED EXAMPLES ■

7.2 A Using the standard basis, find the 4 by 4 matrix P that represents a *cyclic permutation* T from $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ to $T(\boldsymbol{x}) = (x_4, x_1, x_2, x_3)$. Find the matrix for T^2. What is the triple shift $T^3(\boldsymbol{x})$ and why is $T^3 = T^{-1}$?

Find two real independent eigenvectors of P. What are all the eigenvalues of P?

Solution The first vector $(1, 0, 0, 0)$ in the standard basis transforms to $(0, 1, 0, 0)$ which is the second basis vector. So the first column of P is $(0, 1, 0, 0)$. The other three columns come from transforming the other three standard basis vectors:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{Then } P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ copies } T.$$

Since we used the standard basis, T *is ordinary multiplication by* P. The matrix for T^2 is a "double cyclic shift" P^2 and it produces (x_3, x_4, x_1, x_2).

The triple shift T^3 will transform $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ to $T^3(\boldsymbol{x}) = (x_2, x_3, x_4, x_1)$. If we apply T once more we are back to the original $\boldsymbol{x}$. So $T^4 =$ identity transformation and $P^4 =$ identity matrix.

Two real eigenvectors of P are $(1, 1, 1, 1)$ with eigenvalue $\lambda = 1$ and $(1, -1, 1, -1)$ with eigenvalue $\lambda = -1$. The shift leaves $(1, 1, 1, 1)$ unchanged and it reverses signs in $(1, -1, 1, -1)$. The other eigenvalues are i and $-i$. The determinant is $\lambda_1\lambda_2\lambda_3\lambda_4 = -1$.

Notice that the eigenvalues $1, i, -1, -i$ add to zero (the trace of P). They are the 4th roots of 1, since $\det(P - \lambda I) = \lambda^4 - 1$. They are at angles $0°, 90°, 180°, 270°$ in the complex plane. **The Fourier matrix F is the eigenvector matrix for P.**

7.2 B The space of 2 by 2 matrices has these four "vectors" as a basis:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

T is the linear transformation that *transposes* every 2 by 2 matrix. What is the matrix A that represents T in this basis (output basis = input basis)? What is the inverse matrix A^{-1}? What is the transformation T^{-1} that inverts the transpose operation?

Solution Transposing those four "basis matrices" just reverses v_2 and v_3:

$$\begin{aligned} T(v_1) &= v_1 \\ T(v_2) &= v_3 \\ T(v_3) &= v_2 \\ T(v_4) &= v_4 \end{aligned} \qquad \text{gives the four columns of} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The inverse matrix A^{-1} is the same as A. The inverse transformation T^{-1} is the same as T. If we transpose and transpose again, the final output equals the original input.

Problem Set 7.2

Questions 1–4 extend the first derivative example to higher derivatives.

1 The transformation S takes the ***second derivative***. Keep $1, x, x^2, x^3$ as the basis v_1, v_2, v_3, v_4 and also as w_1, w_2, w_3, w_4. Write Sv_1, Sv_2, Sv_3, Sv_4 in terms of the w's. Find the 4 by 4 matrix B for S.

2 What functions have $v'' = \mathbf{0}$? They are in the kernel of the second derivative S. What vectors are in the nullspace of its matrix B in Problem 1?

3 B is not the square of a rectangular first derivative matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ does not allow } A^2.$$

Add a zero row to A, so that output space = input space. Compare A^2 with B. Conclusion: For $B = A^2$ we want output basis = ____ basis. Then $m = n$.

4 (a) The product TS of first and second derivatives produces the *third* derivative. Add zeros to make 4 by 4 matrices, then compute AB.

(b) The matrix B^2 corresponds to S^2 = *fourth* derivative. Why is this zero?

Questions 5–9 are about a particular T and its matrix A.

5 With bases v_1, v_2, v_3 and w_1, w_2, w_3, suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. T is a linear transformation. Find the matrix A and multiply by the vector $(1, 1, 1)$. What is the output from T when the input is $v_1 + v_2 + v_3$?

6 Since $T(v_2) = T(v_3)$, the solutions to $T(v) = \mathbf{0}$ are $v =$ ____. What vectors are in the nullspace of A? Find all solutions to $T(v) = w_2$.

7 Find a vector that is not in the column space of A. Find a combination of w's that is not in the range of T.

8 You don't have enough information to determine T^2. Why is its matrix not necessarily A^2? What more information do you need?

9 Find the rank of A. This is not the dimension of the output space $\mathbf{W}$. It is the dimension of the _____ of T.

Questions 10–13 are about invertible linear transformations.

10 Suppose $T(\boldsymbol{v}_1) = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $T(\boldsymbol{v}_2) = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $T(\boldsymbol{v}_3) = \boldsymbol{w}_3$. Find the matrix A for T using these basis vectors. What input vector $\boldsymbol{v}$ gives $T(\boldsymbol{v}) = \boldsymbol{w}_1$?

11 Invert the matrix A in Problem 10. Also invert the transformation T—what are $T^{-1}(\boldsymbol{w}_1)$ and $T^{-1}(\boldsymbol{w}_2)$ and $T^{-1}(\boldsymbol{w}_3)$?

12 Which of these are true and why is the other one ridiculous?

(a) $T^{-1}T = I$ (b) $T^{-1}(T(\boldsymbol{v}_1)) = \boldsymbol{v}_1$ (c) $T^{-1}(T(\boldsymbol{w}_1)) = \boldsymbol{w}_1$.

13 Suppose the spaces $\mathbf{V}$ and $\mathbf{W}$ have the same basis $\boldsymbol{v}_1, \boldsymbol{v}_2$.

(a) Describe a transformation T (not I) that is its own inverse.

(b) Describe a transformation T (not I) that equals T^2.

(c) Why can't the same T be used for both (a) and (b)?

Questions 14–19 are about changing the basis.

14 (a) What matrix transforms $(1,0)$ into $(2,5)$ and transforms $(0,1)$ to $(1,3)$?

(b) What matrix transforms $(2,5)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

(c) Why does no matrix transform $(2,6)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

15 (a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?

(b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?

(c) What condition on a, b, c, d will make part (b) impossible?

16 (a) How do M and N in Problem 15 yield the matrix that transforms (a,c) to (r,t) and (b,d) to (s,u)?

(b) What matrix transforms $(2,5)$ to $(1,1)$ and $(1,3)$ to $(0,2)$?

17 If you keep the same basis vectors but put them in a different order, the change of basis matrix M is a _____ matrix. If you keep the basis vectors in order but change their lengths, M is a _____ matrix.

18 The matrix that rotates the axis vectors $(1,0)$ and $(0,1)$ through an angle θ is Q. What are the coordinates (a,b) of the original $(1,0)$ using the new (rotated) axes? This *inverse* can be tricky. Draw a figure or solve for a and b:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + b \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

19 The matrix that transforms $(1,0)$ and $(0,1)$ to $(1,4)$ and $(1,5)$ is $M =$ ____. The combination $a(1,4) + b(1,5)$ that equals $(1,0)$ has $(a,b) = (\ ,\ )$. How are those new coordinates of $(1,0)$ related to M or M^{-1}?

Questions 20–23 are about the space of quadratic polynomials $A + Bx + Cx^2$.

20 The parabola $w_1 = \frac{1}{2}(x^2 + x)$ equals one at $x = 1$, and zero at $x = 0$ and $x = -1$. Find the parabolas w_2, w_3, and then find $y(x)$ by linearity.

(a) w_2 equals one at $x = 0$ and zero at $x = 1$ and $x = -1$.

(b) w_3 equals one at $x = -1$ and zero at $x = 0$ and $x = 1$.

(c) $y(x)$ equals 4 at $x = 1$ and 5 at $x = 0$ and 6 at $x = -1$. Use w_1, w_2, w_3.

21 One basis for second-degree polynomials is $v_1 = 1$ and $v_2 = x$ and $v_3 = x^2$. Another basis is w_1, w_2, w_3 from Problem 20. Find two change of basis matrices, from the w's to the v's and from the v's to the w's.

22 What are the three equations for A, B, C if the parabola $Y = A + Bx + Cx^2$ equals 4 at $x = a$ and 5 at $x = b$ and 6 at $x = c$? Find the determinant of the 3 by 3 matrix. That matrix transforms values like 4, 5, 6 to parabolas—or is it the other way?

23 Under what condition on the numbers $m_1, m_2, \ldots, m_9$ do these three parabolas give a basis for the space of all parabolas?

$$v_1 = m_1 + m_2x + m_3x^2, \quad v_2 = m_4 + m_5x + m_6x^2, \quad v_3 = m_7 + m_8x + m_9x^2.$$

24 The Gram-Schmidt process changes a basis a_1, a_2, a_3 to an orthonormal basis q_1, q_2, q_3. These are columns in $A = QR$. Show that R is the change of basis matrix from the a's to the q's (a_2 is what combination of q's when $A = QR$?).

25 Elimination changes the rows of A to the rows of U with $A = LU$. Row 2 of A is what combination of the rows of U? Writing $A^{\mathrm{T}} = U^{\mathrm{T}}L^{\mathrm{T}}$ to work with columns, the change of basis matrix is $M = L^{\mathrm{T}}$. (We have bases provided the matrices are ____.)

26 Suppose v_1, v_2, v_3 are **eigenvectors** for T. This means $T(v_i) = \lambda_i v_i$ for $i = 1, 2, 3$. What is the matrix for T when the input and output bases are the v's?

27 Every invertible linear transformation can have I as its matrix! Choose any input basis $v_1, \ldots, v_n$. For output basis choose $w_i = T(v_i)$. Why must T be invertible?

28 Using $v_1 = w_1$ and $v_2 = w_2$ find the standard matrix for these T's:

(a) $T(v_1) = \mathbf{0}$ and $T(v_2) = 3v_1$ (b) $T(v_1) = v_1$ and $T(v_1 + v_2) = v_1$.

29 Suppose T is reflection across the x axis and S is reflection across the y axis. The domain $\mathbf{V}$ is the xy plane. If $v = (x, y)$ what is $S(T(v))$? Find a simpler description of the product ST.

30 Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $v = (2,1)$ then $T(v) = (1,2)$. Find $S(T(v))$ and $T(S(v))$. This shows that generally $ST \neq TS$.

31 Show that the product ST of two reflections is a rotation. Multiply these reflection matrices to find the rotation angle:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

32 True or false: If we know $T(v)$ for n different nonzero vectors in $\mathbf{R}^n$, then we know $T(v)$ for every vector in $\mathbf{R}^n$.

33 Express $e = (1,0,0,0)$ and $v = (1,-1,1,-1)$ in the wavelet basis, as in equations (8-10). The coefficients c_1, c_2, c_3, c_4 solve $Wc = e$ and $Wc = v$.

34 To represent $v = (7,5,3,1)$ in the wavelet basis, start with $(6,6,2,2)+(1,-1,1,-1)$. Then write $6,6,2,2$ as an overall average plus a difference, using $1,1,1,1$ and $1,1,-1,-1$.

35 What are the eight vectors in the wavelet basis for $\mathbf{R}^8$? They include the long wavelet $(1,1,1,1,-1,-1,-1,-1)$ and the short wavelet $(1,-1,0,0,0,0,0,0)$.

36 Suppose we have two bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ for $\mathbf{R}^n$. If a vector has coefficients b_i in one basis and c_i in the other basis, what is the change of basis matrix in $b = Mc$? Start from

$$b_1 v_1 + \cdots + b_n v_n = Vb = c_1 w_1 + \cdots + c_n w_n = Wc.$$

Your answer represents $T(v) = v$ with input basis of v's and output basis of w's. Because of different bases, the matrix is not I.

Challenge Problems

37 The space $\mathbf{M}$ of 2 by 2 matrices has the basis v_1, v_2, v_3, v_4 in Worked Example **7.2 B**. Suppose T multiplies each matrix by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. What 4 by 4 matrix A represents this transformation T on matrix space?

38 Suppose A is a 3 by 4 matrix of rank $r = 2$, and $T(v) = Av$. Choose input basis vectors v_1, v_2 from the row space of A and v_3, v_4 from the nullspace. Choose output basis $w_1 = Av_1$, $w_2 = Av_2$ in the column space and w_3 from the nullspace of A^{T}. What specially simple matrix represents this T in these special bases?

7.3 Diagonalization and the Pseudoinverse

This section produces better matrices by choosing better bases. When the goal is a diagonal matrix, one way is a basis of *eigenvectors*. The other way is two bases (the input and output bases are different). Those left and right *singular vectors* are orthonormal basis vectors for the four fundamental subspaces of A. They come from the SVD.

By reversing those input and output bases, we will find the "pseudoinverse" of A. This matrix A^+ sends $\mathbf{R}^m$ back to $\mathbf{R}^n$, and it sends column space back to row space.

The truth is that all our great factorizations of A can be regarded as a change of basis. But this is a short section, so we concentrate on the two outstanding examples. In both cases the good matrix is *diagonal*. It is Λ with one basis or Σ with two bases.

1. $S^{-1}AS = \Lambda$ ***when the input and output bases are eigenvectors of*** A.

2. $U^{-1}AV = \Sigma$ ***when those bases are eigenvectors of*** $A^{\mathrm{T}}A$ ***and*** AA^{T}.

You see immediately the difference between Λ and Σ. In Λ the bases are the same. Then $m = n$ and the matrix A must be square. And some square matrices cannot be diagonalized by any S, because they don't have n independent eigenvectors.

In Σ the input and output bases are different. The matrix A can be rectangular. The bases are *orthonormal* because $A^{\mathrm{T}}A$ and AA^{T} are symmetric. Then $U^{-1} = U^{\mathrm{T}}$ and $V^{-1} = V^{\mathrm{T}}$. Every matrix A is allowed, and A has the diagonal form Σ. This is the Singular Value Decomposition (**SVD**) of Section 6.7.

The eigenvector basis is orthonormal only when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ (a "normal" matrix). That includes symmetric and antisymmetric and orthogonal matrices (*special* might be a better word than normal). In this case the singular values in Σ are the absolute values $\sigma_i = |\lambda_i|$, so that $\Sigma = \mathbf{abs}(\Lambda)$. The two diagonalizations are the same when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$, except for possible factors -1 (real) and $e^{i\theta}$ (complex).

I will just note that the Gram-Schmidt factorization $A = QR$ chooses only *one* new basis. That is the orthogonal output basis given by Q. The input uses the standard basis given by I. We don't reach a diagonal Σ, but we do reach a triangular R. The output basis matrix appears on the left and the input basis appears on the right, in $A = QRI$.

We start with input basis equal to output basis. That will produce S and S^{-1}.

Similar Matrices: A and $S^{-1}AS$ and $W^{-1}AW$

Begin with a square matrix and one basis. The input space $\mathbf{V}$ is $\mathbf{R}^n$ and the output space $\mathbf{W}$ is also $\mathbf{R}^n$. The standard basis vectors are the columns of I. The matrix is n by n, and we call it A. The linear transformation T is "multiplication by A".

Most of this book has been about one fundamental problem—*to make the matrix simple*. We made it triangular in Chapter 2 (by elimination) and Chapter 4 (by Gram-Schmidt). We made it diagonal in Chapter 6 (*by eigenvectors*). Now that change from A to Λ comes from a ***change of basis***: ***Eigenvalue matrix from eigenvector basis.***

Here are the main facts in advance. When you change the basis for $\mathbf{V}$, the matrix changes from A to AM. Because $\mathbf{V}$ is the input space, the matrix M goes on the right (to come first). When you change the basis for $\mathbf{W}$, the new matrix is $M^{-1}A$. We are working with the output space so M^{-1} is on the left (to come last).

If you change both bases in the same way, the new matrix is $M^{-1}AM$. The good basis vectors are the eigenvectors of A, when the matrix becomes $S^{-1}AS = \Lambda$.

When the basis contains the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, the matrix for T becomes Λ.

Reason To find column 1 of the matrix, input the first basis vector $\boldsymbol{x}_1$. The transformation multiplies by A. The output is $A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$. This is λ_1 times the first basis vector plus zero times the other basis vectors. Therefore the first column of the matrix is $(\lambda_1, 0, \ldots, 0)$. ***In the eigenvector basis, the matrix is diagonal.***

Example 1 Project onto the line $y = -x$ that goes from northwest to southeast. The vector $(1, 0)$ projects to $(.5, -.5)$ on that line. The projection of $(0, 1)$ is $(-.5, .5)$:

1. **Standard matrix: Project standard basis** $\quad A = \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$.

2. **Find the diagonal matrix Λ in the eigenvector basis.**

Solution The eigenvectors for this projection are $\boldsymbol{x}_1 = (1, -1)$ and $\boldsymbol{x}_2 = (1, 1)$. The first eigenvector lies on the 135° line and the second is perpendicular (on the 45° line). Their projections are $\boldsymbol{x}_1$ and $\mathbf{0}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.

2. **Diagonalized matrix: Project eigenvectors** $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. **Find a third matrix B using another basis $\boldsymbol{v}_1 = \boldsymbol{w}_1 = (2, 0)$ and $\boldsymbol{v}_2 = \boldsymbol{w}_2 = (1, 1)$.**

Solution $\boldsymbol{w}_1$ is not an eigenvector, so the matrix B in this basis will not be diagonal. The first way to compute B follows the rule of Section 7.2:

Find column j of the matrix by writing the projection $T(\boldsymbol{v}_j)$ as a combination of $\boldsymbol{w}$'s.

Apply the projection T to $(2, 0)$. The result is $(1, -1)$ which is $\boldsymbol{w}_1 - \boldsymbol{w}_2$. So the first column of B contains 1 and -1. The second vector $\boldsymbol{w}_2 = (1, 1)$ projects to zero, so the second column of B contains 0 and 0. The eigenvalues must stay at 1 and 0:

3. **Third similar matrix: Project $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$** $\quad B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. $\quad (1)$

The second way to find the same B is more insightful. Use W^{-1} and W to change between the standard basis and the basis of $\boldsymbol{w}$'s. Those change of basis matrices are representing the identity transformation! The product of transformations is just ITI. *The product of matrices is* $B = W^{-1}AW$. This approach shows that B *is* ***similar*** to A.

For any basis $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ find the matrix B in three steps. Change the input basis to the standard basis with W. The matrix in the standard basis is A. Change the output basis back to the $\boldsymbol{w}$'s with W^{-1}. ***Then*** $\boldsymbol{B = W^{-1}AW}$ ***represents*** $\boldsymbol{ITI}$:

$$B_{\boldsymbol{w}\text{'s to }\boldsymbol{w}\text{'s}} = W^{-1}_{\text{standard to }\boldsymbol{w}\text{'s}} \quad A_{\text{standard}} \quad W_{\boldsymbol{w}\text{'s to standard}} \tag{2}$$

A change of basis produces a similarity transformation to $W^{-1}AW$ ***in the matrix.***

Example 2 (continuing with the projection) Apply this $W^{-1}AW$ rule to find B, when the basis $(2,0)$ and $(1,1)$ is in the columns of W:

$$W^{-1}AW = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

The $W^{-1}AW$ rule has produced the same B as in equation (1). The matrices A and B are similar. They have the same eigenvalues (1 and 0). And Λ is similar too.

Notice that the projection matrix keeps the property $A^2 = A$ and $B^2 = B$ and $\Lambda^2 = \Lambda$. The second projection doesn't move the first projection.

The Singular Value Decomposition (SVD)

Now the input basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ can be different from the output basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$. In fact the input space $\mathbf{R}^n$ can be different from the output space $\mathbf{R}^m$. Again the best matrix is diagonal (now m by n). To achieve this diagonal matrix Σ, each input vector $\boldsymbol{v}_j$ must transform into a multiple of the output vector $\boldsymbol{u}_j$. That multiple is the *singular value* σ_j on the main diagonal of Σ:

$$\textbf{SVD} \qquad A\boldsymbol{v}_j = \begin{cases} \sigma_j \boldsymbol{u}_j & \text{for } j \le r \\ \mathbf{0} & \text{for } j > r \end{cases} \quad \text{with orthonormal bases.} \tag{3}$$

The singular values are in the order $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$. The rank r enters because (by definition) singular values are not zero. The second part of the equation says that $\boldsymbol{v}_j$ is in the nullspace for $j = r+1, \ldots, n$. This gives the correct number $n - r$ of basis vectors for the nullspace.

Let me connect the matrices with the linear transformations they represent. A and Σ represent the *same transformation.* $A = U\Sigma V^{\mathrm{T}}$ uses the standard bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. The diagonal Σ uses the input basis of $\boldsymbol{v}$'s and the output basis of $\boldsymbol{u}$'s. The orthogonal matrices V and U give the basis changes; they represent the identity transformations (in $\mathbf{R}^n$ and $\mathbf{R}^m$). The product of transformations is ITI, and it is represented in the $\boldsymbol{v}$ and $\boldsymbol{u}$ bases by $U^{-1}AV$ which is Σ.

The matrix Σ in the $\boldsymbol{u}$ and $\boldsymbol{v}$ bases comes from A in the standard bases by $U^{-1}AV$:

$$\Sigma_{\boldsymbol{v}\text{'s to }\boldsymbol{u}\text{'s}} = U^{-1}_{\textit{standard}\text{ to }\boldsymbol{u}\text{'s}} \quad A_{\textit{standard}} \quad V_{\boldsymbol{v}\text{'s to }\textit{standard}}. \tag{4}$$

The SVD chooses orthonormal bases ($U^{-1} = U^{\mathrm{T}}$ and $V^{-1} = V^{\mathrm{T}}$) that diagonalize A.

The two orthonormal bases in the SVD are the eigenvector bases for $A^{\mathrm{T}}A$ (the $\boldsymbol{v}$'s) and AA^{T} (the $\boldsymbol{u}$'s). Since those are symmetric matrices, their unit eigenvectors are orthonormal. Their eigenvalues are the numbers σ_j^2. Equations (10) and (11) in Section 6.7 proved that those bases diagonalize the standard matrix A to produce Σ.

Polar Decomposition

Every complex number has the polar form $re^{i\theta}$. A nonnegative number r multiplies a number on the unit circle. (Remember that $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$.) Thinking of these numbers as 1 by 1 matrices, $r \geq 0$ corresponds to a *positive semidefinite matrix* (call it H) and $e^{i\theta}$ corresponds to an *orthogonal matrix* Q. The ***polar decomposition*** extends this factorization to matrices: orthogonal times semidefinite, $A = QH$.

Every real square matrix can be factored into $A = \boldsymbol{QH}$, where Q is ***orthogonal*** and H is ***symmetric positive semidefinite***. If A is invertible, H is positive definite.

For the proof we just insert $V^{\mathrm{T}}V = I$ into the middle of the SVD:

$$\textbf{Polar decomposition} \qquad A = U\Sigma V^{\mathrm{T}} = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}}) = (Q)(H). \tag{5}$$

The first factor UV^{T} is Q. The product of orthogonal matrices is orthogonal. The second factor $V\Sigma V^{\mathrm{T}}$ is H. It is positive semidefinite because its eigenvalues are in Σ. If A is invertible then Σ and H are also invertible. ***H is the symmetric positive definite square root of*** $A^{\mathrm{T}}A$. Equation (5) says that $H^2 = V\Sigma^2V^{\mathrm{T}} = A^{\mathrm{T}}A$.

There is also a polar decomposition $A = KQ$ in the reverse order. Q is the same but now $K = U\Sigma U^{\mathrm{T}}$. This is the symmetric positive definite square root of AA^{T}.

Example 3 Find the polar decomposition $A = QH$ from its SVD in Section 6.7:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \\ & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^{\mathrm{T}}.$$

Solution The orthogonal part is $Q = UV^{\mathrm{T}}$. The positive definite part is $H = V\Sigma V^{\mathrm{T}}$. This is also $H = Q^{-1}A$ which is $Q^{\mathrm{T}}A$ because Q is orthogonal:

$$\textbf{Orthogonal} \qquad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\textbf{Positive definite} \qquad H = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}.$$

In mechanics, the polar decomposition separates the ***rotation*** (in Q) from the ***stretching*** (in H). The eigenvalues of H are the singular values of A. They give the stretching factors. The eigenvectors of H are the eigenvectors of $A^{\mathrm{T}}A$. They give the stretching directions (the principal axes). Then Q rotates those axes.

The polar decomposition just splits the key equation $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ into two steps. The "H" part multiplies $\boldsymbol{v}_i$ by σ_i. The "Q" part swings $\boldsymbol{v}_i$ around into $\boldsymbol{u}_i$.

The Pseudoinverse

By choosing good bases, A multiplies $\boldsymbol{v}_i$ in the row space to give $\sigma_i \boldsymbol{u}_i$ in the column space. A^{-1} must do the opposite! If $A\boldsymbol{v} = \sigma \boldsymbol{u}$ then $A^{-1}\boldsymbol{u} = \boldsymbol{v}/\sigma$. The singular values of A^{-1} are $1/\sigma$, just as the eigenvalues of A^{-1} are $1/\lambda$. The bases are reversed. The $\boldsymbol{u}$'s are in the row space of A^{-1}, the $\boldsymbol{v}$'s are in the column space.

Until this moment we would have added "*if A^{-1} exists.*" Now we don't. A matrix that multiplies $\boldsymbol{u}_i$ to produce $\boldsymbol{v}_i/\sigma_i$ *does* exist. It is the pseudoinverse A^+:

Pseudoinverse

$$A^+ = V\Sigma^+U^{\mathrm{T}} = \underset{n \text{ by } n}{\begin{bmatrix} \boldsymbol{v}_1 \cdots \boldsymbol{v}_r \cdots \boldsymbol{v}_n \end{bmatrix}} \underset{n \text{ by } m}{\begin{bmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_r^{-1} \\ & & & \end{bmatrix}} \underset{m \text{ by } m}{\begin{bmatrix} \boldsymbol{u}_1 \cdots \boldsymbol{u}_r \cdots \boldsymbol{u}_m \end{bmatrix}^{\mathrm{T}}}$$

The ***pseudoinverse*** A^+ is an n by m matrix. If A^{-1} exists (we said it again), then A^+ is the same as A^{-1}. In that case $m = n = r$ and we are inverting $U\Sigma V^{\mathrm{T}}$ to get $V\Sigma^{-1}U^{\mathrm{T}}$. The new symbol A^+ is needed when $r < m$ or $r < n$. Then A has no two-sided inverse, but it has a *pseudo*inverse A^+ with that same rank r:

$$A^+\boldsymbol{u}_i = \frac{1}{\sigma_i}\boldsymbol{v}_i \quad \text{for } i \le r \qquad \text{and} \qquad A^+\boldsymbol{u}_i = \boldsymbol{0} \quad \text{for } i > r.$$

The vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ in the column space of A go back to $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ in the row space. The other vectors $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ are in the left nullspace, and A^+ sends them to zero. When we know what happens to each basis vector $\boldsymbol{u}_i$, we know A^+.

Notice the pseudoinverse Σ^+ of the diagonal matrix Σ. Each σ is replaced by σ^{-1}. The product $\Sigma^+\Sigma$ is as near to the identity as we can get (it is a projection matrix, $\Sigma^+\Sigma$ is partly I and partly 0). We get r 1's. We can't do anything about the zero rows and columns. This example has $\sigma_1 = 2$ and $\sigma_2 = 3$:

$$\Sigma^+\Sigma = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix}.$$

The pseudoinverse A^+ is the n by m matrix that makes AA^+ and A^+A into projections:

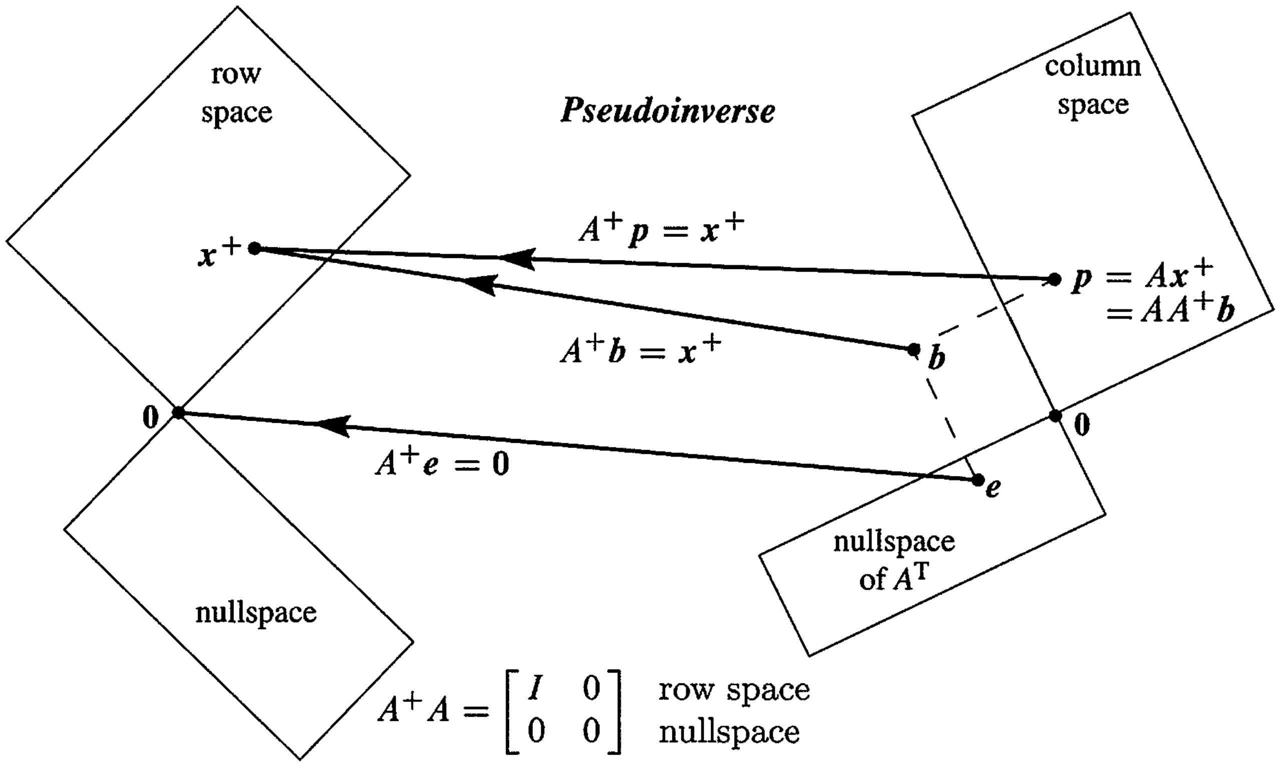


Figure 7.4: Ax^+ in the column space goes back to $A^+Ax^+ = x^+$ in the row space.

Trying for $AA^{-1} = A^{-1}A = I$	$AA^+ =$ projection matrix onto the column space of A $A^+A =$ projection matrix onto the row space of A

Example 4 Find the pseudoinverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. This matrix is not invertible. The rank is 1. The only singular value is $\sqrt{10}$. That is inverted to $1/\sqrt{10}$ in Σ^+:

$$A^+ = V\Sigma^+U^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

A^+ also has rank 1. Its column space is the row space of A. When A takes $(1, 1)$ in the row space to $(4, 2)$ in the column space, A^+ does the reverse. $A^+(4, 2) = (1, 1)$.

Every rank one matrix is a column times a row. With unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, that is $A = \sigma\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Then the best inverse of a rank one matrix is $A^+ = \boldsymbol{v}\boldsymbol{u}^{\mathrm{T}}/\sigma$. The product AA^+ is $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$, the projection onto the line through $\boldsymbol{u}$. The product A^+A is $\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}$.

Application to least squares Chapter 4 found the best solution $\widehat{x}$ to an unsolvable system $A\boldsymbol{x} = \boldsymbol{b}$. The key equation is $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$, with the assumption that $A^{\mathrm{T}}A$ is invertible. The zero vector was alone in the nullspace.

Now A may have dependent columns (rank $< n$). There can be many solutions to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$. ***One solution is*** $\boldsymbol{x}^+ = A^+\boldsymbol{b}$ ***from the pseudoinverse.*** We can check that

A^TAA^+b, is A^Tb, because Figure 7.4 shows that $e = b - AA^+b$ is the part of b in the nullspace of A^T. Any vector in the nullspace of A could be added to x^+, to give another solution $\widehat{x}$ to $A^TA\widehat{x} = A^Tb$. But x^+ will be shorter than any other $\widehat{x}$ (Problem 16):

The shortest least squares solution to $Ax = b$ is $x^+ = A^+b$.

The pseudoinverse A^+ and this best solution x^+ are essential in statistics, because experiments often have a matrix A with **dependent columns**.

■ REVIEW OF THE KEY IDEAS ■

1. Diagonalization $S^{-1}AS = \Lambda$ is the same as a change to the eigenvector basis.
2. The SVD chooses an input basis of v's and an output basis of u's. Those orthonormal bases diagonalize A. This is $Av_i = \sigma_i u_i$, and in matrix form $A = U\Sigma V^T$.
3. Polar decomposition factors A into QH, rotation UV^T times stretching $V\Sigma V^T$.
4. The pseudoinverse $A^+ = V\Sigma^+U^T$ transforms the column space of A back to its row space. A^+A is the identity on the row space (and zero on the nullspace).

■ WORKED EXAMPLES ■

7.3 A If A has rank n (full column rank) then it has a *left inverse* $C = (A^TA)^{-1}A^T$. This matrix C gives $CA = I$. Explain why the pseudoinverse is $A^+ = C$ in this case. If A has rank m (full row rank) then it has a *right inverse* B with $B = A^T(AA^T)^{-1}$. Then $AB = I$. Explain why $A^+ = B$ in this case.

Find B for A_1 and find C for A_2. Find A^+ for all three matrices A_1, A_2, A_3:

$$A_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Solution If A has rank n (independent columns) then A^TA is invertible—this is a key point of Section 4.2. Certainly $C = (A^TA)^{-1}A^T$ multiplies A to give $CA = I$. In the opposite order, $AC = A(A^TA)^{-1}A^T$ is the projection matrix (Section 4.2 again) onto the column space. So C meets the requirements to be A^+: *CA and AC are projections.*

If A has rank m (full row rank) then AA^T is invertible. Certainly A multiplies $B = A^T(AA^T)^{-1}$ to give $AB = I$. In the opposite order, $BA = A^T(AA^T)^{-1}A$ is the projection matrix onto the row space. So B is the pseudoinverse A^+ with rank m.

The example A_1 has full column rank (for C) and A_2 has full row rank (for B):

$$A_1^+ = (A_1^TA_1)^{-1}A_1^T = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & 2 \end{bmatrix} \qquad A_2^+ = A_2^T(A_2A_2^T)^{-1} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Notice $A_1^+A_1 = [1]$ and $A_2A_2^+ = [1]$. But A_3 (rank 1) has no left or right inverse. **Its rank is not full.** Its pseudoinverse is $A_3^+ = \sigma_1^{-1}v_1u_1^T = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]/4$.

Problem Set 7.3

Problems 1–4 compute and use the SVD of a particular matrix (not invertible).

1 (a) Compute A^TA and its eigenvalues and unit eigenvectors v_1 and v_2. Find σ_1.

$$\textbf{Rank one matrix} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

(b) Compute AA^T and its eigenvalues and unit eigenvectors u_1 and u_2.

(c) Verify that $Av_1 = \sigma_1 u_1$. Put numbers into the SVD:

$$A = U\Sigma V^T \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^T.$$

2 (a) From the u's and v's in Problem 1 write down orthonormal bases for the four fundamental subspaces of this matrix A.

(b) Describe all matrices that have those same four subspaces. Multiples of A?

3 From U, V, and Σ in Problem 1 find the orthogonal matrix $Q = UV^T$ and the symmetric matrix $H = V\Sigma V^T$. Verify the polar decomposition $A = QH$. This H is only semidefinite because _____. Test $H^2 = A$.

4 Compute the pseudoinverse $A^+ = V\Sigma^+U^T$. The diagonal matrix Σ^+ contains $1/\sigma_1$. Rename the four subspaces (for A) in Figure 7.4 as four subspaces for A^+. Compute the projections $P_{\text{row}} = A^+A$ and $P_{\text{column}} = AA^+$.

Problems 5–9 are about the SVD of an invertible matrix.

5 Compute A^TA and its eigenvalues and unit eigenvectors v_1 and v_2. What are the singular values σ_1 and σ_2 for this matrix A?

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}.$$

6 AA^{T} has the same eigenvalues σ_1^2 and σ_2^2 as $A^{\mathrm{T}}A$. Find unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Put numbers into the SVD:

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}}.$$

7 In Problem 6, multiply columns times rows to show that $A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$. Prove from $A = U\Sigma V^{\mathrm{T}}$ that every matrix of rank r is the sum of r matrices of rank one.

8 From U, V, and Σ find the orthogonal matrix $Q = UV^{\mathrm{T}}$ and the symmetric matrix $K = U\Sigma U^{\mathrm{T}}$. Verify the polar decomposition in reverse order $A = KQ$.

9 The pseudoinverse of this A is the same as _____ because _____.

Problems 10–11 compute and use the SVD of a 1 by 3 rectangular matrix.

10 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors when the matrix is $A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$. What are the singular values of A?

11 Put numbers into the singular value decomposition of A:

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = [\boldsymbol{u}_1] \begin{bmatrix} \sigma_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}^{\mathrm{T}}.$$

Put numbers into the pseudoinverse $V\Sigma^+U^{\mathrm{T}}$ of A. *Compute AA^+ and A^+A*:

$$\textbf{Pseudoinverse} \quad A^+ = \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \end{bmatrix}^{\mathrm{T}}.$$

12 What is the only 2 by 3 matrix that has no pivots and no singular values? What is Σ for that matrix? A^+ is the zero matrix, but what shape?

13 If $\det A = 0$ why is $\det A^+ = 0$? If A has rank r, why does A^+ have rank r?

14 When are the factors in $U\Sigma V^{\mathrm{T}}$ the same as in $Q\Lambda Q^{\mathrm{T}}$? The eigenvalues λ_i must be positive, to equal the σ_i. Then A must be _____ and positive _____.

Problems 15–18 bring out the main properties of A^+ and $\boldsymbol{x}^+ = A^+\boldsymbol{b}$.

15 All matrices in this problem have rank one. The vector $\boldsymbol{b}$ is (b_1, b_2).

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad A^{\mathrm{T}} = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix} \quad AA^{\mathrm{T}} = \begin{bmatrix} .8 & .4 \\ .4 & .2 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

(a) The equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ has many solutions because $A^{\mathrm{T}}A$ is _____.

(b) Verify that $\boldsymbol{x}^+ = A^+\boldsymbol{b} = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^{\mathrm{T}}A\boldsymbol{x}^+ = A^{\mathrm{T}}\boldsymbol{b}$.

(c) Add $(1,-1)$ to that x^+ to get another solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Show that $\|\widehat{x}\|^2 = \|x^+\|^2 + 2$, and x^+ is shorter.

16 ***The vector $x^+ = A^+b$ is the shortest possible solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.*** Reason: The difference $\widehat{x} - x^+$ is in the nullspace of $A^{\mathrm{T}}A$. This is also the nullspace of A, orthogonal to x^+. Explain how it follows that $\|\widehat{x}\|^2 = \|x^+\|^2 + \|\widehat{x} - x^+\|^2$.

17 Every b in $\mathbf{R}^m$ is $p + e$. This is the column space part plus the left nullspace part. Every x in $\mathbf{R}^n$ is $x_r + x_n =$ (row space part) + (nullspace part). Then

$$AA^+p = \_\_\_\_ \qquad AA^+e = \_\_\_\_ \qquad A^+Ax_r = \_\_\_\_ \qquad A^+Ax_n = \_\_\_\_$$

18 Find A^+ and A^+A and AA^+ and x^+ for this 2 by 1 matrix and these b:

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1] \qquad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

19 A general 2 by 2 matrix A is determined by four numbers. If triangular, it is determined by three. If diagonal, by two. If a rotation, by one. An eigenvector, by one. Check that the total count is four for each factorization of A:

$$\textbf{Four numbers in} \quad LU \quad LDU \quad QR \quad U\Sigma V^{\mathrm{T}} \quad S\Lambda S^{-1}.$$

20 Following Problem 19, check that LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are determined by *three* numbers. This is correct because the matrix A is now _____.

21 From $A = U\Sigma V^{\mathrm{T}}$ and $A^+ = V\Sigma^+U^{\mathrm{T}}$ explain these splittings into rank 1:

$$A = \sum_1^r \sigma_i u_i v_i^{\mathrm{T}} \qquad A^+ = \sum_1^r \frac{v_i u_i^{\mathrm{T}}}{\sigma_i} \qquad A^+A = \sum_1^r v_i v_i^{\mathrm{T}} \qquad AA^+ = \sum_1^r u_i u_i^{\mathrm{T}}$$

Challenge Problems

22 This problem looks for all matrices A with a given column space in $\mathbf{R}^m$ and a given row space in $\mathbf{R}^n$. Suppose $c_1, \ldots, c_r$ and $b_1, \ldots, b_r$ are bases for those two spaces. Make them columns of C and B. The goal is to show that ***$A = CMB^{\mathrm{T}}$ for an r by r invertible matrix M***. *Hint: Start from $A = U\Sigma V^{\mathrm{T}}$*. A must have this form:

The first r columns of U and V must be connected to C and B by invertible matrices, because they contain bases for the same column space and row space.

23 A pair of singular vectors v and u will satisfy $Av = \sigma u$ and $A^{\mathrm{T}}u = \sigma v$. This means that the double vector $x = \begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of what symmetric block matrix? What is the eigenvalue? The SVD of A is equivalent to the diagonalization of that symmetric block matrix.

Chapter 8

Applications

8.1 Matrices in Engineering

This section will show how engineering problems produce symmetric matrices K (often K is positive definite). The "linear algebra reason" for symmetry and positive definiteness is their form $K = A^{\mathrm{T}}A$ and $K = A^{\mathrm{T}}CA$. The "physical reason" is that the expression $\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}K\boldsymbol{u}$ represents *energy*—and energy is never negative. The matrix C, often diagonal, contains positive physical constants like conductance or stiffness or diffusivity.

Our first examples come from mechanical and civil and aeronautical engineering. K is the ***stiffness matrix***, and $K^{-1}\boldsymbol{f}$ is the structure's response to forces $\boldsymbol{f}$ from outside. Section 8.2 turns to electrical engineering—the matrices come from networks and circuits. The exercises involve chemical engineering and I could go on! Economics and management and engineering design come later in this chapter (there the key is optimization).

Engineering leads to linear algebra in two ways, directly and indirectly:

Direct way The physical problem has only a finite number of pieces. The laws connecting their position or velocity are *linear* (movement is not too big or too fast). The laws are expressed by *matrix equations*.

Indirect way The physical system is "continuous". Instead of individual masses, the mass density and the forces and the velocities are functions of x or x, y or x, y, z. The laws are expressed by *differential equations*. ***To find accurate solutions we approximate by finite difference equations or finite element equations.***

Both ways produce matrix equations and linear algebra. I really believe that you cannot do modern engineering without matrices.

Here we present equilibrium equations $K\boldsymbol{u} = \boldsymbol{f}$. With motion, $M d^2\boldsymbol{u}/dt^2 + K\boldsymbol{u} = \boldsymbol{f}$ becomes dynamic. Then we use eigenvalues from $K\boldsymbol{x} = \lambda M\boldsymbol{x}$, or finite differences.

Before explaining the physical examples, may I write down the matrices? The tridiagonal K_0 appears many times in this textbook. Now we will see its applications. These matrices are all symmetric, and the first four are positive definite:

$$K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{2} \end{bmatrix} \qquad A_0^{\mathrm{T}}C_0A_0 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3+c_4 \end{bmatrix}$$

Fixed-fixed **Spring constants included**

$$K_1 = A_1^{\mathrm{T}}A_1 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix}$$

Fixed-free **Spring constants included**

$$K_{\textbf{singular}} = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad K_{\textbf{circular}} = \begin{bmatrix} 2 & -1 & \mathbf{-1} \\ -1 & 2 & -1 \\ \mathbf{-1} & -1 & 2 \end{bmatrix}$$

Free-free

The matrices $K_0, K_1, K_{\textbf{singular}}$, and $K_{\textbf{circular}}$ have $C = I$ for simplicity. This means that all the "spring constants" are $c_i = 1$. We included $A_0^{\mathrm{T}}C_0A_0$ and $A_1^{\mathrm{T}}C_1A_1$ to show how the spring constants enter the matrix (without changing its positive definiteness). Our first goal is to show where these stiffness matrices come from.

A Line of Springs

Figure 8.1 shows three masses m_1, m_2, m_3 connected by a line of springs. One case has four springs, with top and bottom fixed. The fixed-free case has only three springs; the lowest mass hangs freely. The **fixed-fixed** problem will lead to K_0 and $A_0^{\mathrm{T}}C_0A_0$. The **fixed-free** problem will lead to K_1 and $A_1^{\mathrm{T}}C_1A_1$. A **free-free** problem, with no support at either end, produces the matrix $K_{\textbf{singular}}$.

We want equations for the mass movements $\boldsymbol{u}$ and the tensions (or compressions) $\boldsymbol{y}$:

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3) = \textbf{movements of the masses} \text{ (down or up)} \\ \boldsymbol{y} &= (y_1, y_2, y_3, y_4) \text{ or } (y_1, y_2, y_3) = \textbf{tensions in the springs} \end{aligned}$$

When a mass moves downward, its displacement is positive ($u_i > 0$). For the springs, tension is positive and compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward. Each spring is controlled by its own Hooke's Law $y = c\,e$: (*stretching force*) = (*spring constant*) times (*stretching distance*).

Our job is to link these one-spring equations $y = ce$ into a vector equation $K\boldsymbol{u} = f$ for the whole system. The force vector f comes from gravity. The gravitational constant g will multiply each mass to produce forces $f = (m_1g, m_2g, m_3g)$.

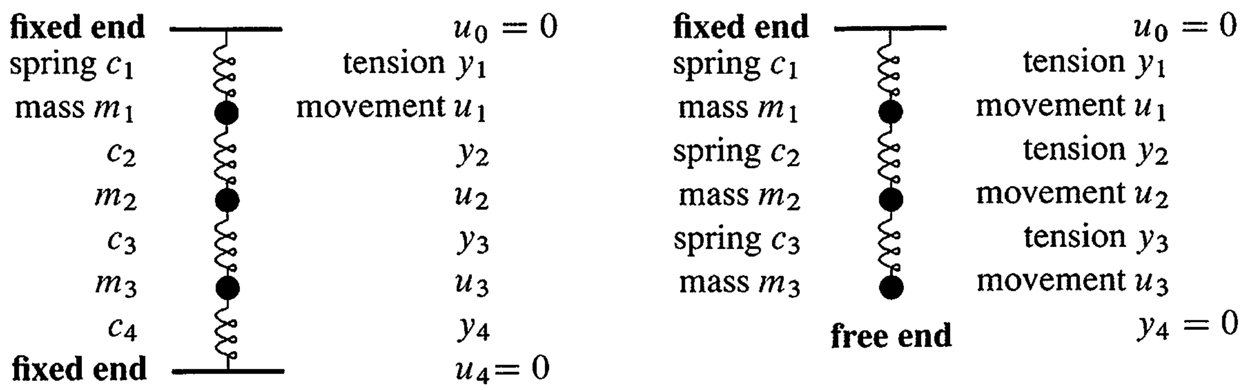


Figure 8.1: Lines of springs and masses: **fixed-fixed** and **fixed-free** ends.

The real problem is to find the stiffness matrix (**fixed-fixed** and **fixed-free**). The best way to create K is in three steps, not one. Instead of connecting the movements $\boldsymbol{u}_i$ directly to the forces, it is much better to connect each vector to the next in this list:

$$\begin{array}{lcllcl} u & = & \textit{Movements} \text{ of } n \text{ masses} & & = & (u_1, \ldots, u_n) \\ e & = & \textit{Elongations} \text{ of } m \text{ springs} & & = & (e_1, \ldots, e_m) \\ y & = & \textit{Internal forces} \text{ in } m \text{ springs} & & = & (y_1, \ldots, y_m) \\ f & = & \textit{External forces} \text{ on } n \text{ masses} & & = & (f_1, \ldots, f_n) \end{array}$$

The framework that connects $\boldsymbol{u}$ to $\boldsymbol{e}$ to $\boldsymbol{y}$ to f looks like this:

$$\begin{array}{ccccc} \boxed{u} & & \boxed{f} & \quad e = Au & \quad A \text{ is } m \text{ by } n \\ A\downarrow & & \uparrow A^{\mathrm{T}} & \quad y = Ce & \quad C \text{ is } m \text{ by } m \\ \boxed{e} & \xrightarrow{C} & \boxed{y} & \quad f = A^{\mathrm{T}}y & \quad A^{\mathrm{T}} \text{ is } n \text{ by } m \end{array}$$

We will write down the matrices A and C and A^{T} for the two examples, first with fixed ends and then with the lower end free. Forgive the simplicity of these matrices, it is their form that is so important. Especially the appearance of A together with A^{T}.

The *elongation* $\boldsymbol{e}$ is the stretching distance—how far the springs are extended. Originally there is no stretching—the system is lying on a table. When it becomes vertical and upright, gravity acts. The masses move down by distances u_1, u_2, u_3. Each spring is stretched or compressed by $e_i = u_i - u_{i-1}$, *the difference in displacements of its ends*:

	First spring:	$e_1 = u_1$	(the top is fixed so $u_0 = 0$)
Stretching of	Second spring:	$e_2 = u_2 - u_1$	
each spring	Third spring:	$e_3 = u_3 - u_2$	
	Fourth spring:	$e_4 = \quad - u_3$	(the bottom is fixed so $u_4 = 0$)

If both ends move the same distance, that spring is not stretched: $u_i = u_{i-1}$ and $e_i = 0$. The matrix in those four equations is a 4 by 3 *difference matrix* A, and $\boldsymbol{e} = A\boldsymbol{u}$:

$$\begin{array}{c}\textbf{Stretching}\\ \textbf{distances}\\ \textbf{(elongations)}\end{array} \qquad \boldsymbol{e} = A\boldsymbol{u} \quad \text{is} \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{1}$$

The next equation $\boldsymbol{y} = C\boldsymbol{e}$ connects spring elongation $\boldsymbol{e}$ with spring tension $\boldsymbol{y}$. *This is Hooke's Law $y_i = c_i e_i$ for each separate spring.* It is the "constitutive law" that depends on the material in the spring. A soft spring has small c, so a moderate force y can produce a large stretching e. Hooke's linear law is nearly exact for real springs, before they are overstretched and the material becomes plastic.

Since each spring has its own law, the matrix in $\boldsymbol{y} = C\boldsymbol{e}$ is a diagonal matrix C:

$$\begin{array}{c}\textbf{Hooke's}\\ \textbf{Law}\\ \boldsymbol{y} = C\boldsymbol{e}\end{array} \quad \begin{array}{rcl} y_1 &=& c_1e_1 \\ y_2 &=& c_2e_2 \\ y_3 &=& c_3e_3 \\ y_4 &=& c_4e_4 \end{array} \quad \text{is} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{2}$$

Combining $\boldsymbol{e} = A\boldsymbol{u}$ with $\boldsymbol{y} = C\boldsymbol{e}$, the spring forces are $\boldsymbol{y} = CA\boldsymbol{u}$.

Finally comes the *balance equation*, the most fundamental law of applied mathematics. The internal forces from the springs balance the external forces on the masses. Each mass is pulled or pushed by the spring force y_j above it. From below it feels the spring force y_{j+1} plus f_j from gravity. Thus $y_j = y_{j+1} + f_j$ or $f_j = y_j - y_{j+1}$:

$$\begin{array}{c}\textbf{Force}\\ \textbf{balance}\\ \boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}\end{array} \quad \begin{array}{rcl} f_1 &=& y_1 - y_2 \\ f_2 &=& y_2 - y_3 \\ f_3 &=& y_3 - y_4 \end{array} \quad \text{is} \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{3}$$

That matrix is** A^{T}. The equation for balance of forces is $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$.* Nature transposes the rows and columns of the $\boldsymbol{e} - \boldsymbol{u}$ matrix to produce the $\boldsymbol{f} - \boldsymbol{y}$ matrix. This is the beauty of the framework, that A^{T} appears along with A. The three equations combine into $K\boldsymbol{u} = \boldsymbol{f}$, where the ***stiffness matrix is $K = A^{\mathrm{T}}CA$:

$$\left\{\begin{array}{rcl} \boldsymbol{e} &=& A\boldsymbol{u} \\ \boldsymbol{y} &=& C\boldsymbol{e} \\ \boldsymbol{f} &=& A^{\mathrm{T}}\boldsymbol{y} \end{array}\right\} \quad \text{combine into} \quad A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f} \quad \text{or} \quad K\boldsymbol{u} = \boldsymbol{f}.$$

In the language of elasticity, $\boldsymbol{e} = A\boldsymbol{u}$ is the **kinematic** equation (for displacement). The force balance $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ is the **static** equation (for equilibrium). The **constitutive law** is $\boldsymbol{y} = C\boldsymbol{e}$ (from the material). Then $A^{\mathrm{T}}CA$ is n by n = (n by m)(m by m)(m by n).

Finite element programs spend major effort on assembling $K = A^{\mathrm{T}}CA$ from thousands of smaller pieces. We find K for four springs (**fixed-fixed**) by multiplying A^{T} times CA:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ -c_2 & c_2 & 0 \\ 0 & -c_3 & c_3 \\ 0 & 0 & -c_4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

If all springs are identical, with $c_1 = c_2 = c_3 = c_4 = 1$, then $C = I$. The stiffness matrix reduces to $A^{\mathrm{T}}A$. It becomes the special $-1, 2, -1$ matrix:

$$\textbf{With } C = I \qquad K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{4}$$

Note the difference between $A^{\mathrm{T}}A$ from engineering and LL^{T} from linear algebra. The matrix A from four springs is 4 by 3. The triangular matrix L from elimination is square. The stiffness matrix K is assembled from $A^{\mathrm{T}}A$, and then broken up into LL^{T}. One step is applied mathematics, the other is computational mathematics. Each K is built from rectangular matrices and factored into square matrices.

May I list some properties of $K = A^{\mathrm{T}}CA$? You know almost all of them:

1. K is **tridiagonal**, because mass 3 is not connected to mass 1.
2. K is **symmetric**, because C is symmetric and A^{T} comes with A.
3. K is **positive definite**, because $c_i > 0$ and A has **independent columns**.
4. K^{-1} is a full matrix in equation (5) with **all positive entries**.

That last property leads to an important fact about $\boldsymbol{u} = K^{-1}\boldsymbol{f}$: *If all forces act downwards* ($f_j > 0$) *then all movements are downwards* ($u_j > 0$). Notice that "positiveness" is different from "positive definiteness". Here K^{-1} is positive (K is not). Both K and K^{-1} are positive definite.

Example 1 Suppose all $c_i = c$ and $m_j = m$. Find the movements $\boldsymbol{u}$ and tensions $\boldsymbol{y}$.

All springs are the same and all masses are the same. But all movements and elongations and tensions will *not* be the same. K^{-1} includes $\frac{1}{c}$ because $A^{\mathrm{T}}CA$ includes c:

$$\boldsymbol{u} = K^{-1}\boldsymbol{f} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{2} \\ \mathbf{3/2} \end{bmatrix} \tag{5}$$

The displacement u_2, for the mass in the middle, is greater than u_1 and u_3. The units are correct: the force mg divided by force per unit length c gives a length u. Then

$$\boldsymbol{e} = A\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{1/2} \\ \mathbf{-1/2} \\ \mathbf{-3/2} \end{bmatrix}.$$

Those elongations add to zero because the ends of the line are fixed. (The sum $u_1 + (u_2 - u_1) + (u_3 - u_2) + (-u_3)$ is certainly zero.) For each spring force y_i we just multiply e_i by c. So y_1, y_2, y_3, y_4 are $\frac{3}{2}mg$, $\frac{1}{2}mg$, $-\frac{1}{2}mg$, $-\frac{3}{2}mg$. The upper two springs are stretched, the lower two springs are compressed.

Notice how $\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{y}$ are computed in that order. We assembled $K = A^{\mathrm{T}}CA$ from rectangular matrices. To find $\boldsymbol{u} = K^{-1}\boldsymbol{f}$, we work with the whole matrix and not its three pieces! The rectangular matrices A and A^{T} do not have (two-sided) inverses.

$$\boxed{\textbf{Warning: } \textit{Normally you cannot write} \quad K^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}.}$$

The three matrices are mixed together by $A^{\mathrm{T}}CA$, and they cannot easily be untangled. In general, $A^{\mathrm{T}}y = f$ has many solutions. And four equations $Au = e$ would usually have no solution with three unknowns. But $A^{\mathrm{T}}CA$ gives the correct solution to all three equations in the framework. Only when $m = n$ and the matrices are square can we go from $y = (A^{\mathrm{T}})^{-1}f$ to $e = C^{-1}y$ to $u = A^{-1}e$. We will see that now.

Fixed End and Free End

Remove the fourth spring. All matrices become 3 by 3. The pattern does not change! The matrix A loses its fourth row and (of course) A^{T} loses its fourth column. The new stiffness matrix K_1 becomes a product of square matrices:

$$A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The missing column of A^{T} and row of A multiplied the missing c_4. So the quickest way to find the new $A^{\mathrm{T}}CA$ is to set $c_4 = 0$ in the old one:

FIXED
FREE

$$K_1 = A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}. \qquad (6)$$

If $c_1 = c_2 = c_3 = 1$ and $C = I$, this is the $-1, 2, -1$ tridiagonal matrix, except the last entry is 1 instead of 2. The spring at the bottom is free.

Example 2 All $c_i = c$ and all $m_j = m$ in the fixed-free hanging line of springs. Then

$$K_1 = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad K_1^{-1} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

The forces mg from gravity are the same. But the movements change from the previous example because the stiffness matrix has changed:

$$u = K_1^{-1}f = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

Those movements are greater in this fixed-free case. The number 3 appears in u_1 because all three masses are pulling the first spring down. The next mass moves by that 3 plus an additional 2 from the masses below it. The third mass drops even more $(3 + 2 + 1 = 6)$. The elongations $e = Au$ in the springs display those numbers $3, 2, 1$:

$$e = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{mg}{c} \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Multiplying by c, the forces y in the three springs are $3mg$ and $2mg$ and mg. And the special point of square matrices is that y can be found directly from f! The balance equation $A^{\mathrm{T}}y = f$ determines y immediately, because $m = n$ and A^{T} is square. We are allowed to write $(A^{\mathrm{T}}CA)^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$:

$$y = (A^{\mathrm{T}})^{-1}f \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}.$$

Two Free Ends: K is Singular

The first line of springs in Figure 8.2 is free at *both ends*. This means trouble (the whole line can move). The matrix A is 2 by 3, short and wide. Here is $\boldsymbol{e} = A\boldsymbol{u}$:

$$\textbf{FREE-FREE} \qquad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{7}$$

Now there is a nonzero solution to $A\boldsymbol{u} = \boldsymbol{0}$. ***The masses can move with no stretching of the springs.*** The whole line can shift by $\boldsymbol{u} = (1,1,1)$ and this leaves $\boldsymbol{e} = (0,0)$. A has *dependent columns* and the vector $(1,1,1)$ is in its nullspace:

$$A\boldsymbol{u} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \textbf{no stretching}. \tag{8}$$

$A\boldsymbol{u} = \boldsymbol{0}$ certainly leads to $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{0}$. So $A^{\mathrm{T}}CA$ is only *positive semidefinite*, without c_1 and c_4. The pivots will be c_2 and c_3 and *no third pivot*. The rank is only 2:

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{9}$$

Two eigenvalues will be positive but $\boldsymbol{x} = (1,1,1)$ is an eigenvector for $\lambda = 0$. We can solve $A^{\mathrm{T}}CA\boldsymbol{u} = f$ only for special vectors f. The forces have to add to $f_1 + f_2 + f_3 = 0$, or the whole line of springs (with both ends free) will take off like a rocket.

Circle of Springs

A third spring will complete the circle from mass 3 back to mass 1. This doesn't make K invertible—the new matrix is still singular. That stiffness matrix $K_{circular}$ is not tridiagonal, but it is symmetric (always) and *semidefinite*:

$$A^{\mathrm{T}}_{\textbf{circular}}A_{\textbf{circular}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{10}$$

The only pivots are 2 and $\frac{3}{2}$. The eigenvalues are 3 and 3 and 0. The determinant is zero. The nullspace still contains $\boldsymbol{x} = (1,1,1)$, when all the masses move together.

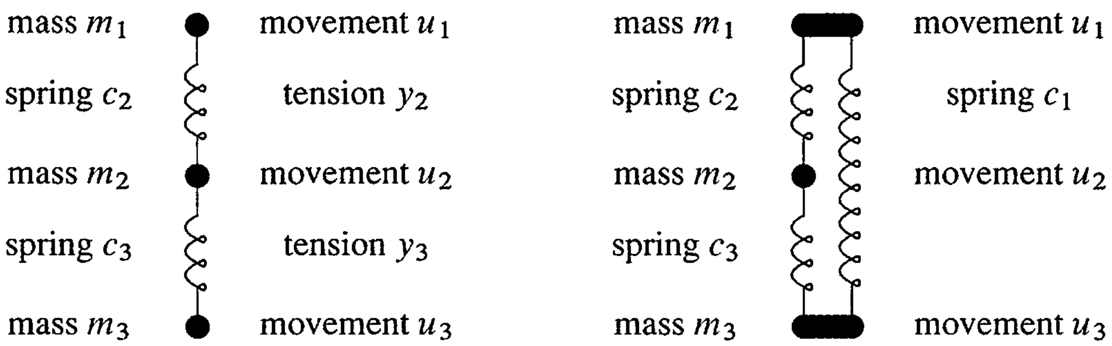


Figure 8.2: **Free-free ends**: A line of springs and a "circle" of springs: *Singular* K's. The masses can move without stretching the springs so $Au = \mathbf{0}$ has nonzero solutions.

This movement vector $(1, 1, 1)$ is in the nullspace of $A_{\textbf{circular}}$ and $K_{\textbf{circular}}$, even after the diagonal matrix C of spring constants is included: the springs are not stretched.

$$(A^{\mathrm{T}}CA)_{\textbf{circular}} = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_1 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_1 & -c_3 & c_3 + c_1 \end{bmatrix}. \tag{11}$$

Continuous Instead of Discrete

Matrix equations are discrete. Differential equations are continuous. We will see the differential equation that corresponds to the tridiagonal $-1, 2, -1$ matrix $A^{\mathrm{T}}A$. And it is a pleasure to see the boundary conditions that go with K_0 and K_1.

The matrices A and $\boldsymbol{A^{\mathrm{T}}}$ ***correspond to the derivatives*** $\boldsymbol{d/dx}$ ***and*** $\boldsymbol{-d/dx}$! Remember that $\boldsymbol{e} = A\boldsymbol{u}$ took differences $u_i - u_{i-1}$, and $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ took differences $y_i - y_{i+1}$. Now the springs are infinitesimally short, and those differences become derivatives:

$$\frac{u_i - u_{i-1}}{\Delta x} \text{ is like } \frac{du}{dx} \qquad \frac{y_i - y_{i+1}}{\Delta x} \text{ is like } -\frac{dy}{dx}$$

The factor Δx didn't appear earlier—we imagined the distance between masses was 1. To approximate a continuous solid bar, we take many more masses (smaller and closer). Let me jump to the three steps A, C, A^{T} in the continuous model, when there is stretching and Hooke's Law and force balance at every point x:

$$e(x) = Au = \frac{du}{dx} \qquad y(x) = c(x)e(x) \qquad A^{\mathrm{T}}y = -\frac{dy}{dx} = f(x)$$

Combining those equations into $A^{\mathrm{T}}CAu(x) = f(x)$, we have a differential equation not a matrix equation. The line of springs becomes an elastic bar:

$$\textbf{Solid Elastic Bar} \quad A^{\mathrm{T}}CAu(x) = f(x) \quad \text{is} \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \tag{12}$$

$A^{\mathrm{T}}A$ corresponds to a second derivative. A is a "difference matrix" and $A^{\mathrm{T}}A$ is a "second difference matrix". ***The matrix has*** $-1, 2, -1$ ***and the equation has*** $-d^2u/dx^2$:

$$-u_{i+1} + 2u_i - u_{i-1} \text{ is a } \textbf{second difference} \qquad -\frac{d^2u}{dx^2} \text{ is a } \textbf{second derivative}.$$

Now we see why this symmetric matrix is a favorite. When we meet a first derivative du/dx, we have three choices (*forward, backward, and centered differences*):

$$\frac{du}{dx} \simeq \frac{u(x+\Delta x) - u(x)}{\Delta x} \text{ or } \frac{u(x) - u(x-\Delta x)}{\Delta x} \text{ or } \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}.$$

When we meet d^2u/dx^2, the natural choice is $u(x+\Delta x) - 2u(x) + u(x-\Delta x)$, divided by $(\Delta x)^2$. *Why reverse these signs to* $-1, 2, -1$? Because the positive definite matrix has $+2$ on the diagonal. First derivatives are *anti*symmetric; the transpose has a minus sign. So second differences are negative definite, and we change to $-d^2u/dx^2$.

We have moved from vectors to functions. Scientific computing moves the other way. It starts with a differential equation like (12). Sometimes there is a formula for the solution $u(x)$, more often not. In reality we *create* the discrete matrix K by approximating the continuous problem. Watch how the boundary conditions on u come in! By missing u_0 we treat it (correctly) as zero:

$$\begin{matrix}\textbf{FIXED}\\ \textbf{FIXED}\end{matrix} \qquad Au = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \approx \frac{du}{dx} \quad \text{with} \quad \begin{matrix} \boldsymbol{u_0 = 0} \\ \boldsymbol{u_4 = 0} \end{matrix} \qquad (13)$$

Fixing the top end gives the boundary condition $u_0 = 0$. What about the free end, when the bar hangs in the air? Row 4 of A is gone and so is u_4. The boundary condition must come from A^{T}. It is the missing y_4 that we are treating (correctly) as zero:

$$\begin{matrix}\textbf{FIXED}\\ \textbf{FREE}\end{matrix} \qquad A^{\mathrm{T}}y = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx -\frac{dy}{dx} \quad \text{with} \quad \begin{matrix} \boldsymbol{u_0 = 0} \\ \boldsymbol{y_4 = 0} \end{matrix} \qquad (14)$$

The boundary condition $y_4 = 0$ *at the free end becomes* $du/dx = 0$, since $y = Au$ corresponds to du/dx. The force balance $A^{\mathrm{T}}y = f$ at that end (in the air) is $0 = 0$. The last row of $K_1u = f$ has entries $-1, 1$ to reflect this condition $du/dx = 0$.

May I summarize this section? I hope this example will help you turn calculus into linear algebra, replacing differential equations by difference equations. If your step Δx is small enough, you will have a totally satisfactory solution.

$$\textbf{The equation is } -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \textbf{ with } u(0) = 0 \textbf{ and } \left[u(1) \text{ or } \frac{du}{dx}(1)\right] = 0$$

Divide the bar into N pieces of length Δx. Replace du/dx by Au and $-dy/dx$ by $A^{\mathrm{T}}y$. Now A and A^{T} include $1/\Delta x$. The end conditions are $u_0 = 0$ and $[u_N = 0$ or $y_N = 0]$.

The three steps $-d/dx$ and $c(x)$ and d/dx correspond to A^{T} and C and A:

$$f = A^{\mathrm{T}}y \text{ and } y = Ce \text{ and } e = Au \text{ give } A^{\mathrm{T}}CAu = f.$$

This is a fundamental example in computational science and engineering. Our book concentrates on Step 3 in that process (linear algebra). Now we have taken Step 2.

1. Model the problem by a differential equation
2. Discretize the differential equation to a difference equation
3. Understand and solve the difference equation (and boundary conditions!)
4. Interpret the solution; visualize it; redesign if needed.

Numerical simulation has become a third branch of science, together with experiment and deduction. Designing the Boeing 777 was much less expensive on a computer than in a wind tunnel. Our discussion still has to move from ordinary to partial differential equations, and from linear to nonlinear.

The texts *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) develop this whole subject further—see the course page **math.mit.edu/18085** with video lectures (also on **ocw.mit.edu**). The principles remain the same, and I hope this book helps you to see the framework behind the computations.

Problem Set 8.1

1 Show that $\det A_0^{\mathrm{T}}C_0A_0 = c_1c_2c_3 + c_1c_3c_4 + c_1c_2c_4 + c_2c_3c_4$. Find also $\det A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example.

2 Invert $A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example by multiplying $A_1^{-1}C_1^{-1}(A_1^{\mathrm{T}})^{-1}$.

3 In the free-free case when $A^{\mathrm{T}}CA$ in equation (9) is singular, add the three equations $A^{\mathrm{T}}CAu = f$ to show that we need $f_1 + f_2 + f_3 = 0$. Find a solution to $A^{\mathrm{T}}CAu = f$ when the forces $f = (-1, 0, 1)$ balance themselves. Find all solutions!

4 Both end conditions for the free-free differential equation are $du/dx = 0$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \quad \text{with} \quad \frac{du}{dx} = 0 \text{ at both ends.}$$

Integrate both sides to show that the force $f(x)$ must balance itself, $\int f(x)\,dx = 0$, or there is no solution. The complete solution is one particular solution $u(x)$ plus any constant. The constant corresponds to $u = (1, 1, 1)$ in the nullspace of $A^{\mathrm{T}}CA$.

5 In the fixed-free problem, the matrix A is square and invertible. We can solve $A^{\mathrm{T}}y = f$ separately from $Au = e$. Do the same for the differential equation:

$$\text{Solve } -\frac{dy}{dx} = f(x) \text{ with } y(1) = 0. \quad \text{Graph } y(x) \text{ if } f(x) = 1.$$

6 The 3 by 3 matrix $K_1 = A_1^{\mathrm{T}} C_1 A_1$ in equation (6) splits into three "element matrices" $c_1 E_1 + c_2 E_2 + c_3 E_3$. Write down those pieces, one for each c. Show how they come from *column times row* multiplication of $A_1^{\mathrm{T}} C_1 A_1$. This is how finite element stiffness matrices are actually assembled.

7 For five springs and four masses with both ends fixed, what are the matrices A and C and K? With $C = I$ solve $K\boldsymbol{u} = \text{ones}(4)$.

8 Compare the solution $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ in Problem 7 to the solution of the continuous problem $-u'' = 1$ with $u(0) = 0$ and $u(1) = 0$. The parabola $u(x)$ should correspond at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ to $\boldsymbol{u}$—is there a $(\Delta x)^2$ factor to account for?

9 Solve the fixed-free problem $-u'' = mg$ with $u(0) = 0$ and $u'(1) = 0$. Compare $u(x)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ with the vector $\boldsymbol{u} = (3mg, 5mg, 6mg)$ in Example 2.

10 Suppose $c_1 = c_2 = c_3 = c_4 = 1$, $m_1 = 2$ and $m_2 = m_3 = 1$. Solve $A^{\mathrm{T}} C A\,\boldsymbol{u} = (2, 1, 1)$ for this fixed-fixed line of springs. Which mass moves the most (largest u)?

11 (MATLAB) Find the displacements $u(1), \ldots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of $\boldsymbol{u}$ with **fixed-fixed** and **fixed-free** ends. Note that diag(ones(n, 1), d) is a matrix with n ones along diagonal d. This print command will graph a vector u:

```
plot(u, '+');    xlabel('mass number');    ylabel('movement');    print
```

12 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\Delta x = \frac{1}{8}$. Graph your three numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\,\frac{du}{dx} = 1 \quad \text{with} \quad u(0) = u(1) = 0.$$

This ***convection-diffusion equation*** appears everywhere. It transforms to the Black-Scholes equation for option prices in mathematical finance.

Problem 12 is developed into the first MATLAB homework in my 18.085 course on Computational Science and Engineering at MIT. Videos on *ocw.mit.edu*.

8.2 Graphs and Networks

Over the years I have seen one model so often, and I found it so basic and useful, that I always put it first. The model consists of ***nodes connected by edges***. This is called a ***graph***.

Graphs of the usual kind display functions $f(x)$. Graphs of this node-edge kind lead to matrices. This section is about the ***incidence matrix*** of a graph—which tells how the n nodes are connected by the m edges. Normally $m > n$, there are more edges than nodes.

For any m by n matrix there are two fundamental subspaces in $\mathbf{R}^n$ and two in $\mathbf{R}^m$. They are the row spaces and nullspaces of A and A^{T}. Their *dimensions* are related by the most important theorem in linear algebra. The second part of that theorem is the *orthogonality* of the subspaces. Our goal is to show how examples from graphs illuminate the Fundamental Theorem of Linear Algebra.

We review the four subspaces (for any matrix). Then we construct a ***directed graph*** and its ***incidence matrix***. The dimensions will be easy to discover. But we want the subspaces themselves—this is where orthogonality helps. It is essential to connect the subspaces to the graph they come from. By specializing to incidence matrices, the laws of linear algebra become Kirchhoff's laws. Please don't be put off by the words "current" and "voltage" and "Kirchhoff." These rectangular matrices are the best.

Every entry of an incidence matrix is 0 or 1 or -1. This continues to hold during elimination. All pivots and multipliers are ± 1. Therefore both factors in $A = LU$ also contain $0, 1, -1$. So do the nullspace matrices! All four subspaces have basis vectors with these exceptionally simple components. The matrices are not concocted for a textbook, they come from a model that is absolutely essential in pure and applied mathematics.

Here is a first incidence matrix. Notice -1 and 1 in each row. This matrix takes ***differences in voltage***, across six edges of a graph. The voltages are x_1, x_2, x_3, x_4 at the four nodes in Figure 8.4—where we will construct this matrix A. Its echelon form is U:

$$\begin{array}{l}\textbf{Incidence}\\ \textbf{matrix}\\ \textbf{6 edges}\\ \textbf{4 nodes}\end{array} \quad A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{reduces to} \quad U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nullspace of A and U is the line through $\boldsymbol{x} = (1, 1, 1, 1)$. The column spaces of A and U have dimension $r = 3$. The pivot rows are a basis for the row space.

Figure 8.3 shows more—the subspaces are orthogonal. ***Every vector in the nullspace is perpendicular to every vector in the row space.*** This comes directly from the m equations $A\boldsymbol{x} = \mathbf{0}$. For A and U above, $\boldsymbol{x} = (1, 1, 1, 1)$ is perpendicular to all rows and thus to the whole row space. Equal voltages produce no current!

I would like to review the Four Fundamental Subspaces before using them. The whole point will be to see their meaning on the network.

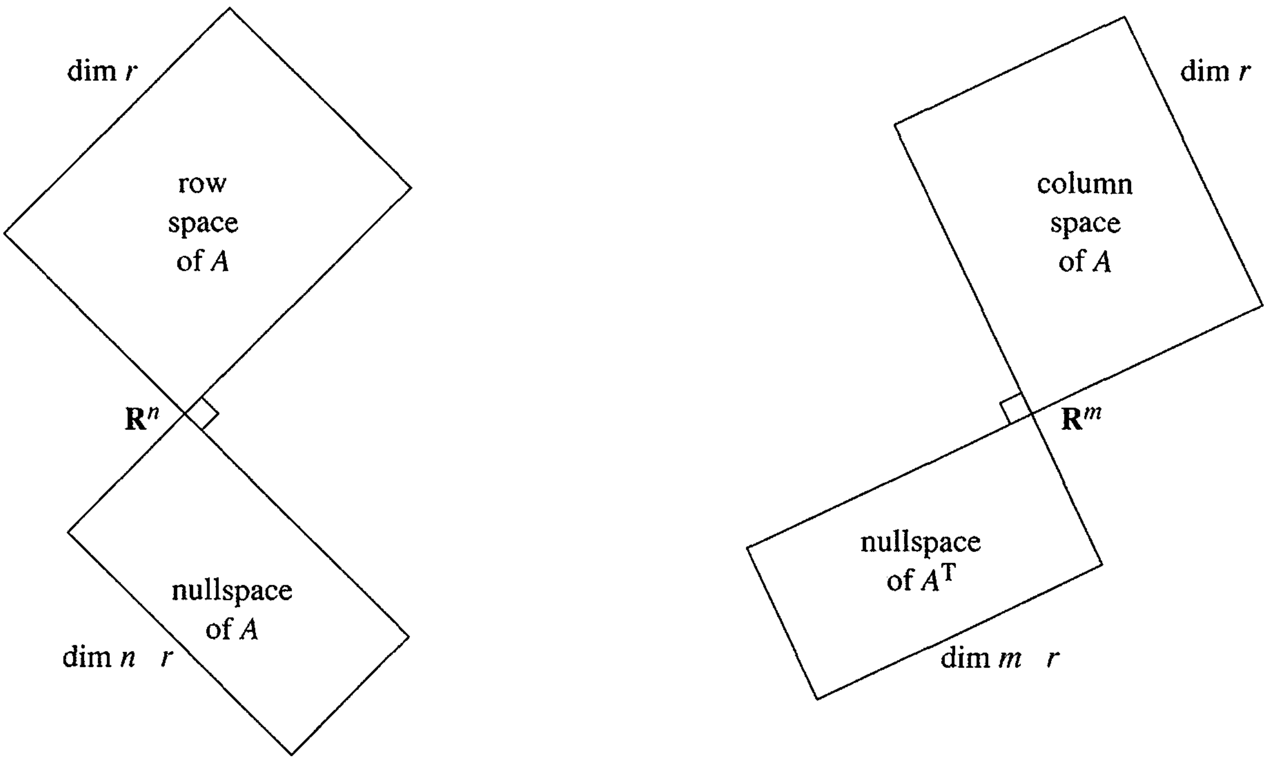


Figure 8.3: **Big picture:** The four subspaces with their dimensions and orthogonality.

Start with an m by n matrix. Its columns are vectors in $\mathbf{R}^m$. Their linear combinations produce the ***column space*** $C(A)$, a subspace of $\mathbf{R}^m$. Those combinations are exactly the matrix-vector products Ax.

The rows of A are vectors in $\mathbf{R}^n$ (or they would be, if they were column vectors). Their linear combinations produce the ***row space***. To avoid any inconvenience with rows, we transpose the matrix. The row space becomes $C(A^{\mathrm{T}})$, the column space of A^{T}.

The central questions of linear algebra come from these two ways of looking at the same numbers, by columns and by rows.

The ***nullspace*** $N(A)$ contains every x that satisfies $Ax = \mathbf{0}$—this is a subspace of $\mathbf{R}^n$. The ***"left" nullspace*** contains all solutions to $A^{\mathrm{T}}y = \mathbf{0}$. Now y has m components, and $N(A^{\mathrm{T}})$ is a subspace of $\mathbf{R}^m$. Written as $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$, we are combining rows of A to produce the zero row. The four subspaces are illustrated by Figure 8.3, which shows $\mathbf{R}^n$ on one side and $\mathbf{R}^m$ on the other. The link between them is A.

The information in that figure is crucial. First come the dimensions, which obey the two central laws of linear algebra:

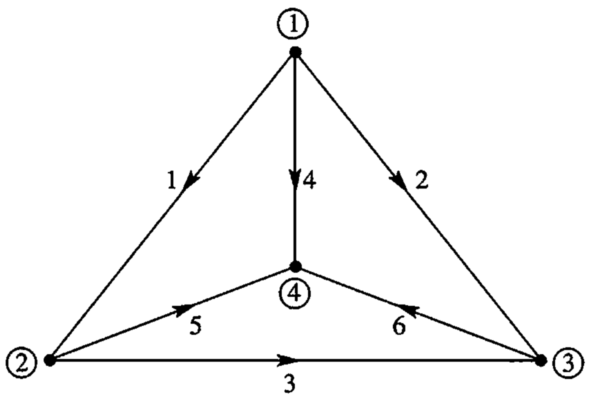
$$\dim C(A) = \dim C(A^{\mathrm{T}}) \quad \text{and} \quad \dim C(A) + \dim N(A) = n.$$

When the row space has dimension r, the nullspace has dimension $n - r$. Elimination leaves these two spaces unchanged, and the echelon form U gives the dimension count. There are r rows and columns with pivots. There are $n - r$ free columns without pivots, and those lead to vectors in the nullspace.

This review of the subspaces applies to any matrix A—only the example was special. Now we concentrate on that example. It is the incidence matrix for a particular graph, and we look to the graph for the meaning of every subspace.

Directed Graphs and Incidence Matrices

Figure 8.4 displays a *graph* with $m = 6$ edges and $n = 4$ nodes, so the matrix A is 6 by 4. It tells which nodes are connected by which edges. The entries -1 and $+1$ also tell the direction of each arrow (this is a *directed* graph). The first row $-1, 1, 0, 0$ of A gives a record of the first edge from node 1 to node 2:



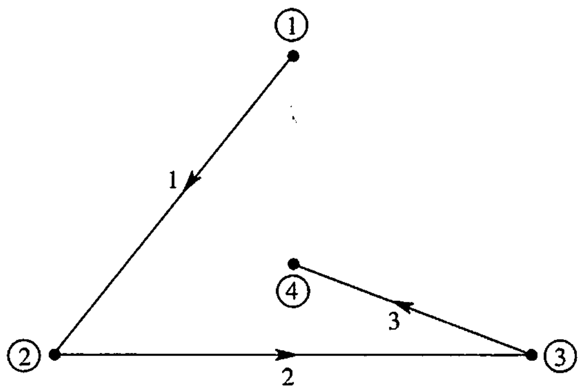
$$A = \begin{array}{c} \textbf{node} \\ \begin{matrix} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{matrix} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \quad \textbf{edge}$$

Figure 8.4a: Complete graph with $m = 6$ edges and $n = 4$ nodes.

Row numbers are edge numbers, column numbers are node numbers.

You can write down A immediately by looking at the graph.

The second graph has the same four nodes but only three edges. Its incidence matrix is 3 by 4:



$$B = \begin{array}{c} \textbf{node} \\ \begin{matrix} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \textbf{edge}$$

Figure 8.4b: Tree with 3 edges and 4 nodes and no loops.

The first graph is *complete*—every pair of nodes is connected by an edge. The second graph is a ***tree***—the graph has ***no closed loops***. Those graphs are the two extremes, the maximum number of edges is $\frac{1}{2}n(n-1)$ and the minimum (a tree) is $m = n - 1$.

The rows of B match the nonzero rows of U—the echelon form found earlier. ***Elimination reduces every graph to a tree.*** The loops produce zero rows in U. Look at the loop from edges 1, 2, 3 in the first graph, which leads to a zero row:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Those steps are typical. When two edges share a node, elimination produces the "shortcut edge" without that node. If the graph already has this shortcut edge, elimination gives a row of zeros. When the dust clears we have a tree.

An idea suggests itself: ***Rows are dependent when edges form a loop.*** Independent rows come from trees. This is the key to the row space. We are assuming that the graph is connected, and it makes no fundamental difference which way the arrows go. On each edge, flow with the arrow is "positive." Flow in the opposite direction counts as negative. The flow might be a current or a signal or a force—or even oil or gas or water.

For the column space we look at Ax, which is a vector of differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}. \tag{1}$$

The unknowns x_1, x_2, x_3, x_4 represent ***potentials*** or ***voltages*** at the nodes. Then Ax gives the ***potential differences*** or ***voltage differences*** across the edges. It is these differences that cause flows. We now examine the meaning of each subspace.

1 The ***nullspace*** contains the solutions to $Ax = \mathbf{0}$. All six potential differences are zero. This means: *All four potentials are equal.* Every x in the nullspace is a constant vector (c, c, c, c). The nullspace of A is a line in $\mathbf{R}^n$—its dimension is $n - r = 1$.

The second incidence matrix B has the same nullspace. It contains $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$:

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can raise or lower all potentials by the same amount c, without changing the differences. There is an "arbitrary constant" in the potentials. Compare this with the same statement for functions. We can raise or lower $f(x)$ by the same amount C, without changing its derivative. There is an arbitrary constant C in the integral.

Calculus adds "$+C$" to indefinite integrals. Graph theory adds (c, c, c, c) to the vector x of potentials. Linear algebra adds any vector x_n in the nullspace to one particular solution of $Ax = b$.

The "$+C$" disappears in calculus when the integral starts at a known point $x = a$. Similarly the nullspace disappears when we set $x_4 = 0$. The unknown x_4 is removed and so are the fourth columns of A and B. Electrical engineers would say that node 4 has been "grounded."

2 The ***row space*** contains all combinations of the six rows. Its dimension is certainly not six. The equation $r + (n - r) = n$ must be $3 + 1 = 4$. The rank is $r = 3$, as we also saw from elimination. After 3 edges, we start forming loops! The new rows are not independent.

How can we tell if $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$ is in the row space? The slow way is to combine rows. The quick way is by orthogonality:

$\boldsymbol{v}$ is in the row space if and only if it is perpendicular to $(\mathbf{1, 1, 1, 1})$ in the nullspace.

The vector $\boldsymbol{v} = (0, 1, 2, 3)$ fails this test—its components add to 6. The vector $(-6, 1, 2, 3)$ passes the test. It lies in the row space because its components add to zero. It equals $6(\text{row }1) + 5(\text{row }3) + 3(\text{row }6)$.

Each row of A adds to zero. This must be true for every vector in the row space.

3 The ***column space*** contains all combinations of the four columns. We expect three independent columns, since there were three independent rows. The first three columns are independent (so are any three). But the four columns add to the zero vector, which says again that $(1, 1, 1, 1)$ is in the nullspace. ***How can we tell if a particular vector $\boldsymbol{b}$ is in the column space of an incidence matrix?***

First answer Try to solve $A\boldsymbol{x} = \boldsymbol{b}$. That misses all the insight. As before, orthogonality gives a better answer. We are now coming to Kirchhoff's two famous laws of circuit theory—the voltage law and current law. Those are natural expressions of "laws" of linear algebra. It is especially pleasant to see the key role of the left nullspace.

Second answer $A\boldsymbol{x}$ is the vector of differences in equation (1). If we add differences around a closed loop in the graph, the cancellation leaves zero. Around the big triangle formed by edges $1, 3, -2$ (the arrow goes backward on edge 2) the differences cancel:

$$\textbf{Voltage Law} \qquad (x_2 - x_1) + (x_3 - x_2) - (x_3 - x_1) = 0.$$

The components of $A\boldsymbol{x}$ add to zero around every loop. When $\boldsymbol{b}$ is in the column space of A, it must obey the same law:

$$\textbf{\textit{Kirchhoff's Law:}} \qquad b_1 + b_3 - b_2 = 0.$$

By testing each loop, we decide whether $\boldsymbol{b}$ is in the column space. $A\boldsymbol{x} = \boldsymbol{b}$ can be solved exactly when the components of $\boldsymbol{b}$ satisfy all the same dependencies as the rows of A. Then elimination leads to $0 = 0$, and $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

4 The ***left nullspace*** contains the solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Its dimension is $m - r = 6 - 3$:

Current Law (KCL)
$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (2)$$

The true number of equations is $r = 3$ and not $n = 4$. Reason: The four equations add to $0 = 0$. The fourth equation follows automatically from the first three.

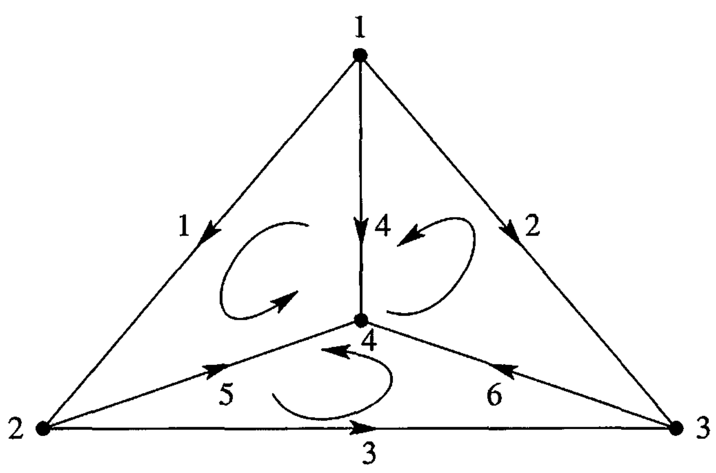
What do the equations mean? The first equation says that $-y_1 - y_2 - y_4 = 0$. ***The net flow into node* 1 *is zero***. The fourth equation says that $y_4 + y_5 + y_6 = 0$. ***Flow into the node minus flow out is zero***. The equations $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are famous and fundamental:

Kirchhoff's Current Law: ***Flow in equals flow out at each node.***

This law deserves first place among the equations of applied mathematics. It expresses "*conservation*" and "*continuity*" and "*balance*." Nothing is lost, nothing is gained. When currents or forces are in equilibrium, the equation to solve is $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Notice the beautiful fact that the matrix in this balance equation is the transpose of the incidence matrix A.

What are the actual solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? The currents must balance themselves. The easiest way is to **flow around a loop**. If a unit of current goes around the big triangle (forward on edge 1, forward on 3, backward on 2), the vector is $\boldsymbol{y} = (1, -1, 1, 0, 0, 0)$. This satisfies $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. *Every loop current is a solution to the Current Law.* Around the loop, flow in equals flow out at every node. A smaller loop goes forward on edge 1, forward on 5, back on 4. Then $\boldsymbol{y} = (1, 0, 0, -1, 1, 0)$ is also in the left nullspace.

We expect three independent $\boldsymbol{y}$'s, since $6 - 3 = 3$. The three small loops in the graph are independent. The big triangle seems to give a fourth $\boldsymbol{y}$, but it is the sum of flows around the small loops. The small loops give a basis for the left nullspace.



$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{small loops}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{big loop}}$$

Summary The incidence matrix A comes from a connected graph with n nodes and m edges. The row space and column space have dimensions $n-1$. The nullspaces of A and A^{T} have dimension 1 and $m-n+1$:

1 The constant vectors $(c, c, \ldots, c)$ make up the nullspace of A.

2 There are $r = n-1$ independent rows, using edges from any tree.

3 ***Voltage law***: The components of Ax add to zero around every loop.

4 ***Current law***: $A^{\mathrm{T}}y = \mathbf{0}$ is solved by loop currents. $N(A^{\mathrm{T}})$ has dimension $m-r$. ***There are $m - r = m - n + 1$ independent loops in the graph.***

For every graph in a plane, linear algebra yields ***Euler's formula***:

$$(\textit{number of nodes}) - (\textit{number of edges}) + (\textit{number of small loops}) = \mathbf{1}.$$

This is $\boldsymbol{n - m + (m - n + 1) = 1}$. The graph in our example has $4 - 6 + 3 = 1$.

A single triangle has (3 nodes) $-$ (3 edges) $+$ (1 loop). On a 10-node tree with 9 edges and no loops, Euler's count is $10 - 9 + 0$. All planar graphs lead to the answer 1.

Networks and $A^{\mathrm{T}}CA$

In a real network, the current y along an edge is the product of two numbers. One number is the difference between the potentials x at the ends of the edge. This difference is Ax and it drives the flow. The other number is the "***conductance***" c—which measures how easily flow gets through.

In physics and engineering, c is decided by the material. For electrical currents, c is high for metal and low for plastics. For a superconductor, c is nearly infinite. If we consider elastic stretching, c might be low for metal and higher for plastics. In economics, c measures the capacity of an edge or its cost.

To summarize, the graph is known from its "connectivity matrix" A. This tells the connections between nodes and edges. A ***network*** goes further, and assigns a conductance c to each edge. These numbers $c_1, \ldots, c_m$ go into the "conductance matrix" C—which is diagonal.

For a network of resistors, the conductance is $c = 1/(\text{resistance})$. In addition to Kirchhoff's Laws for the whole system of currents, we have Ohm's Law for each particular current. Ohm's Law connects the current y_1 on edge 1 to the potential difference $x_2 - x_1$ between the nodes:

Ohm's Law: Current along edge = conductance times potential difference.

Ohm's Law for all m currents is $y = -CAx$. The vector Ax gives the potential differences, and C multiplies by the conductances. Combining Ohm's Law with Kirchhoff's Current

Law $A^{\mathrm{T}}y = \mathbf{0}$, we get $A^{\mathrm{T}}CAx = \mathbf{0}$. This is *almost* the central equation for network flows. The only thing wrong is the zero on the right side! The network needs power from outside—a voltage source or a current source—to make something happen.

Note about signs In circuit theory we change from Ax to $-Ax$. The flow is from higher potential to lower potential. There is (positive) current from node 1 to node 2 when $x_1 - x_2$ is positive—whereas Ax was constructed to yield $x_2 - x_1$. The minus sign in physics and electrical engineering is a plus sign in mechanical engineering and economics. Ax versus $-Ax$ is a general headache but unavoidable.

Note about applied mathematics Every new application has its own form of Ohm's law. For elastic structures $y = CAx$ is Hooke's law. The stress y is (elasticity C) times (stretching Ax). For heat conduction, Ax is a temperature gradient. For oil flows it is a pressure gradient. There is a similar law in Section 8.6 for least squares regression in statistics.

My textbooks *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) are practically built on $A^{\mathrm{T}}CA$. This is the key to equilibrium in matrix equations and also in differential equations. Applied mathematics is more organized than it looks. *I have learned to watch for $A^{\mathrm{T}}CA$.*

We now give an example with a current source. Kirchhoff's Law changes from $A^{\mathrm{T}}y = \mathbf{0}$ to $A^{\mathrm{T}}y = f$, to balance the source f from outside. *Flow into each node still equals flow out.* Figure 8.5 shows the network with its conductances $c_1, \ldots, c_6$, and it shows the current source going into node 1. The source comes out at node 4 to keep the balance (in = out). The problem is: ***Find the currents $y_1, \ldots, y_6$ on the six edges.***

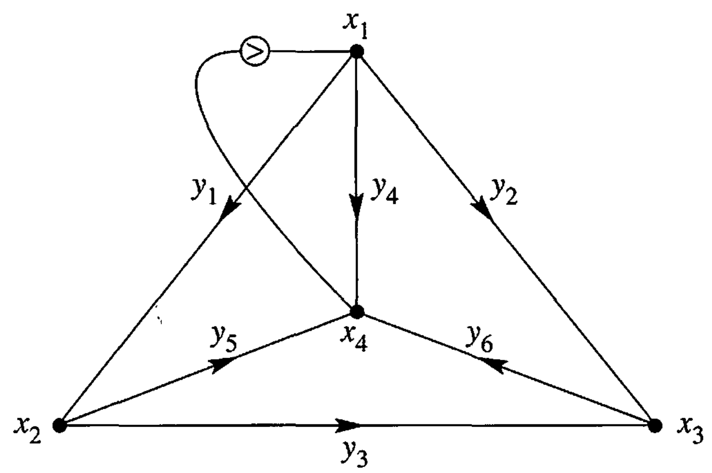


Figure 8.5: The currents in a network with a source S into node 1.

Example 1 All conductances are $c = 1$, so that $C = I$. A current y_4 travels directly from node 1 to node 4. Other current goes the long way from node 1 to node 2 to node 4 (this is $y_1 = y_5$). Current also goes from node 1 to node 3 to node 4 (this is $y_2 = y_6$). We can find the six currents by using special rules for symmetry, or we can do it right by using

$A^{\mathrm{T}}CA$. Since $C = I$, this matrix is $A^{\mathrm{T}}A$, the ***graph Laplacian matrix***:

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

That last matrix is not invertible! We cannot solve for all four potentials because $(1, 1, 1, 1)$ is in the nullspace. One node has to be grounded. Setting $x_4 = 0$ removes the fourth row and column, and this leaves a 3 by 3 invertible matrix. Now we solve $A^{\mathrm{T}}CAx = f$ for the unknown potentials x_1, x_2, x_3, with source S into node 1:

$$\textbf{Voltages} \qquad \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

Ohm's Law $y = -CAx$ yields the six currents. Remember $C = I$ and $x_4 = 0$:

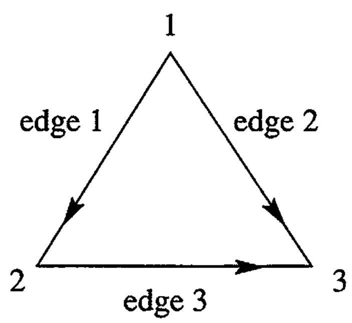
$$\textbf{Currents} \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} S/2 \\ S/4 \\ S/4 \\ 0 \end{bmatrix} = \begin{bmatrix} S/4 \\ S/4 \\ 0 \\ S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

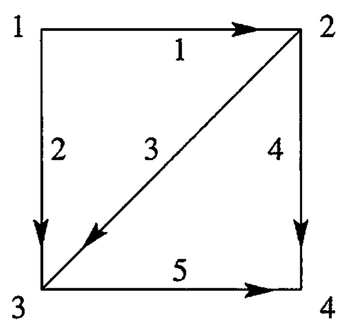
Half the current goes directly on edge 4. That is $y_4 = S/2$. No current crosses from node 2 to node 3. Symmetry indicated $y_3 = 0$ and now the solution proves it.

The same matrix $A^{\mathrm{T}}A$ appears in least squares. Nature distributes the currents to minimize the heat loss. Statistics chooses $\widehat{x}$ to minimize the least squares error.

Problem Set 8.2

Problems 1–7 and 8–14 are about the incidence matrices for these graphs.





1 Write down the 3 by 3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (x_1, x_2, x_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

2 Write down A^{T} for the triangle graph. Find a vector $\boldsymbol{y}$ in its nullspace. The components of $\boldsymbol{y}$ are currents on the edges—how much current is going around the triangle?

3 Eliminate x_1 and x_2 from the third equation to find the echelon matrix U. What tree corresponds to the two nonzero rows of U?

$$\begin{aligned} -x_1 + x_2 &= b_1 \\ -x_1 + x_3 &= b_2 \\ -x_2 + x_3 &= b_3. \end{aligned}$$

4 Choose a vector (b_1, b_2, b_3) for which $A\boldsymbol{x} = \boldsymbol{b}$ can be solved, and another vector $\boldsymbol{b}$ that allows no solution. How are those $\boldsymbol{b}$'s related to $\boldsymbol{y} = (1, -1, 1)$?

5 Choose a vector (f_1, f_2, f_3) for which $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ can be solved, and a vector $\boldsymbol{f}$ that allows no solution. How are those $\boldsymbol{f}$'s related to $\boldsymbol{x} = (1, 1, 1)$? The equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is Kirchhoff's _____ law.

6 Multiply matrices to find $A^{\mathrm{T}}A$. Choose a vector $\boldsymbol{f}$ for which $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{x}$. Put those potentials $\boldsymbol{x}$ and the currents $\boldsymbol{y} = -A\boldsymbol{x}$ and current sources $\boldsymbol{f}$ onto the triangle graph. Conductances are 1 because $C = I$.

7 With conductances $c_1 = 1$ and $c_2 = c_3 = 2$, multiply matrices to find $A^{\mathrm{T}}CA$. For $\boldsymbol{f} = (1, 0, -1)$ find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$. Write the potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the triangle graph, when the current source $\boldsymbol{f}$ goes into node 1 and out from node 3.

8 Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{x} = \mathbf{0}$ and two solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$.

9 Find two requirements on the b's for the five differences $x_2 - x_1, x_3 - x_1, x_3 - x_2, x_4 - x_2, x_4 - x_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's _____ law around the two _____ in the graph.

10 Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

11 Multiply matrices to find $A^{\mathrm{T}}A$ and guess how its entries come from the graph:

(a) The diagonal of $A^{\mathrm{T}}A$ tells how many _____ into each node.

(b) The off-diagonals -1 or 0 tell which pairs of nodes are _____.

12 Why is each statement true about $A^{\mathrm{T}}A$? *Answer for $A^{\mathrm{T}}A$ not A.*

(a) Its nullspace contains $(1, 1, 1, 1)$. Its rank is $n - 1$.

(b) It is positive semidefinite but not positive definite.

(c) Its four eigenvalues are real and their signs are _____ .

13 With conductances $c_1 = c_2 = 2$ and $c_3 = c_4 = c_5 = 3$, multiply the matrices $A^{\mathrm{T}}CA$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f} = (1, 0, 0, -1)$. Write these potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the nodes and edges of the square graph.

14 The matrix $A^{\mathrm{T}}CA$ is not invertible. What vectors $\boldsymbol{x}$ are in its nullspace? Why does $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ have a solution if and only if $f_1 + f_2 + f_3 + f_4 = 0$?

15 A connected graph with 7 nodes and 7 edges has how many loops?

16 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$. If you connect it to two old nodes, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$.

17 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on $\boldsymbol{f}$ makes it possible to solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$?

(c) The diagonal entries of $A^{\mathrm{T}}A$ give the number of edges into each node. What is the sum of those diagonal entries?

18 Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A tree connecting 6 nodes has _____ edges.

Note The ***stoichiometric matrix*** in chemistry is an important "generalized" incidence matrix. Its entries show how much of each chemical species (each column) goes into each reaction (each row).

8.3 Markov Matrices, Population, and Economics

This section is about ***positive matrices***: every $a_{ij} > 0$. The key fact is quick to state: ***The largest eigenvalue is real and positive and so is its eigenvector.*** In economics and ecology and population dynamics and random walks, that fact leads a long way:

Markov $\lambda_{\max} = 1$ **Population** $\lambda_{\max} > 1$ **Consumption** $\lambda_{\max} < 1$

$\lambda_{\max}$ controls the powers of A. We will see this first for $\lambda_{\max} = 1$.

Markov Matrices

Suppose we multiply a positive vector $\boldsymbol{u}_0 = (a, 1-a)$ again and again by this A:

Markov matrix $$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad \boldsymbol{u}_1 = A\boldsymbol{u}_0 \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$$

After k steps we have $A^k\boldsymbol{u}_0$. The vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots$ will approach a "*steady state*" $\boldsymbol{u}_\infty = (.6, .4)$. This final outcome does not depend on the starting vector: ***For every $\boldsymbol{u}_0$ we converge to the same $\boldsymbol{u}_\infty$.*** The question is why.

The steady state equation $A\boldsymbol{u}_\infty = \boldsymbol{u}_\infty$ makes $\boldsymbol{u}_\infty$ ***an eigenvector with eigenvalue*** **1**:

Steady state $$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

Multiplying by A does not change $\boldsymbol{u}_\infty$. But this does not explain why all vectors $\boldsymbol{u}_0$ lead to $\boldsymbol{u}_\infty$. Other examples might have a steady state, but it is not necessarily attractive:

Not Markov $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has the unattractive steady state $B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In this case, the starting vector $\boldsymbol{u}_0 = (0, 1)$ will give $\boldsymbol{u}_1 = (0, 2)$ and $\boldsymbol{u}_2 = (0, 4)$. The second components are doubled. In the language of eigenvalues, B has $\lambda = 1$ but also $\lambda = 2$— this produces instability. The component of $\boldsymbol{u}$ along that unstable eigenvector is multiplied by λ, and $|\lambda| > 1$ means blowup.

This section is about two special properties of A that guarantee a stable steady state. These properties define a ***Markov matrix***, and A above is one particular example:

Markov matrix
1. ***Every entry of A is nonnegative.***
2. ***Every column of A adds to*** **1.**

B did not have Property **2**. When A is a Markov matrix, two facts are immediate:

1. Multiplying a nonnegative $\boldsymbol{u}_0$ by A produces a nonnegative $\boldsymbol{u}_1 = A\boldsymbol{u}_0$.

2. If the components of $\boldsymbol{u}_0$ add to 1, so do the components of $\boldsymbol{u}_1 = A\boldsymbol{u}_0$.

Reason: The components of u_0 add to 1 when $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}u_0 = 1$. This is true for each column of A by Property 2. Then by matrix multiplication $[1 \;\ldots\; 1]A = [1 \;\ldots\; 1]$:

$$\textbf{Components of } A u_0 \textbf{ add to 1} \qquad [1 \;\cdots\; 1]Au_0 = [1 \;\cdots\; 1]u_0 = 1.$$

The same facts apply to $u_2 = Au_1$ and $u_3 = Au_2$. ***Every vector $A^k u_0$ is nonnegative with components adding to*** 1. These are "***probability vectors***." The limit u_∞ is also a probability vector—but we have to prove that there is a limit. We will show that $\lambda_{\max} = 1$ for a positive Markov matrix.

Example 1 The fraction of rental cars in Denver starts at $\frac{1}{50} = .02$. The fraction outside Denver is .98. Every month, 80% of the Denver cars stay in Denver (and 20% leave). Also 5% of the outside cars come in (95% stay outside). This means that the fractions $u_0 = (.02, .98)$ are multiplied by A:

$$\textbf{First month} \qquad A = \begin{bmatrix} .80 & .05 \\ .20 & .95 \end{bmatrix} \quad \text{leads to} \quad u_1 = Au_0 = A\begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .065 \\ .935 \end{bmatrix}.$$

Notice that $.065 + .935 = 1$. All cars are accounted for. Each step multiplies by A:

$$\textbf{Next month} \qquad u_2 = Au_1 = (.09875,\ .90125). \text{ This is } A^2u_0.$$

All these vectors are positive because A is positive. Each vector u_k will have its components adding to 1. The first component has grown from .02 and cars are moving toward Denver. What happens in the long run?

This section involves powers of matrices. The understanding of A^k was our first and best application of diagonalization. Where A^k can be complicated, the diagonal matrix Λ^k is simple. The eigenvector matrix S connects them: A^k equals $S\Lambda^k S^{-1}$. The new application to Markov matrices uses the eigenvalues (in Λ) and the eigenvectors (in S). We will show that u_∞ is an eigenvector corresponding to $\lambda = 1$.

Since every column of A adds to 1, nothing is lost or gained. We are moving rental cars or populations, and no cars or people suddenly appear (or disappear). The fractions add to 1 and the matrix A keeps them that way. The question is how they are distributed after k time periods—which leads us to A^k.

Solution $A^k u_0$ gives the fractions in and out of Denver after k steps. We diagonalize A to understand A^k. The eigenvalues are $\lambda = 1$ and .75 (the trace is 1.75).

$$Ax = \lambda x \qquad A\begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} \quad \textbf{and} \quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = .75\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The starting vector u_0 combines x_1 and x_2, in this case with coefficients 1 and .18:

$$\textbf{Combination of eigenvectors} \qquad u_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now multiply by A to find u_1. The eigenvectors are multiplied by $\lambda_1 = 1$ and $\lambda_2 = .75$:

$$\textbf{Each } x \textbf{ is multiplied by } \lambda \qquad u_1 = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Every month, another .75 multiplies the vector x_2. The eigenvector x_1 is unchanged:

$$\textbf{After } k \textbf{ steps} \qquad u_k = A^k u_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)^k(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This equation reveals what happens. ***The eigenvector x_1 with $\lambda = 1$ is the steady state.*** The other eigenvector x_2 disappears because $|\lambda| < 1$. The more steps we take, the closer we come to $u_\infty = (.2, .8)$. In the limit, $\frac{2}{10}$ of the cars are in Denver and $\frac{8}{10}$ are outside. This is the pattern for Markov chains, even starting from $u_0 = (0, 1)$:

If A is a *positive* Markov matrix (entries $a_{ij} > 0$, each column adds to **1**), then $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvector x_1 is the ***steady state***:

$$u_k = x_1 + c_2(\lambda_2)^k x_2 + \cdots + c_n(\lambda_n)^k x_n \quad \textit{always approaches} \quad u_\infty = x_1.$$

The first point is to see that $\lambda = 1$ is an eigenvalue of A. *Reason*: Every column of $A - I$ adds to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. Those rows are linearly dependent, so $A - I$ is singular. Its determinant is zero and $\lambda = 1$ is an eigenvalue.

The second point is that no eigenvalue can have $|\lambda| > 1$. With such an eigenvalue, the powers A^k would grow. But A^k is also a Markov matrix! A^k has nonnegative entries still adding to 1—and that leaves no room to get large.

A lot of attention is paid to the possibility that another eigenvalue has $|\lambda| = 1$.

Example 2 $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside, and vice versa. The powers A^k alternate between A and I. The second eigenvector $x_2 = (-1, 1)$ will be multiplied by $\lambda_2 = -1$ at every step—and does not become smaller: No steady state.

Suppose the entries of A or any power of A are all *positive*—zero is not allowed. In this "regular" or "primitive" case, $\lambda = 1$ is strictly larger than any other eigenvalue. The powers A^k approach the rank one matrix that has the steady state in every column.

Example 3 (**"Everybody moves"**) Start with three groups. At each time step, half of group 1 goes to group 2 and the other half goes to group 3. The other groups also *split in half and move*. Take one step from the starting populations p_1, p_2, p_3:

$$\textbf{New populations} \qquad u_1 = Au_0 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_2 \end{bmatrix}.$$

A is a Markov matrix. Nobody is born or lost. A contains zeros, which gave trouble in Example 2. But after two steps in this new example, the zeros disappear from A^2:

$$\textbf{Two-step matrix} \qquad u_2 = A^2 u_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ (because A is Markov) and $\lambda_2 = \lambda_3 = -\frac{1}{2}$. For $\lambda = 1$, ***the eigenvector*** $\boldsymbol{x}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ***will be the steady state.*** When three equal populations split in half and move, the populations are again equal. Starting from $\boldsymbol{u}_0 = (8, 16, 32)$, the Markov chain approaches its steady state:

$$\boldsymbol{u}_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \qquad \boldsymbol{u}_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 16 \\ 18 \\ 22 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}.$$

The step to $\boldsymbol{u}_4$ will split some people in half. This cannot be helped. The total population is $8 + 16 + 32 = 56$ at every step. The steady state is 56 times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. You can see the three populations approaching, but never reaching, their final limits $56/3$.

Challenge Problem 6.7.16 created a Markov matrix A from the number of links between websites. The steady state $\boldsymbol{u}$ will give the Google rankings. ***Google finds*** $\boldsymbol{u}_\infty$ ***by a random walk that follows links*** (*random surfing*). That eigenvector comes from counting the fraction of visits to each website—a quick way to compute the steady state.

The size $|\lambda_2|$ of the next largest eigenvalue controls the speed of convergence to steady state.

Perron-Frobenius Theorem

One matrix theorem dominates this subject. The Perron-Frobenius Theorem applies when all $a_{ij} \geq 0$. There is no requirement that columns add to 1. We prove the neatest form, when all $a_{ij} > 0$.

Perron-Frobenius for $A > 0$ ***All numbers in $A\boldsymbol{x} = \lambda_{\max}\boldsymbol{x}$ are strictly positive.***

Proof The key idea is to look at all numbers t such that $A\boldsymbol{x} \geq t\boldsymbol{x}$ for some nonnegative vector $\boldsymbol{x}$ (other than $\boldsymbol{x} = \boldsymbol{0}$). We are allowing inequality in $A\boldsymbol{x} \geq t\boldsymbol{x}$ in order to have many positive candidates t. For the largest value $t_{\max}$ (which is attained), we will show that ***equality holds***: $A\boldsymbol{x} = t_{\max}\boldsymbol{x}$.

Otherwise, if $A\boldsymbol{x} \geq t_{\max}\boldsymbol{x}$ is not an equality, multiply by A. Because A is positive that produces a strict inequality $A^2\boldsymbol{x} > t_{\max}A\boldsymbol{x}$. Therefore the positive vector $\boldsymbol{y} = A\boldsymbol{x}$ satisfies $A\boldsymbol{y} > t_{\max}\boldsymbol{y}$, and $t_{\max}$ could be increased. This contradiction forces the equality $A\boldsymbol{x} = t_{\max}\boldsymbol{x}$, and we have an eigenvalue. Its eigenvector $\boldsymbol{x}$ is positive because on the left side of that equality, $A\boldsymbol{x}$ is sure to be positive.

To see that no eigenvalue can be larger than $t_{\max}$, suppose $A\boldsymbol{z} = \lambda\boldsymbol{z}$. Since λ and $\boldsymbol{z}$ may involve negative or complex numbers, we take absolute values: $|\lambda||\boldsymbol{z}| = |A\boldsymbol{z}| \leq A|\boldsymbol{z}|$ by the "triangle inequality." This $|\boldsymbol{z}|$ is a nonnegative vector, so $|\lambda|$ is one of the possible candidates t. Therefore $|\lambda|$ cannot exceed $t_{\max}$—which must be $\lambda_{\max}$.

Population Growth

Divide the population into three age groups: age < 20, age 20 to 39, and age 40 to 59. At year T the sizes of those groups are n_1, n_2, n_3. Twenty years later, the sizes have changed for two reasons:

1. **Reproduction** $n_1^{\text{new}} = F_1 n_1 + F_2 n_2 + F_3 n_3$ gives a new generation
2. **Survival** $n_2^{\text{new}} = P_1 n_1$ and $n_3^{\text{new}} = P_2 n_2$ gives the older generations

The fertility rates are F_1, F_2, F_3 (F_2 largest). The *Leslie matrix* A might look like this:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{\text{new}} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} .04 & \mathbf{1.1} & .01 \\ .98 & 0 & 0 \\ 0 & \mathbf{.92} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

This is population projection in its simplest form, the same matrix A at every step. In a realistic model, A will change with time (from the environment or internal factors). Professors may want to include a fourth group, age ≥ 60, but we don't allow it.

The matrix has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem still applies because $A^3 > 0$. The largest eigenvalue is $\lambda_{\max} \approx 1.06$. You can watch the generations move, starting from $n_2 = 1$ in the middle generation:

$$\textbf{eig}(A) = \begin{matrix} \mathbf{1.06} \\ -1.01 \\ -0.01 \end{matrix} \qquad A^2 = \begin{bmatrix} 1.08 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{1.08} & .01 \\ 0.90 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0.10 & \mathbf{1.19} & .01 \\ 0.06 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{0.99} & .01 \end{bmatrix}.$$

A fast start would come from $\boldsymbol{u}_0 = (0, 1, 0)$. That middle group will reproduce 1.1 and also survive .92. The newest and oldest generations are in $\boldsymbol{u}_1 = (1.1, 0, .92) =$ column 2 of A. Then $\boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$ is the second column of A^2. The early numbers (transients) depend a lot on $\boldsymbol{u}_0$, but ***the asymptotic growth rate*** $\lambda_{\max}$ ***is the same from every start.*** Its eigenvector $\boldsymbol{x} = (.63, .58, .51)$ shows all three groups growing steadily together.

Caswell's book on *Matrix Population Models* emphasizes sensitivity analysis. The model is never exactly right. If the F's or P's in the matrix change by 10%, does $\lambda_{\max}$ go below 1 (which means extinction)? Problem 19 will show that a matrix change ΔA produces an eigenvalue change $\Delta\lambda = \boldsymbol{y}^{\mathrm{T}}(\Delta A)\boldsymbol{x}$. Here $\boldsymbol{x}$ and $\boldsymbol{y}^{\mathrm{T}}$ are the right and left eigenvectors of A. So $\boldsymbol{x}$ is a column of S and $\boldsymbol{y}^{\mathrm{T}}$ is a row of S^{-1}.

Linear Algebra in Economics: The Consumption Matrix

A long essay about linear algebra in economics would be out of place here. A short note about one matrix seems reasonable. The ***consumption matrix*** tells how much of each input goes into a unit of output. This describes the manufacturing side of the economy.

Consumption matrix We have n industries like chemicals, food, and oil. To produce a unit of chemicals may require .2 units of chemicals, .3 units of food, and .4 units of oil. Those numbers go into row 1 of the consumption matrix A:

$$\begin{bmatrix} \text{chemical output} \\ \text{food output} \\ \text{oil output} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{chemical input} \\ \text{food input} \\ \text{oil input} \end{bmatrix}.$$

Row 2 shows the inputs to produce food—a heavy use of chemicals and food, not so much oil. Row 3 of A shows the inputs consumed to refine a unit of oil. The real consumption matrix for the United States in 1958 contained 83 industries. The models in the 1990's are much larger and more precise. We chose a consumption matrix that has a convenient eigenvector.

Now comes the question: Can this economy meet demands y_1, y_2, y_3 for chemicals, food, and oil? To do that, the inputs p_1, p_2, p_3 will have to be higher—because part of $\boldsymbol{p}$ is consumed in producing $\boldsymbol{y}$. The input is $\boldsymbol{p}$ and the consumption is $A\boldsymbol{p}$, which leaves the output $\boldsymbol{p} - A\boldsymbol{p}$. This net production is what meets the demand $\boldsymbol{y}$:

Problem Find a vector $\boldsymbol{p}$ such that $\quad \boldsymbol{p} - A\boldsymbol{p} = \boldsymbol{y} \quad$ or $\quad \boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}.$

Apparently the linear algebra question is whether $I - A$ is invertible. But there is more to the problem. The demand vector $\boldsymbol{y}$ is nonnegative, and so is A. *The production levels in* $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$ *must also be nonnegative*. The real question is:

When is $(I - A)^{-1}$ a nonnegative matrix?

This is the test on $(I - A)^{-1}$ for a productive economy, which can meet any positive demand. If A is small compared to I, then $A\boldsymbol{p}$ is small compared to $\boldsymbol{p}$. There is plenty of output. If A is too large, then production consumes more than it yields. In this case the external demand $\boldsymbol{y}$ cannot be met.

"Small" or "large" is decided by the largest eigenvalue λ_1 of A (which is positive):

If $\lambda_1 > 1$ then $(I - A)^{-1}$ has negative entries
If $\lambda_1 = 1$ then $(I - A)^{-1}$ fails to exist
If $\lambda_1 < 1$ then $(I - A)^{-1}$ is nonnegative as desired.

The main point is that last one. The reasoning uses a nice formula for $(I - A)^{-1}$, which we give now. The most important infinite series in mathematics is the ***geometric series*** $1 + x + x^2 + \cdots$. This series adds up to $1/(1 - x)$ provided x lies between -1 and 1. When $x = 1$ the series is $1 + 1 + 1 + \cdots = \infty$. When $|x| \geq 1$ the terms x^n don't go to zero and the series has no chance to converge.

The nice formula for $(I - A)^{-1}$ is the ***geometric series of matrices***:

Geometric series $\quad (I - A)^{-1} = I + A + A^2 + A^3 + \cdots.$

If you multiply the series $S = I + A + A^2 + \cdots$ by A, you get the same series except for I. Therefore $S - AS = I$, which is $(I - A)S = I$. The series adds to $S = (I - A)^{-1}$ if it converges. ***And it converges if all eigenvalues of A have $|\lambda| < 1$.***

In our case $A \geq 0$. All terms of the series are nonnegative. Its sum is $(I - A)^{-1} \geq 0$.

Example 4 $A = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix}$ has $\lambda_{\max} = .9$ and $(I - A)^{-1} = \frac{1}{93}\begin{bmatrix} 41 & 25 & 27 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$.

This economy is productive. A is small compared to I, because $\lambda_{\max}$ is .9. To meet the demand $\boldsymbol{y}$, start from $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$. Then $A\boldsymbol{p}$ is consumed in production, leaving $\boldsymbol{p} - A\boldsymbol{p}$. This is $(I - A)\boldsymbol{p} = \boldsymbol{y}$, and the demand is met.

Example 5 $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has $\lambda_{\max} = 2$ and $(I - A)^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

This consumption matrix A is too large. Demands can't be met, because production consumes more than it yields. The series $I + A + A^2 + \ldots$ does not converge to $(I - A)^{-1}$ because $\lambda_{\max} > 1$. The series is growing while $(I - A)^{-1}$ is actually negative.

In the same way $1 + 2 + 4 + \cdots$ is not really $1/(1 - 2) = -1$. But not entirely false!

Problem Set 8.3

Questions 1–12 are about Markov matrices and their eigenvalues and powers.

1 Find the eigenvalues of this Markov matrix (their sum is the trace):

$$A = \begin{bmatrix} .90 & .15 \\ .10 & .85 \end{bmatrix}.$$

What is the steady state eigenvector for the eigenvalue $\lambda_1 = 1$?

2 Diagonalize the Markov matrix in Problem 1 to $A = S\Lambda S^{-1}$ by finding its other eigenvector:

$$A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & \\ & .75 \end{bmatrix}\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

What is the limit of $A^k = S\Lambda^k S^{-1}$ when $\Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & .75^k \end{bmatrix}$ approaches $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

3 What are the eigenvalues and steady state eigenvectors for these Markov matrices?

$$A = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \quad A = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

4 For every 4 by 4 Markov matrix, what eigenvector of A^{T} corresponds to the (known) eigenvalue $\lambda = 1$?

5 Every year 2% of young people become old and 3% of old people become dead. (No births.) Find the steady state for

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} .98 & .00 & 0 \\ .02 & .97 & 0 \\ .00 & .03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_k .$$

6 For a Markov matrix, the sum of the components of $\boldsymbol{x}$ equals the sum of the components of $A\boldsymbol{x}$. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\lambda \neq 1$, prove that the components of this non-steady eigenvector $\boldsymbol{x}$ add to zero.

7 Find the eigenvalues and eigenvectors of A. Explain why A^k approaches A^∞:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Challenge problem: Which Markov matrices produce that steady state $(.6, .4)$?

8 The steady state eigenvector of a permutation matrix is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This is *not* approached when $\boldsymbol{u}_0 = (0, 0, 0, 1)$. What are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$? What are the four eigenvalues of P, which solve $\lambda^4 = 1$?

Permutation matrix = Markov matrix $$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

9 Prove that the square of a Markov matrix is also a Markov matrix.

10 If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a Markov matrix, its eigenvalues are 1 and _____. The steady state eigenvector is $\boldsymbol{x}_1 =$ _____.

11 Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\boldsymbol{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ - & - & - \end{bmatrix}.$$

12 A Markov differential equation is not $d\boldsymbol{u}/dt = A\boldsymbol{u}$ but $d\boldsymbol{u}/dt = (A - I)\boldsymbol{u}$. The diagonal is negative, the rest of $A - I$ is positive. The columns add to zero.

Find the eigenvalues of $B = A - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$. Why does $A - I$ have $\lambda = 0$?

When $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ multiply $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, what is the steady state as $t \to \infty$?

Questions 13–15 are about linear algebra in economics.

13 Each row of the consumption matrix in Example 4 adds to .9. Why does that make $\lambda = .9$ an eigenvalue, and what is the eigenvector?

14 Multiply $I + A + A^2 + A^3 + \cdots$ by $I - A$ to show that the series adds to _____. For $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, find A^2 and A^3 and use the pattern to add up the series.

15 For which of these matrices does $I + A + A^2 + \cdots$ yield a nonnegative matrix $(I - A)^{-1}$? Then the economy can meet any demand:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}.$$

If the demands are $\boldsymbol{y} = (2, 6)$, what are the vectors $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$?

16 (Markov again) This matrix has zero determinant. What are its eigenvalues?

$$A = \begin{bmatrix} .4 & .2 & .3 \\ .2 & .4 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

Find the limits of $A^k\boldsymbol{u}_0$ starting from $\boldsymbol{u}_0 = (1, 0, 0)$ and then $\boldsymbol{u}_0 = (100, 0, 0)$.

17 If A is a Markov matrix, does $I + A + A^2 + \cdots$ add up to $(I - A)^{-1}$?

18 For the Leslie matrix show that $\det(A - \lambda I) = 0$ gives $F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2 = \lambda^3$. The right side λ^3 is larger as $\lambda \longrightarrow \infty$. The left side is larger at $\lambda = 1$ if $F_1 + F_2P_1 + F_3P_1P_2 > 1$. In that case the two sides are equal at an eigenvalue $\lambda_{\max} > 1$: *growth.*

19 **Sensitivity of eigenvalues**: A matrix change ΔA produces eigenvalue changes $\Delta\Lambda$. *The formula for those changes* $\Delta\lambda_1, \ldots, \Delta\lambda_n$ *is* **diag**$(S^{-1}\,\Delta A\,S)$. **Challenge**:

Start from $AS = S\Lambda$. The eigenvectors and eigenvalues change by ΔS and $\Delta\Lambda$:

$$(A{+}\Delta A)(S{+}\Delta S) = (S{+}\Delta S)(\Lambda{+}\Delta\Lambda) \text{ becomes } A(\Delta S){+}(\Delta A)S = S(\Delta\Lambda){+}(\Delta S)\Lambda.$$

Small terms $(\Delta A)(\Delta S)$ and $(\Delta S)(\Delta\Lambda)$ are ignored. *Multiply the last equation by* S^{-1}. From the inner terms, the diagonal part of $S^{-1}(\Delta A)S$ gives $\Delta\Lambda$ as we want. *Why do the outer terms* $S^{-1}\,A\,\Delta S$ *and* $S^{-1}\,\Delta S\,\Lambda$ *cancel on the diagonal?*

$$\text{Explain } S^{-1}A = \Lambda S^{-1} \text{ and then } \quad \textbf{diag}(\Lambda\,S^{-1}\,\Delta S) = \textbf{diag}(S^{-1}\,\Delta S\,\Lambda).$$

20 Suppose $B > A > 0$, meaning that each $b_{ij} > a_{ij} > 0$. How does the Perron-Frobenius discussion show that $\lambda_{\max}(B) > \lambda_{\max}(A)$?

8.4 Linear Programming

Linear programming is linear algebra plus two new ideas: ***inequalities*** and ***minimization***. The starting point is still a matrix equation $Ax = b$. But the only acceptable solutions are *nonnegative*. We require $x \geq 0$ (meaning that no component of x can be negative). The matrix has $n > m$, more unknowns than equations. If there are any solutions $x \geq \mathbf{0}$ to $Ax = b$, there are probably a lot. Linear programming picks the solution $x^* \geq \mathbf{0}$ that minimizes the cost:

The cost is $c_1x_1 + \cdots + c_nx_n$. The winning vector x^* is the nonnegative solution of $Ax = b$ that has smallest cost.

Thus a linear programming problem starts with a matrix A and two vectors b and c:

i) A has $n > m$: for example $A = [1 \ \ 1 \ \ 2]$ (one equation, three unknowns)

ii) b has m components for m equations $Ax = b$: for example $b = [4]$

iii) The ***cost vector c*** has n components: for example $c = [5 \ \ 3 \ \ 8]$.

Then the problem is to minimize $c \cdot x$ subject to the requirements $Ax = b$ and $x \geq 0$:

$$\textit{Minimize}\quad 5x_1 + 3x_2 + 8x_3 \quad \textit{subject to}\quad x_1 + x_2 + 2x_3 = 4 \quad \textit{and}\quad x_1, x_2, x_3 \geq 0.$$

We jumped right into the problem, without explaining where it comes from. Linear programming is actually the most important application of mathematics to management. Development of the fastest algorithm and fastest code is highly competitive. You will see that finding x^* is harder than solving $Ax = b$, because of the extra requirements: $x^* \geq 0$ and minimum cost $c^{\mathrm{T}}x^*$. We will explain the background, and the famous *simplex method*, and *interior point methods*, after solving the example.

Look first at the "constraints": $Ax = b$ and $x \geq 0$. The equation $x_1 + x_2 + 2x_3 = 4$ gives a plane in three dimensions. The nonnegativity $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ chops the plane down to a triangle. The solution x^* must lie in the triangle PQR in Figure 8.6.

Inside that triangle, all components of x are positive. On the edges of PQR, one component is zero. At the corners P and Q and R, two components are zero. ***The optimal solution x^* will be one of those corners!*** We will now show why.

The triangle contains all vectors x that satisfy $Ax = b$ and $x \geq 0$. Those x's are called *feasible points*, and the triangle is the ***feasible set***. These points are the allowed candidates in the minimization of $c \cdot x$, which is the final step:

Find x^* in the triangle PQR to minimize the cost $5x_1 + 3x_2 + 8x_3$.

The vectors that have *zero* cost lie on the plane $5x_1 + 3x_2 + 8x_3 = 0$. That plane does not meet the triangle. We cannot achieve zero cost, while meeting the requirements on x. So increase the cost C until the plane $5x_1 + 3x_2 + 8x_3 = C$ does meet the triangle. As C increases, we have *parallel planes moving toward the triangle*.

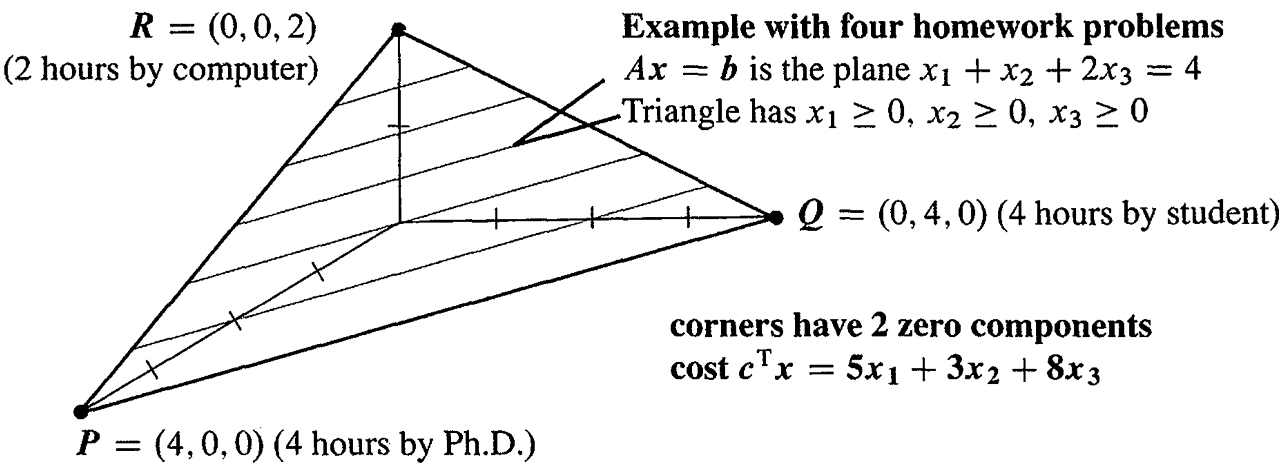


Figure 8.6: The triangle contains all nonnegative solutions: $Ax = b$ and $x \geq 0$. The lowest cost solution x^* is a corner P, Q, or R of this feasible set.

The first plane $5x_1 + 3x_2 + 8x_3 = C$ to touch the triangle has minimum cost C. ***The point where it touches is the solution x^*.*** This touching point must be one of the corners P or Q or R. A moving plane could not reach the inside of the triangle before it touches a corner! So check the cost $5x_1 + 3x_2 + 8x_3$ at each corner:

$$P = (4, 0, 0) \text{ costs } 20 \qquad Q = (0, 4, 0) \text{ costs } 12 \qquad R = (0, 0, 2) \text{ costs } 16.$$

The winner is Q. Then $x^* = (0, 4, 0)$ solves the linear programming problem.

If the cost vector c is changed, the parallel planes are tilted. For small changes, Q is still the winner. For the cost $c \cdot x = 5x_1 + 4x_2 + 7x_3$, the optimum x^* moves to $R = (0, 0, 2)$. The minimum cost is now $7 \cdot 2 = 14$.

Note 1 Some linear programs *maximize profit* instead of minimizing cost. The mathematics is almost the same. The parallel planes start with a large value of C, instead of a small value. They move toward the origin (instead of away), as C gets smaller. *The first touching point is still a corner.*

Note 2 The requirements $Ax = b$ and $x \geq 0$ could be impossible to satisfy. The equation $x_1 + x_2 + x_3 = -1$ cannot be solved with $x \geq 0$. *That feasible set is empty.*

Note 3 It could also happen that the feasible set is *unbounded.* If the requirement is $x_1 + x_2 - 2x_3 = 4$, the large positive vector $(100, 100, 98)$ is now a candidate. So is the larger vector $(1000, 1000, 998)$. The plane $Ax = b$ is no longer chopped off to a triangle. The two corners P and Q are still candidates for x^*, but R moved to infinity.

Note 4 With an unbounded feasible set, the minimum cost could be $-\infty$ (*minus infinity*). Suppose the cost is $-x_1 - x_2 + x_3$. Then the vector $(100, 100, 98)$ costs $C = -102$. The vector $(1000, 1000, 998)$ costs $C = -1002$. We are being paid to include x_1 and x_2, instead of paying a cost. In realistic applications this will not happen. But it is theoretically possible that A, b, and c can produce unexpected triangles and costs.

The Primal and Dual Problems

This first problem will fit A, b, c in that example. The unknowns x_1, x_2, x_3 represent hours of work by a Ph.D. and a student and a machine. The costs per hour are \$5, \$3, and \$8. (*I apologize for such low pay.*) The number of hours cannot be negative: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. The Ph.D. and the student get through one homework problem per hour. *The machine solves two problems in one hour.* In principle they can share out the homework, which has four problems to be solved: $x_1 + x_2 + 2x_3 = 4$.

The problem is to finish the four problems at minimum cost $c^{\mathrm{T}}x$.

If all three are working, the job takes one hour: $x_1 = x_2 = x_3 = 1$. The cost is $5 + 3 + 8 = 16$. But certainly the Ph.D. should be put out of work by the student (who is just as fast and costs less—this problem is getting realistic). When the student works two hours and the machine works one, the cost is $6 + 8$ and all four problems get solved. We are on the edge QR of the triangle because the Ph.D. is not working: $x_1 = 0$. But the best point is all work by student (at Q) or all work by machine (at R). In this example the student solves four problems in four hours for \$12—the minimum cost.

With only one equation in $Ax = b$, the corner $(0, 4, 0)$ has only one nonzero component. ***When $Ax = b$ has m equations, corners have m nonzeros.*** We solve $Ax = b$ for those m variables, with $n - m$ free variables set to zero. But unlike Chapter 3, ***we don't know which m variables to choose.***

The number of possible corners is the number of ways to choose m components out of n. This number "n choose m" is heavily involved in gambling and probability. With $n = 20$ unknowns and $m = 8$ equations (still small numbers), the "feasible set" can have $20!/8!12!$ corners. That number is $(20)(19)\cdots(13) = 5{,}079{,}110{,}400$.

Checking three corners for the minimum cost was fine. Checking five billion corners is not the way to go. The simplex method described below is much faster.

The Dual Problem In linear programming, problems come in pairs. There is a minimum problem and a maximum problem—the original and its "dual." The original problem was specified by a matrix A and two vectors b and c. The dual problem transposes A and switches b and c: ***Maximize $b \cdot y$***. Here is the dual to our example:

> **A cheater offers to solve homework problems by selling the answers.** The charge is y dollars per problem, or $4y$ altogether. (Note how $b = 4$ has gone into the cost.) The cheater must be as cheap as the Ph.D. or student or machine: $y \leq 5$ and $y \leq 3$ and $2y \leq 8$. (Note how $c = (5, 3, 8)$ has gone into inequality constraints). The cheater maximizes the income $4y$.

Dual Problem ***Maximize $b \cdot y$ subject to $A^{\mathrm{T}}y \leq c$***.

The maximum occurs when $y = 3$. The income is $4y = 12$. The maximum in the dual problem (\$12) equals the minimum in the original (\$12). *Max = min* is duality.

If either problem has a best vector (x^* or y^*) then so does the other.
Minimum cost $c \cdot x^*$ equals maximum income $b \cdot y^*$

This book started with a row picture and a column picture. The first "duality theorem" was about rank: The number of independent rows equals the number of independent columns. That theorem, like this one, was easy for small matrices. Minimum cost = maximum income is proved in our text *Linear Algebra and Its Applications*. One line will establish the easy half of the theorem: ***The cheater's income $b^{\mathrm{T}}y$ cannot exceed the honest cost***:

$$\textit{If}\ \ Ax = b, x \geq 0, A^{\mathrm{T}}y \leq c \quad \textit{then} \quad b^{\mathrm{T}}y = (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y) \leq x^{\mathrm{T}}c. \tag{1}$$

The full duality theorem says that when $b^{\mathrm{T}}y$ reaches its maximum and $x^{\mathrm{T}}c$ reaches its minimum, they are equal: $b \cdot y^* = c \cdot x^*$. Look at the last step in (1), with $\leq$ sign:

The dot product of $x \geq 0$ and $s = c - A^{\mathrm{T}}y \geq 0$ gave $x^{\mathrm{T}}s \geq 0$. This is $x^{\mathrm{T}}A^{\mathrm{T}}y \leq x^{\mathrm{T}}c$.

Equality needs $x^{\mathrm{T}}s = 0$ So the optimal solution has $x_j^* = 0$ or $s_j^* = 0$ for each j.

The Simplex Method

Elimination is the workhorse for linear equations. The simplex method is the workhorse for linear inequalities. We cannot give the simplex method as much space as elimination, but the idea can be clear. *The simplex method goes from one corner to a neighboring corner of lower cost.* Eventually (and quite soon in practice) it reaches the corner of minimum cost.

A ***corner*** is a vector $x \geq 0$ that satisfies the m equations $Ax = b$ with at most m positive components. *The other $n - m$ components are zero.* (Those are the free variables. Back substitution gives the m basic variables. All variables must be nonnegative or x is a false corner.) For a *neighboring corner*, one zero component of x becomes positive and one positive component becomes zero.

The simplex method must decide which component "enters" by becoming positive, and which component "leaves" by becoming zero. That exchange is chosen so as to lower the total cost. This is one step of the simplex method, moving toward x^*.

Here is the overall plan. Look at each zero component at the current corner. If it changes from 0 to 1, the other nonzeros have to adjust to keep $Ax = b$. Find the new x by back substitution and compute the change in the total cost $c \cdot x$. This change is the "reduced cost" r of the new component. The ***entering variable*** is the one that gives the ***most negative*** r. This is the greatest cost reduction for a single unit of a new variable.

Example 1 Suppose the current corner is $P = (4, 0, 0)$, with the Ph.D. doing all the work (the cost is \$20). If the student works one hour, the cost of $x = (3, 1, 0)$ is down to \$18. The reduced cost is $r = -2$. If the machine works one hour, then $x = (2, 0, 1)$ also costs \$18. The reduced cost is also $r = -2$. In this case the simplex method can choose either the student or the machine as the entering variable.

Even in this small example, the first step may not go immediately to the best x^*. The method chooses the entering variable before it knows how much of that variable to include. We computed r when the entering variable changes from 0 to 1, but one unit may be too much or too little. The method now chooses the leaving variable (the Ph.D.). It moves to corner Q or R in the figure.

The more of the entering variable we include, the lower the cost. This has to stop when one of the positive components (which are adjusting to keep $Ax = b$) hits zero. ***The leaving variable** is the first positive x_i to reach zero.* When that happens, a neighboring corner has been found. Then start again (from the new corner) to find the next variables to enter and leave.

When all reduced costs are positive, the current corner is the optimal x^*. No zero component can become positive without increasing $c \cdot x$. No new variable should enter. The problem is solved (and we can show that y^* is found too).

Note Generally x^* is reached in αn steps, where α is not large. But examples have been invented which use an exponential number of simplex steps. Eventually a different approach was developed, which is guaranteed to reach x^* in fewer (but more difficult) steps. The new methods travel through the *interior* of the feasible set.

Example 2 Minimize the cost $c \cdot x = 3x_1 + x_2 + 9x_3 + x_4$. The constraints are $x \geq 0$ and two equations $Ax = b$:

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 4 \qquad m = 2 \quad \text{equations} \\ x_2 + x_3 - x_4 &= 2 \qquad n = 4 \quad \text{unknowns.} \end{aligned}$$

A starting corner is $x = (4, 2, 0, 0)$ which costs $c \cdot x = 14$. It has $m = 2$ nonzeros and $n - m = 2$ zeros. The zeros are x_3 and x_4. The question is whether x_3 or x_4 should enter (become nonzero). Try one unit of each of them:

$$\begin{aligned} &\text{If } x_3 = 1 \text{ and } x_4 = 0, \quad \text{then } x = (2, 1, 1, 0) \text{ costs } 16. \\ &\text{If } x_4 = 1 \text{ and } x_3 = 0, \quad \text{then } x = (3, 3, 0, 1) \text{ costs } 13. \end{aligned}$$

Compare those costs with 14. The reduced cost of x_3 is $r = 2$, positive and useless. The reduced cost of x_4 is $r = -1$, negative and helpful. *The entering variable is x_4.*

How much of x_4 can enter? One unit of x_4 made x_1 drop from 4 to 3. Four units will make x_1 drop from 4 to zero (while x_2 increases all the way to 6). *The leaving variable is x_1.* The new corner is $x = (0, 6, 0, 4)$, which costs only $c \cdot x = 10$. This is the optimal x^*, but to know that we have to try another simplex step from $(0, 6, 0, 4)$. Suppose x_1 or x_3 tries to enter:

Start from the	If $x_1 = 1$ and $x_3 = 0$,	then $x = (1, 5, 0, 3)$ costs 11.
corner $(0, 6, 0, 4)$	If $x_3 = 1$ and $x_1 = 0$,	then $x = (0, 3, 1, 2)$ costs 14.

Those costs are higher than 10. Both r's are positive—it does not pay to move. The current corner $(0, 6, 0, 4)$ is the solution x^*.

These calculations can be streamlined. Each simplex step solves three linear systems with the same matrix B. (This is the m by m matrix that keeps the m basic columns of A.) When a column enters and an old column leaves, there is a quick way to update B^{-1}. That is how most codes organize the simplex method.

Our text on *Computational Science and Engineering* includes a short code with comments. (The code is also on **math.mit.edu/cse**) The best $\boldsymbol{y}^*$ solves m equations $A^{\mathrm{T}}\boldsymbol{y}^* = \boldsymbol{c}$ in the m components that are nonzero in $\boldsymbol{x}^*$. Then we have optimality $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s} = 0$ and this is duality: *Either* $x_j^* = 0$ *or the "slack" in* $\boldsymbol{s}^* = \boldsymbol{c} - A^{\mathrm{T}}y^*$ *has* $s_j^* = 0$.

When $\boldsymbol{x}^* = (0,4,0)$ was the optimal corner $\boldsymbol{Q}$, the cheater's price was set by $y^* = 3$.

Interior Point Methods

The simplex method moves along the edges of the feasible set, eventually reaching the optimal corner $\boldsymbol{x}^*$. ***Interior point methods move inside the feasible set*** (where $\boldsymbol{x} > \boldsymbol{0}$). These methods hope to go more directly to $\boldsymbol{x}^*$. They work well.

One way to stay inside is to put a barrier at the boundary. Add extra cost as a *logarithm that blows up* when any variable x_j touches zero. The best vector has $\boldsymbol{x} > \boldsymbol{0}$. The number θ is a small parameter that we move toward zero.

Barrier problem **Minimize** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\,(\log x_1 + \cdots + \log x_n)$ **with** $A\boldsymbol{x} = \boldsymbol{b}$ (2)

This cost is nonlinear (but linear programming is already nonlinear from inequalities). The constraints $x_j \geq 0$ are not needed because $\log x_j$ becomes infinite at $x_j = 0$.

The barrier gives an *approximate problem* for each θ. The m constraints $A\boldsymbol{x} = \boldsymbol{b}$ have Lagrange multipliers $y_1, \ldots, y_m$. This is the good way to deal with constraints.

$\boldsymbol{y}$ from Lagrange
$$L(x, y, \theta) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\left(\textstyle\sum \log x_i\right) - \boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{x} - \boldsymbol{b}) \tag{3}$$

$\partial L/\partial y = 0$ brings back $A\boldsymbol{x} = \boldsymbol{b}$. The derivatives $\partial L/\partial x_j$ are interesting!

Optimality in barrier pbm
$$\frac{\partial L}{\partial x_j} = c_j - \frac{\theta}{x_j} - (A^{\mathrm{T}}y)_j = 0 \quad \text{which is} \quad \boldsymbol{x_j s_j = \theta}\,. \tag{4}$$

The true problem has $x_j s_j = 0$. The barrier problem has $x_j s_j = \theta$. The solutions $\boldsymbol{x}^*(\theta)$ lie on the ***central path*** to $\boldsymbol{x}^*(0)$. Those n optimality equations $x_j s_j = \theta$ are nonlinear, and we solve them iteratively by Newton's method.

The current $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}$ will satisfy $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ and $A^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$, *but not* $x_j s_j = \theta$. Newton's method takes a step $\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}$. By ignoring the second-order term $\Delta\boldsymbol{x}\,\Delta\boldsymbol{s}$ in $(\boldsymbol{x} + \Delta\boldsymbol{x})(\boldsymbol{s} + \Delta\boldsymbol{s}) = \boldsymbol{\theta}$, the corrections in $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}$ come from linear equations:

Newton step
$$\begin{aligned} A\,\Delta\boldsymbol{x} &= 0 \\ A^{\mathrm{T}}\Delta\boldsymbol{y} + \Delta\boldsymbol{s} &= 0 \\ s_j\Delta x_j + x_j\Delta s_j &= \theta - x_j s_j \end{aligned} \tag{5}$$

Newton iteration has quadratic convergence for each θ, and then θ approaches zero. The duality gap $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s}$ generally goes below 10^{-8} after 20 to 60 steps. The explanation in my *Computational Science and Engineering* textbook takes one Newton step in detail, for the example with four homework problems. I didn't intend that the student should end up doing all the work, but $\boldsymbol{x}^*$ turned out that way.

This interior point method is used almost "as is" in commercial software, for a large class of linear and nonlinear optimization problems.

Problem Set 8.4

1 Draw the region in the xy plane where $x + 2y = 6$ and $x \geq 0$ and $y \geq 0$. Which point in this "feasible set" minimizes the cost $c = x + 3y$? Which point gives maximum cost? Those points are at corners.

2 Draw the region in the xy plane where $x + 2y \leq 6,\ 2x + y \leq 6,\ x \geq 0,\ y \geq 0$. It has four corners. Which corner minimizes the cost $c = 2x - y$?

3 What are the corners of the set $x_1 + 2x_2 - x_3 = 4$ with x_1, x_2, x_3 all ≥ 0? Show that the cost $x_1 + 2x_3$ can be very negative in this feasible set. This is an example of unbounded cost: no minimum.

4 Start at $\boldsymbol{x} = (0, 0, 2)$ where the machine solves all four problems for \$16. Move to $\boldsymbol{x} = (0, 1, \quad)$ to find the reduced cost r (the savings per hour) for work by the student. Find r for the Ph.D. by moving to $\boldsymbol{x} = (1, 0, \quad)$ with 1 hour of Ph.D. work.

5 Start Example 1 from the Ph.D. corner $(4, 0, 0)$ with $\boldsymbol{c}$ changed to $[5 \ \ 3 \ \ 7]$. Show that r is better for the machine even when the total cost is lower for the student. The simplex method takes two steps, first to the machine and then to the student for $\boldsymbol{x}^*$.

6 Choose a different cost vector $\boldsymbol{c}$ so the Ph.D. gets the job. Rewrite the dual problem (maximum income to the cheater).

7 A six-problem homework on which the Ph.D. is fastest gives a second constraint $2x_1 + x_2 + x_3 = 6$. Then $\boldsymbol{x} = (2, 2, 0)$ shows two hours of work by Ph.D. and student on each homework. Does this $\boldsymbol{x}$ minimize the cost $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $\boldsymbol{c} = (5, 3, 8)$?

8 These two problems are also dual. Prove weak duality, that always $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$:

Primal problem	Minimize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $A\boldsymbol{x} \geq \boldsymbol{b}$ and $\boldsymbol{x} \geq \mathbf{0}$.
Dual problem	Maximize $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ with $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$ and $\boldsymbol{y} \geq \mathbf{0}$.

8.5 Fourier Series: Linear Algebra for Functions

This section goes from finite dimensions to *infinite* dimensions. I want to explain linear algebra in infinite-dimensional space, and to show that it still works. First step: look back. This book began with vectors and dot products and linear combinations. We begin by converting those basic ideas to the infinite case—then the rest will follow.

What does it mean for a vector to have infinitely many components? There are two different answers, both good:

1. The vector becomes $v = (v_1, v_2, v_3, \ldots)$. It could be $(1, \frac{1}{2}, \frac{1}{4}, \ldots)$.

2. The vector becomes a function $f(x)$. It could be $\sin x$.

We will go both ways. Then the idea of Fourier series will connect them.

After vectors come *dot products.* The natural dot product of two infinite vectors $(v_1, v_2, \ldots)$ and $(w_1, w_2, \ldots)$ is an infinite series:

$$\textbf{Dot product} \qquad \boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots. \tag{1}$$

This brings a new question, which never occurred to us for vectors in $\mathbf{R}^n$. Does this infinite sum add up to a finite number? Does the series converge? Here is the first and biggest difference between finite and infinite.

When $\boldsymbol{v} = \boldsymbol{w} = (1, 1, 1, \ldots)$, the sum certainly does not converge. In that case $\boldsymbol{v} \cdot \boldsymbol{w} = 1 + 1 + 1 + \cdots$ is infinite. Since $\boldsymbol{v}$ equals $\boldsymbol{w}$, we are really computing $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 =$ length squared. The vector $(1, 1, 1, \ldots)$ has infinite length. *We don't want that vector.* Since we are making the rules, we don't have to include it. The only vectors to be allowed are those with finite length:

DEFINITION The vector $(v_1, v_2, \ldots)$ is in our infinite-dimensional "***Hilbert space***" if and only if its length $\|\boldsymbol{v}\|$ is finite:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + v_3^2 + \cdots \text{ must add to a finite number.}$$

Example 1 The vector $\boldsymbol{v} = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$ is included in Hilbert space, because its length is $2/\sqrt{3}$. We have a geometric series that adds to $4/3$. The length of $\boldsymbol{v}$ is the square root:

$$\textbf{Length squared} \qquad \boldsymbol{v} \cdot \boldsymbol{v} = 1 + \tfrac{1}{4} + \tfrac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{4}} = \tfrac{4}{3}.$$

Question If $\boldsymbol{v}$ and $\boldsymbol{w}$ have finite length, how large can their dot product be?

Answer The sum $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots$ also adds to a finite number. We can safely take dot products. The Schwarz inequality is still true:

$$\textbf{Schwarz inequality} \qquad |\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|. \tag{2}$$

The ratio of $\boldsymbol{v} \cdot \boldsymbol{w}$ to $\|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ is still the cosine of θ (the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$). Even in infinite-dimensional space, $|\cos \theta|$ is not greater than 1.

Now change over to functions. Those are the "vectors." The space of functions $f(x)$, $g(x), h(x), \ldots$ defined for $0 \le x \le 2\pi$ must be somehow bigger than $\mathbf{R}^n$. ***What is the dot product of $f(x)$ and $g(x)$? What is the length of $f(x)$?***

Key point in the continuous case: *Sums are replaced by integrals.* Instead of a sum of v_j times w_j, the dot product is an integral of $f(x)$ times $g(x)$. Change the "dot" to parentheses with a comma, and change the words "dot product" to *inner product*:

DEFINITION The ***inner product*** of $f(x)$ and $g(x)$, and the ***length squared***, are

$$(f, g) = \int_0^{2\pi} f(x)g(x)\,dx \quad \text{and} \quad \|f\|^2 = \int_0^{2\pi} \left(f(x)\right)^2 dx. \tag{3}$$

The interval $[0, 2\pi]$ where the functions are defined could change to a different interval like $[0, 1]$ or $(-\infty, \infty)$. We chose 2π because our first examples are $\sin x$ and $\cos x$.

Example 2 The length of $f(x) = \sin x$ comes from its inner product with itself:

$$(f, f) = \int_0^{2\pi} (\sin\ x)^2\,dx = \pi. \quad \text{The length of } \sin x \text{ is } \sqrt{\pi}.$$

That is a standard integral in calculus—not part of linear algebra. By writing $\sin^2 x$ as $\frac{1}{2} - \frac{1}{2}\cos 2x$, we see it go above and below its average value $\frac{1}{2}$. Multiply that average by the interval length 2π to get the answer π.

More important: $\sin x$ ***and*** $\cos x$ ***are orthogonal in function space***:

Inner product is zero
$$\int_0^{2\pi} \sin x \cos x\,dx = \int_0^{2\pi} \tfrac{1}{2}\sin 2x\,dx = \left[-\tfrac{1}{4}\cos 2x\right]_0^{2\pi} = 0. \tag{4}$$

This zero is no accident. It is highly important to science. The orthogonality goes beyond the two functions $\sin x$ and $\cos x$, to an infinite list of sines and cosines. The list contains $\cos 0x$ (which is 1), $\sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots$.

Every function in that list is orthogonal to every other function in the list.

Fourier Series

The Fourier series of a function $y(x)$ is its expansion into sines and cosines:

$$y(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots. \tag{5}$$

We have an orthogonal basis! The vectors in "function space" are combinations of the sines and cosines. On the interval from $x = 2\pi$ to $x = 4\pi$, all our functions repeat what they did from 0 to 2π. They are "*periodic*." The distance between repetitions is the period 2π.

Remember: The list is infinite. The Fourier series is an infinite series. We avoided the vector $v = (1, 1, 1, \ldots)$ because its length is infinite, now we avoid a function like $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots$. (*Note*: This is π times the famous **delta function** $\delta(x)$. It is an infinite "spike" above a single point. At $x = 0$ its height $\frac{1}{2} + 1 + 1 + \cdots$ is infinite. At all points inside $0 < x < 2\pi$ the series adds in some average way to zero.) The integral of $\delta(x)$ is 1. But $\int \delta^2(x) = \infty$, so delta functions are excluded from Hilbert space.

Compute the length of a typical sum $f(x)$:

$$\begin{aligned}(f, f) &= \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots)^2 \, dx \\ &= \int_0^{2\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \cdots) \, dx \\ \|f\|^2 &= 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + \cdots).\end{aligned} \tag{6}$$

The step from line 1 to line 2 used orthogonality. All products like $\cos x \cos 2x$ integrate to give zero. Line 2 contains what is left—the integrals of each sine and cosine squared. Line 3 evaluates those integrals. (The integral of 1^2 is 2π, when all other integrals give π.) If we divide by their lengths, our functions become *orthonormal*:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \textit{ is an orthonormal basis for our function space.}$$

These are unit vectors. We could combine them with coefficients $A_0, A_1, B_1, A_2, \ldots$ to yield a function $F(x)$. Then the 2π and the π's drop out of the formula for length.

Function length = vector length $\quad \|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \cdots. \quad (7)$

Here is the important point, for $f(x)$ as well as $F(x)$. *The function has finite length exactly when the vector of coefficients has finite length.* Fourier series gives us a perfect match between function space and infinite-dimensional Hilbert space. The function is in L^2, its Fourier coefficients are in ℓ^2.

The function space contains $f(x)$ exactly when the Hilbert space contains the vector $v = (a_0, a_1, b_1, \ldots)$ of Fourier coefficients. Both $f(x)$ and v have finite length.

Example 3 Suppose $f(x)$ is a "square wave," equal to 1 for $0 \le x < \pi$. Then $f(x)$ drops to -1 for $\pi \le x < 2\pi$. The $+1$ and -1 repeats forever. This $f(x)$ is an odd function like the sines, and all its cosine coefficients are zero. We will find its Fourier series, containing only sines:

Square wave $$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]. \tag{8}$$

The length is $\sqrt{2\pi}$, because at every point $\big(f(x)\big)^2$ is $(-1)^2$ or $(+1)^2$:

$$\|f\|^2 = \int_0^{2\pi} \big(f(x)\big)^2 \, dx = \int_0^{2\pi} 1 \, dx = 2\pi.$$

At $x = 0$ the sines are zero and the Fourier series gives zero. This is half way up the jump from -1 to $+1$. The Fourier series is also interesting when $x = \frac{\pi}{2}$. At this point the square wave equals 1, and the sines in (8) alternate between $+1$ and -1:

$$\textbf{Formula for } \pi \qquad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \tag{9}$$

Multiply by π to find a magical formula $4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$ for that famous number.

The Fourier Coefficients

How do we find the a's and b's which multiply the cosines and sines? For a given function $f(x)$, we are asking for its Fourier coefficients:

$$\textbf{Fourier series} \qquad f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots.$$

Here is the way to find a_1. ***Multiply both sides by*** $\cos x$. ***Then integrate from 0 to*** 2π. The key is orthogonality! All integrals on the right side are zero, except for $\cos^2 x$:

$$\textbf{Coefficient } a_1 \qquad \int_0^{2\pi} f(x) \cos x \, dx = \int_0^{2\pi} a_1 \cos^2 x \, dx = \pi a_1. \tag{10}$$

Divide by π and you have a_1. To find any other a_k, multiply the Fourier series by $\cos kx$. Integrate from 0 to 2π. Use orthogonality, so only the integral of $a_k \cos^2 kx$ is left. That integral is πa_k, and divide by π:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \quad \text{and similarly} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx. \tag{11}$$

The exception is a_0. This time we multiply by $\cos 0x = 1$. The integral of 1 is 2π:

$$\textbf{Constant term} \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 \, dx = \textbf{\textit{average value}} \text{ of } f(x). \tag{12}$$

I used those formulas to find the Fourier coefficients for the square wave. The integral of $f(x) \cos kx$ was zero. The integral of $f(x) \sin kx$ was $4/k$ for odd k.

Compare Linear Algebra in $\mathbf{R}^n$

The point to emphasize is how this infinite-dimensional case is so much like the n-dimensional case. Suppose the nonzero vectors $v_1, \ldots, v_n$ are orthogonal. We want to write the vector b (instead of the function $f(x)$) as a combination of those v's:

$$\textbf{Finite orthogonal series} \quad b = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n. \tag{13}$$

Multiply both sides by v_1^{T}. Use orthogonality, so $v_1^{\mathrm{T}} v_2 = 0$. Only the c_1 term is left:

$$\textbf{Coefficient } c_1 \quad v_1^{\mathrm{T}} b = c_1 v_1^{\mathrm{T}} v_1 + 0 + \cdots + 0. \quad \text{Therefore } c_1 = v_1^{\mathrm{T}} b / v_1^{\mathrm{T}} v_1. \tag{14}$$

The denominator $v_1^{\mathrm{T}} v_1$ is the length squared, like π in equation 11. The numerator $v_1^{\mathrm{T}} b$ is the inner product like $\int f(x) \cos kx \, dx$. ***Coefficients are easy to find when the basis***

vectors are orthogonal. We are just doing one-dimensional projections, to find the components along each basis vector.

The formulas are even better when the vectors are orthonormal. Then we have unit vectors. The denominators $v_k^{\mathrm{T}} v_k$ are all 1. You know $c_k = v_k^{\mathrm{T}} b$ in another form:

$$\textbf{Equation for } c\textbf{'s} \quad c_1 v_1 + \cdots + c_n v_n = b \quad \text{or} \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = b.$$

The v's are in an orthogonal matrix Q. Its inverse is Q^{T}. That gives the c's:

$$Qc = b \quad \text{yields} \quad c = Q^{\mathrm{T}} b. \quad \text{Row by row this is } c_k = q_k^{\mathrm{T}} b.$$

Fourier series is like having a matrix with infinitely many orthogonal columns. Those columns are the basis functions $1, \cos x, \sin x, \ldots$. After dividing by their lengths we have an "infinite orthogonal matrix." Its inverse is its transpose. Orthogonality is what reduces a series of terms to one single term.

Problem Set 8.5

1 Integrate the trig identity $2\cos jx \cos kx = \cos(j+k)x + \cos(j-k)x$ to show that $\cos jx$ is orthogonal to $\cos kx$, provided $j \neq k$. What is the result when $j = k$?

2 Show that $1, x$, and $x^2 - \frac{1}{3}$ are orthogonal, when the integration is from $x = -1$ to $x = 1$. Write $f(x) = 2x^2$ as a combination of those orthogonal functions.

3 Find a vector $(w_1, w_2, w_3, \ldots)$ that is orthogonal to $v = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$. Compute its length $\|w\|$.

4 The first three *Legendre polynomials* are $1, x$, and $x^2 - \frac{1}{3}$. Choose c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. All integrals go from -1 to 1.

5 For the square wave $f(x)$ in Example 3, show that

$$\int_0^{2\pi} f(x)\cos x\,dx = 0 \qquad \int_0^{2\pi} f(x)\sin x\,dx = 4 \qquad \int_0^{2\pi} f(x)\sin 2x\,dx = 0.$$

Which three Fourier coefficients come from those integrals?

6 The square wave has $\|f\|^2 = 2\pi$. Then (6) gives what remarkable sum for π^2?

7 Graph the square wave. Then graph by hand the sum of two sine terms in its series, or graph by machine the sum of 2, 3, and 10 terms. The famous ***Gibbs phenomenon*** is the oscillation that overshoots the jump (this doesn't die down with more terms).

8 Find the lengths of these vectors in Hilbert space:

(a) $v = \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \ldots\right)$

(b) $v = (1, a, a^2, \dots)$

(c) $f(x) = 1 + \sin x$.

9 Compute the Fourier coefficients a_k and b_k for $f(x)$ defined from 0 to 2π:

(a) $f(x) = 1$ for $0 \le x \le \pi$, $f(x) = 0$ for $\pi < x < 2\pi$

(b) $f(x) = x$.

10 When $f(x)$ has period 2π, why is its integral from $-\pi$ to π the same as from 0 to 2π? If $f(x)$ is an *odd* function, $f(-x) = -f(x)$, show that $\int_0^{2\pi} f(x)\,dx$ is zero. Odd functions only have sine terms, even functions have cosines.

11 From trig identities find the only two terms in the Fourier series for $f(x)$:

(a) $f(x) = \cos^2 x$ (b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$ (c) $f(x) = \sin^3 x$

12 The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$ are a basis for Hilbert space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5 by 5 "differentiation matrix" for these functions?

13 Find the Fourier coefficients a_k and b_k of the square pulse $F(x)$ centered at $x = 0$: $F(x) = 1/h$ for $|x| \le h/2$ and $F(x) = 0$ for $h/2 < |x| \le \pi$.

As $h \to 0$, this $F(x)$ approaches a delta function. Find the limits of a_k and b_k.

The Fourier Series section 4.1 of *Computational Science and Engineering* explains the sine series, cosine series, complete series, and complex series $\Sigma c_k e^{ikx}$ on **math.mit.edu/cse.**

8.6 Linear Algebra for Statistics and Probability

Statistics deals with data, often in large quantities. Since data tends to go into rectangular matrices, we expect to see $A^{\mathrm{T}}A$. The least squares problem $A\widehat{x} \approx \boldsymbol{b}$ is ***linear regression.*** The best solution $\widehat{x}$ fits m observations by $n < m$ parameters. This is a fundamental application of linear algebra to statistics.

This section goes beyond $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$. These unweighted equations assume that the measurements $b_1, \ldots, b_m$ are equally reliable. When there is good reason to expect higher accuracy (lower variance) in some b_i, those equations should be weighted more heavily. *With what weights* $w_1, \cdots, w_m$? And if the b_i are not independent, a ***covariance matrix*** Σ gives the statistics of the errors. Here are key topics in this section:

1. Weighted least squares and $A^{\mathrm{T}}CA\widehat{x} = A^{\mathrm{T}}C\boldsymbol{b}$
2. Variances $\sigma_1^2, \ldots, \sigma_m^2$ and the covariance matrix Σ
3. Important probability distributions: binomial, Poisson, and normal
4. Principal Component Analysis (PCA) to find combinations with greatest variance.

Weighted Least Squares

To include weights in the m equations $A\boldsymbol{x} = \boldsymbol{b}$, multiply each equation i by a weight w_i. Put those m weights into a diagonal matrix W. *We are replacing* $A\boldsymbol{x} = \boldsymbol{b}$ *by* $WA\boldsymbol{x} = W\boldsymbol{b}$. The equations are no more and no less solvable—we expect to use least squares.

The least squares equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$ changes to $(WA)^{\mathrm{T}}WA\widehat{x} = (WA)^{\mathrm{T}}W\boldsymbol{b}$. The matrix $C = W^{\mathrm{T}}W$ is inside $(WA)^{\mathrm{T}}WA$, in the middle of weighted least squares.

Weighted least squares ***$C = W^{\mathrm{T}}W$ is in the n equations for $\widehat{x}$*** $\quad A^{\mathrm{T}}CA\widehat{x} = A^{\mathrm{T}}C\boldsymbol{b} \quad$ (1)

When $n = 1$ and A = column of 1's, $\widehat{x}$ changes from an average to a weighted average:

Simplest case $$\widehat{x} = \frac{b_1 + \cdots + b_m}{m} \quad \text{changes to} \quad \widehat{x}_W = \frac{w_1^2 b_1 + \cdots + w_m^2 b_m}{w_1^2 + \cdots + w_m^2}. \quad (2)$$

This average $\widehat{x}_W$ gives greatest weight to the observations b_i that have the largest w_i. We always assume that errors have *zero mean.* (Subtract the mean if necessary, so there is no one-sided bias in the measurements.)

How should we choose the weights w_i? This depends on the reliability of b_i. If that observation has variance σ_i^2, then the root mean square error in b_i is σ_i. **When we divide the equations by** $\sigma_1, \ldots \sigma_m$ (left side together with right side), **all variances will equal 1.** So the weight is $w_i = 1/\sigma_i$ and the diagonal of $C = W^{\mathrm{T}}W$ contains the numbers $1/\sigma_i^2$.

The statistically correct matrix is $C = \operatorname{diag}(1/\sigma_1^2, \ldots, 1/\sigma_m^2)$.

This is correct provided the errors e_i and e_j in different equations are statistically independent. If the errors are dependent, off-diagonal entries show up in the covariance matrix Σ. The good choice is still $C = \Sigma^{-1}$ as described in this section.

Mean and Variance

The two crucial numbers for a random variable are its **mean** m and its **variance** σ^2. The "expected value" E[e] is found from the probabilities $p_1, p_2, \ldots$ of the possible errors $e_1, e_2, \ldots$ (and the variance σ^2 is always measured around the mean).

For a discrete random variable, the error e_j has probability p_j (the p_j add to 1):

$$\textbf{Mean } m = \mathrm{E}[e] = \sum e_j p_j \quad \textbf{Variance } \sigma^2 = \mathrm{E}[(e-m)^2] = \sum (e_j - m)^2 p_j \tag{3}$$

Example 1 Flip a fair coin. The result is 1 (for heads) or 0 (for tails). Those events have equal probabilities $p_0 = p_1 = 1/2$. The mean is $m = 1/2$ and the variance is $\sigma^2 = 1/4$:

$$\text{Mean} = (0)\,\frac{1}{2} + (1)\,\frac{1}{2} \qquad \text{Variance} = \left(0 - \frac{1}{2}\right)^2 \frac{1}{2} + \left(1 - \frac{1}{2}\right)^2 \frac{1}{2} = \frac{1}{4}.$$

Example 2 **(Binomial)** Flip the fair coin N times and count heads. With 3 flips, we see $M = 0, 1, 2,$ or 3 heads. The chances are $1/8, 3/8, 3/8, 1/8$. There are three ways to see $M = 2$ heads: HHT, HTH, and THH, and only HHH for $M = 3$ heads.

For all N, the number of ways to see M heads is the binomial coefficient "N *choose* M". Divide by the total number 2^N of all possible outcomes to get the probability for each M:

$$\begin{array}{l}\boldsymbol{M} \textbf{ heads in} \\ \boldsymbol{N} \textbf{ coin flips}\end{array} \quad p_M = \frac{1}{2^N}\binom{N}{M} = \frac{1}{2^N}\,\frac{N!}{M!(N-M)!} \qquad \text{Check } \frac{1}{2^3}\,\frac{3!}{2!\,1!} = \frac{3}{8} \tag{4}$$

Gamblers know this instinctively. The probabilities p_M add to $\left(\frac{1}{2} + \frac{1}{2}\right)^N = 1$. The mean value of the number of heads is $m = N/2$. The variance around m turns out to be $\sigma^2 = N/4$. The standard deviation $\sigma = \sqrt{N}/2$ measures the expected spread around the mean.

Example 3 **(Poisson)** A very unfair coin (small $p << \frac{1}{2}$) is flipped very often (large N). **The product $\lambda = pN$ is kept fixed.** The high probability of tails is $1 - p$ each time. So the chance p_0 of no heads in N flips (tails every time) is $(1-p)^N = (1 - \lambda/N)^N$. For large N this approaches $e^{-\lambda}$. The probability p_j of j heads in N very unfair flips comes out neatly in terms of the crucial number $\lambda = pN$:

$$\textbf{Poisson probabilities} \quad p_j = \frac{\lambda^j}{j!}e^{-\lambda} \quad \textbf{Mean } m = \lambda \quad \textbf{Variance } \sigma^2 = \lambda \tag{5}$$

Poisson applies to counting infrequent events (low p) over a long time T. Then $\lambda = pT$.

A *continuous* random variable will have a probability *density* function $p(x)$ instead of $p_1, p_2, \ldots$. "An outcome between x and $x + dx$ has probability $p(x)\,dx$." The total probability is $\int p(x)\,dx = 1$, since some outcome must happen. Sums become integrals:

$$\textbf{Mean } m = \textbf{Expected value} = \int x p(x)\,dx \quad \textbf{Variance } \sigma^2 = \int (x-m)^2 p(x)\,dx. \tag{6}$$

The outstanding example of a probability density function $p(x)$ (called the pdf) is the ***normal distribution*** $\mathrm{N}(0, \sigma)$. This has mean zero by symmetry. Its variance is σ^2:

$$\textbf{Normal (Gaussian)} \qquad p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2} \quad \text{with} \quad \int_{-\infty}^{\infty} p(x)\,dx = 1. \tag{7}$$

The graph of $p(x)$ is the famous bell-shaped curve. The integral of $p(x)$ from $-\sigma$ to σ is the probability that a random sample is less than one standard deviation σ from the mean. This is near $2/3$. MATLAB's **randn** uses the normal distribution with $\sigma = 1$.

This normal $p(x)$ appears everywhere because of the ***Central Limit Theorem***: The average over many independent trials of another distribution (like binomial) will approach a normal distribution as $N \to \infty$. A shift produces $m = 0$ and rescaling produces $\sigma = 1$.

$$\textbf{Normalized headcount} \qquad x = \frac{M - \text{mean}}{\sigma} = \frac{M - N/2}{\sqrt{N}/2} \longrightarrow \text{Normal } \mathrm{N}(0, 1).$$

The Covariance Matrix

Now run m different experiments at once. They might be independent, or there might be some correlation between them. Each measurement $\boldsymbol{b}$ is now a *vector* with m components. Those components are the outputs b_i from the m experiments.

If we measure distances from the means m_i, each error $e_i = b_i - m_i$ has *mean zero*. If two errors e_i and e_j are *independent* (no relation between them), their product $e_i e_j$ also has mean zero. But if the measurements are by the same observer at nearly the same time, the errors e_i and e_j could tend to have the same sign or the same size. ***The errors in the m experiments could be correlated.*** The products $e_i e_j$ are weighted by p_{ij} (their probability): *covariance* $\sigma_{ij} = \sum\sum p_{ij} e_i e_j$. The sum of $e_i^2 p_{ii}$ is the variance σ_i^2:

$$\textbf{Covariance} \qquad \sigma_{ij} = \sigma_{ji} = \mathrm{E}[e_i e_j] = \textbf{expected value of } (e_i \textbf{ times } e_j). \tag{8}$$

This is the (i, j) and (j, i) entry of the ***covariance matrix*** $\boldsymbol{\Sigma}$. The (i, i) entry is $\sigma_{ii} = \sigma_i^2$.

Example 4 **(Multivariate normal)** For m random variables, the probability density function moves from $p(x)$ to $p(\boldsymbol{b}) = p(b_1, \ldots, b_m)$. The normal distribution with mean zero was controlled by one positive number σ^2. Now $p(\boldsymbol{b})$ is controlled by an m by m positive definite matrix $\boldsymbol{\Sigma}$. This is the covariance matrix and its determinant is $|\boldsymbol{\Sigma}|$:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-x^2/2\sigma^2} \quad \text{becomes} \quad p(\boldsymbol{b}) = \frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}\, e^{-\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{b}/2}$$

The integral of $p(\boldsymbol{b})$ over m-dimensional space is 1. The integral of $\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} p(\boldsymbol{b})$ is $\boldsymbol{\Sigma}$.

The good way to handle that exponent $-\boldsymbol{b}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}/2$ is to use the eigenvalues and orthonormal eigenvectors of Σ (*linear algebra enters here*). When $\Sigma = Q\Lambda Q^{\mathrm{T}} = Q\Lambda Q^{-1}$, replacing $\boldsymbol{b}$ by $Q\boldsymbol{c}$ will split $p(\boldsymbol{b})$ into m one-dimensional normal distributions:

$$\exp\left(-\boldsymbol{b}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}/2\right) = \exp\left(-\boldsymbol{c}^{\mathrm{T}}\Lambda^{-1}\boldsymbol{c}/2\right) = \left(e^{-c_1^2/2\lambda_1}\right)\cdots\left(e^{-c_m^2/2\lambda_m}\right).$$

The determinant has $|\Sigma|^{1/2} = |\Lambda|^{1/2} = (\lambda_1 \cdots \lambda_m)^{1/2}$. Each integral over $-\infty < c_i < \infty$ is back to one dimension, where $\lambda = \sigma^2$. Notice the wonderful fact that after any linear transformation (here $\boldsymbol{c} = Q^{-1}\boldsymbol{b}$), we still have a multivariate normal distribution.

We could even reach variances $= 1$ by including $\sqrt{\Lambda}$ in the change from $\boldsymbol{b}$ to z:

Standard normal $\qquad \boldsymbol{b} = \sqrt{\Lambda}Qz$ changes $p(\boldsymbol{b})d\boldsymbol{b}$ to $p(z)dz = \dfrac{e^{-z^{\mathrm{T}}z/2}}{(2\pi)^{m/2}}dz$

This tells us the right weight matrix W to bring $A\boldsymbol{x} = \boldsymbol{b}$ back to ordinary least squares for $WA\boldsymbol{x} = W\boldsymbol{b}$. We want $W\boldsymbol{b}$ to become the standard normal z. So W will be the inverse of $\sqrt{\Lambda}Q$. Better than that, ***$C = W^{\mathrm{T}}W$ is the inverse of $Q\Lambda Q^{\mathrm{T}}$ which is Σ.***

Summary For independent errors, Σ is the diagonal matrix $\mathbf{diag}(\sigma_1^2, \ldots, \sigma_m^2)$. This is the usual choice. The right weights w_i for the equations $A\boldsymbol{x} = \boldsymbol{b}$ are $1/\sigma_1, \ldots, 1/\sigma_m$ (this will equalize all variances to 1). The right matrix $C = W^{\mathrm{T}}W$ in the middle of the weighted least squares equations is exactly Σ^{-1}:

Weighted least squares $\qquad A^{\mathrm{T}}\Sigma^{-1}A\widehat{x} = A^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b} \qquad (9)$

This choice of weighting returns $A\boldsymbol{x} = \boldsymbol{b}$ to a least squares problem $WA\boldsymbol{x} = W\boldsymbol{b}$ with equally reliable and independent errors. The usual equation $(WA)^{\mathrm{T}}WA\widehat{x} = (WA)^{\mathrm{T}}W\boldsymbol{b}$ is the same as (9).

It was Gauss who found this *best linear unbiased estimate* $\widehat{x}$. Unbiased because the mean of $\boldsymbol{x} - \widehat{x}$ is zero, linear because of equation (9), best because the covariance of $\boldsymbol{x} - \widehat{x}$ is as small as possible. That covariance (for error in $\widehat{x}$, not error in $\boldsymbol{b}$!) is important:

Covariance of the best $\widehat{x}$ $\qquad P = \mathrm{E}\left[(x - \widehat{x})(x - \widehat{x})^{\mathrm{T}}\right] = \left(A^{\mathrm{T}}\Sigma^{-1}A\right)^{-1}. \qquad (10)$

Example 5 Your pulse rate is measured ten times by independent doctors, all equally reliable. The mean error of each b_i is zero, and each variance is σ^2. Then $\Sigma = \sigma^2 I$. The ten equations $x = b_i$ produce the 10 by 1 matrix A of all ones. The best estimate $\widehat{x}$ is the average of the ten b_i. *The variance of that average value $\widehat{x}$ is the number P*:

$$P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1} = \sigma^2/10 \quad \text{so averaging reduces the variance.}$$

This matrix $P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ tells how reliable is the result $\widehat{x}$ of the experiment (Problem 6). P does not depend on the b's in the actual experiment! Those b's have probability distributions. Each experiment produces a sample value of $\widehat{x}$ from a sample $\boldsymbol{b}$.

When a small Σ gives good reliability of the inputs $\boldsymbol{b}$, a small P gives good reliability of the outputs $\widehat{\boldsymbol{x}}$. The key formula $P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ connects those covariances.

Principal Component Analysis

These paragraphs are about finding useful information in a data matrix A. Start by measuring m properties (m features) of n samples. These could be grades in m courses for n students (a row for each course, a column for each student). From each row, subtract its average so the sample means are zero. We look for a ***combination of courses*** and/or ***combination of students*** for which the data provides the most information.

Information is "distance from randomness" and it is measured by **variance**. A large variance in course grades means greater information than a small variance.

The key matrix idea is the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$. We are back again to $A^{\mathrm{T}}A$ and AA^{T}, because their unit eigenvectors are the singular vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ in V and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in U. The singular values in the diagonal matrix Σ (not the covariance) are in decreasing order and σ_1 is the most important. Weighting the m courses by the components of $\boldsymbol{u}_1$ gives a "***master course***" or "***eigencourse***" with the most significant grades.

Example 6 Suppose the grades A, B, C, F are worth $\mathbf{4, 2, 0, -6}$ points. If each course and each student has one of each grade, then all means are zero. Here is the grade matrix A with $(1, 1, 1, 1)$ in its nullspace (rank 3). To keep integers, the SVD of A will be written as $2U$ times $\Sigma/4$ times $(2V)^{\mathrm{T}}$. So the σ's are $12, 8, 4$:

$$\begin{bmatrix} -6 & 2 & 0 & 4 \\ 0 & 4 & -6 & 2 \\ 4 & 0 & 2 & -6 \\ 2 & -6 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Weighting the rows (the courses) by $\boldsymbol{u}_1 = \frac{1}{2}(-1, -1, 1, 1)$ will give the *eigencourse*. Weighting the columns (the students) by $\boldsymbol{v}_1 = \frac{1}{2}(1, -1, 1, -1)$ gives the *eigenstudent*. The fraction of the grade matrix that is "explained" by that one course and student is $\sigma_1^2/(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = 9/14$. The σ's in the SVD are the variances σ^2.

I guess this master course is what a Director of Admissions is looking for. If all grades in gym are the same, that row of A will be all zero—and gym is not part of the master course. Probably calculus is a part, but what about students who don't take calculus? The problem of ***missing data*** (holes in the matrix A) is extremely difficult for social sciences and the census and so much of the statistics of experiments.

Gene expression data Determining the functions of genes, and combinations of genes, is a central problem of genetics. Which genes combine to give which properties? Which genes malfunction to give which diseases?

We now have an incredibly fast way to find gene expression data in the lab. A gene microarray is often packed onto an Affymetrix chip, measuring tens of thousands of genes from one sample (one person). The understanding of genetic data (*bioinformatics*) has become a tremendous application of linear algebra.

Problem Set 8.6

1 Which line $Ct + D$ is the best fit to the three independent measurements $1, 2, 4$ at times $t = 0, 1, 2$ if the variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are $1, 1, 2$? Use weights $w_i = 1/\sigma_i$.

2 In Problem 1, suppose that the third measurement is **totally unreliable**. The variance σ_3^2 becomes infinite. Then the best line will not use _____. Find the line that goes through the first two points and solves the first two equations in $A\boldsymbol{x} = \boldsymbol{b}$ exactly.

3 In Problem 1, suppose that the third measurement is **totally reliable**. The variance σ_3^2 approaches zero. Now the best line will go through the third point exactly. Choose that line to minimize the sum of squares of the first two errors.

4 A single flip of a fair coin (0 or 1) has mean $m = 1/2$ and variance $\sigma^2 = 1/4$. This was Example 1. For the sum of two flips, the mean is $m = 1$. Compute the variance σ^2 around this mean, using the outcomes $0, 1, 2$ with their probabilities.

5 Instead of adding the flip results, make them two independent experiments. The outcome is $(0, 0), (1, 0), (0, 1)$ or $(1, 1)$. What is the covariance matrix Σ?

6 Change Example 1 so that the coin flip can be unfair. The probability is p for heads and $1 - p$ for tails. Find the mean m and the variance σ^2 of this distribution.

7 For two independent measurements $x = b_1$ and $x = b_2$, the best $\widehat{x}$ should be some weighted average $\widehat{x} = ab_1 + (1 - a)b_2$. When b_1 and b_2 have mean zero and variances σ_1^2 and σ_2^2, the variance of $\widehat{x}$ will be $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$. ***Choose the number a that minimizes P***: $dP/da = 0$.

Show that this a gives the $\widehat{x}$ in equation (2) which the text claimed is best, using weights $w_1 = 1/\sigma_1$ and $w_2 = 1/\sigma_2$.

8 The least squares estimate correctly weighted by Σ^{-1} is $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}$. ***Call that*** $\widehat{\boldsymbol{x}} = L\boldsymbol{b}$. If $\boldsymbol{b}$ contains an error vector $\boldsymbol{e}$, then $\widehat{\boldsymbol{x}}$ contains the error $L\boldsymbol{e}$.

The covariance matrix of those output errors $L\boldsymbol{e}$ is their expected value (average value) $P = \mathrm{E}\left[(L\boldsymbol{e})(L\boldsymbol{e})^{\mathrm{T}}\right] = L\mathrm{E}\left[\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}\right]L^{\mathrm{T}} = L\Sigma L^{\mathrm{T}}$. **Problem**: Do the multiplication $L\Sigma L^{\mathrm{T}}$ to show that P equals $(A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ as predicted in equation (10).

9 Change the grades to $3, 1, -1, -3$ for A, B, C, F. Show that the SVD of this grade matrix has the same $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$ (same eigencourses) as in Example 5, but now A has rank 2.

$$\textbf{Grade matrix} \qquad A = \begin{bmatrix} 3 & -1 & 1 & -3 \\ -1 & 3 & -3 & 1 \\ -3 & 1 & -1 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

Notes One way to deal with missing entries in A is to complete the matrix to have minimum rank. And statistics makes major use of the pseudoinverse A^+ (which is exactly the left inverse $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ from the normal equation when $A^{\mathrm{T}}A$ is invertible).

8.7 Computer Graphics

Computer graphics deals with images. The images are moved around. Their scale is changed. Three dimensions are projected onto two dimensions. All the main operations are done by matrices—but the shape of these matrices is surprising.

The transformations of three-dimensional space are done with 4 *by* 4 *matrices.* You would expect 3 by 3. The reason for the change is that one of the four key operations cannot be done with a 3 by 3 matrix multiplication. Here are the four operations:

Translation	**(shift the origin to another point $P_0 = (x_0, y_0, z_0)$)**
Rescaling	**(by c in all directions or by different factors c_1, c_2, c_3)**
Rotation	**(around an axis through the origin or an axis through P_0)**
Projection	**(onto a plane through the origin or a plane through P_0).**

Translation is the easiest—just add (x_0, y_0, z_0) to every point. But this is not linear! No 3 by 3 matrix can move the origin. So we change the coordinates of the origin to $(0, 0, 0, 1)$. This is why the matrices are 4 by 4. The "*homogeneous coordinates*" of the point (x, y, z) are $(x, y, z, 1)$ and we now show how they work.

1. Translation Shift the whole three-dimensional space along the vector v_0. The origin moves to (x_0, y_0, z_0). This vector v_0 is added to every point v in $\mathbf{R}^3$. Using homogeneous coordinates, the 4 by 4 matrix T shifts the whole space by v_0:

$$\textbf{\textit{Translation matrix}} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{bmatrix}.$$

Important: *Computer graphics works with row vectors.* We have row times matrix instead of matrix times column. You can quickly check that $[0\ \ 0\ \ 0\ \ 1]\,T = [x_0\ \ y_0\ \ z_0\ \ 1]$.

To move the points $(0, 0, 0)$ and (x, y, z) by v_0, change to homogeneous coordinates $(0, 0, 0, 1)$ and $(x, y, z, 1)$. Then multiply by T. A row vector times T gives a row vector. ***Every v moves to $v + v_0$***: $[x\ \ y\ \ z\ \ 1]\,T = [x + x_0\ \ y + y_0\ \ z + z_0\ \ 1]$.

The output tells where any v will move. (It goes to $v + v_0$.) Translation is now achieved by a matrix, which was impossible in $\mathbf{R}^3$.

2. Scaling To make a picture fit a page, we change its width and height. A Xerox copier will rescale a figure by 90%. In linear algebra, we multiply by .9 times the identity matrix. That matrix is normally 2 by 2 for a plane and 3 by 3 for a solid. In computer graphics, with homogeneous coordinates, the matrix is *one size larger:*

$$\textbf{\textit{Rescale the plane:}}\quad S = \begin{bmatrix} .9 & & \\ & .9 & \\ & & 1 \end{bmatrix} \qquad \textbf{\textit{Rescale a solid:}}\quad S = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Important: S is not cI. We keep the "1" in the lower corner. Then $[x, y, 1]$ times S is the correct answer in homogeneous coordinates. The origin stays in its normal position because $[0\ 0\ 1]S = [0\ 0\ 1]$.

If we change that 1 to c, the result is strange. ***The point* (cx, cy, cz, c) *is the same as*** $(x, y, z, 1)$. The special property of homogeneous coordinates is that *multiplying by cI does not move the point.* The origin in $\mathbf{R}^3$ has homogeneous coordinates $(0, 0, 0, 1)$ and $(0, 0, 0, c)$ for every nonzero c. This is the idea behind the word "homogeneous."

Scaling can be different in different directions. To fit a full-page picture onto a half-page, scale the y direction by $\frac{1}{2}$. To create a margin, scale the x direction by $\frac{3}{4}$. The graphics matrix is diagonal but not 2 by 2. It is 3 by 3 to rescale a plane and 4 by 4 to rescale a space:

$$\textbf{\textit{Scaling matrices}} \quad S = \begin{bmatrix} \frac{3}{4} & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & 1 \end{bmatrix}.$$

That last matrix S rescales the x, y, z directions by positive numbers c_1, c_2, c_3. The extra column in all these matrices leaves the extra 1 at the end of every vector.

Summary The scaling matrix S is the same size as the translation matrix T. They can be multiplied. To translate and then rescale, multiply $\boldsymbol{v}TS$. To rescale and then translate, multiply $\boldsymbol{v}ST$. Are those different? *Yes.*

The point (x, y, z) in $\mathbf{R}^3$ has homogeneous coordinates $(x, y, z, 1)$ in $\mathbf{P}^3$. This "projective space" is not the same as $\mathbf{R}^4$. It is still three-dimensional. To achieve such a thing, (cx, cy, cz, c) is the same point as $(x, y, z, 1)$. Those points of projective space $\mathbf{P}^3$ are really lines through the origin in $\mathbf{R}^4$.

Computer graphics uses ***affine*** transformations, *linear plus shift.* An affine transformation T is executed on $\mathbf{P}^3$ by a 4 by 4 matrix with a special fourth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} = \begin{bmatrix} T(1,0,0) & 0 \\ T(0,1,0) & 0 \\ T(0,0,1) & 0 \\ T(0,0,0) & 1 \end{bmatrix}.$$

The usual 3 by 3 matrix tells us three outputs, this tells four. The usual outputs come from the inputs $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. When the transformation is linear, three outputs reveal everything. When the transformation is affine, the matrix also contains the output from $(0, 0, 0)$. Then we know the shift.

3. Rotation A rotation in $\mathbf{R}^2$ or $\mathbf{R}^3$ is achieved by an orthogonal matrix Q. The determinant is $+1$. (With determinant -1 we get an extra reflection through a mirror.) Include the extra column when you use homogeneous coordinates!

$$\textbf{\textit{Plane rotation}} \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix rotates the plane around the origin. ***How would we rotate around a different point*** $(4,5)$? The answer brings out the beauty of homogeneous coordinates. ***Translate*** $(4,5)$ ***to*** $(0,0)$***, then rotate by*** θ***, then translate*** $(0,0)$ ***back to*** $(4,5)$:

$$vT_-RT_+ = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -5 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}.$$

I won't multiply. The point is to apply the matrices one at a time: v translates to vT_-, then rotates to vT_-R, and translates back to vT_-RT_+. Because each point $\begin{bmatrix} x & y & 1 \end{bmatrix}$ is a row vector, T_- acts first. The center of rotation $(4,5)$—otherwise known as $(4,5,1)$—moves first to $(0,0,1)$. Rotation doesn't change it. Then T_+ moves it back to $(4,5,1)$. All as it should be. The point $(4,6,1)$ moves to $(0,1,1)$, then turns by θ and moves back.

In three dimensions, every rotation Q turns around an axis. The axis doesn't move—it is a line of eigenvectors with $\lambda = 1$. Suppose the axis is in the z direction. The 1 in Q is to leave the z axis alone, the extra 1 in R is to leave the origin alone:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} & & & 0 \\ & Q & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now suppose the rotation is around the unit vector $\boldsymbol{a} = (a_1, a_2, a_3)$. With this axis $\boldsymbol{a}$, the rotation matrix Q which fits into R has three parts:

$$Q = (\cos\theta)I + (1-\cos\theta)\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}. \quad (1)$$

The axis doesn't move because $\boldsymbol{a}Q = \boldsymbol{a}$. When $\boldsymbol{a} = (0,0,1)$ is in the z direction, this Q becomes the previous Q—for rotation around the z axis.

The linear transformation Q always goes in the upper left block of R. Below it we see zeros, because rotation leaves the origin in place. When those are not zeros, the transformation is affine and the origin moves.

4. Projection In a linear algebra course, most planes go through the origin. In real life, most don't. A plane through the origin is a vector space. The other planes are affine spaces, sometimes called "flats." An affine space is what comes from translating a vector space.

We want to project three-dimensional vectors onto planes. Start with a plane through the origin, whose unit normal vector is $\boldsymbol{n}$. (We will keep $\boldsymbol{n}$ as a column vector.) The vectors in the plane satisfy $\boldsymbol{n}^{\mathrm{T}}v = 0$. ***The usual projection onto the plane is the matrix*** $\boldsymbol{I - nn^{\mathrm{T}}}$. To project a vector, multiply by this matrix. The vector $\boldsymbol{n}$ is projected to zero, and the in-plane vectors v are projected onto themselves:

$$(I - \boldsymbol{nn}^{\mathrm{T}})\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n}) = \boldsymbol{0} \quad \text{and} \quad (I - \boldsymbol{nn}^{\mathrm{T}})v = v - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}v) = v.$$

In homogeneous coordinates the projection matrix becomes 4 by 4 (but the origin doesn't move):

$$\textbf{\textit{Projection onto the plane}} \quad n^{\mathrm{T}}v = 0 \qquad P = \begin{bmatrix} & & & 0 \\ & I - nn^{\mathrm{T}} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now project onto a plane $n^{\mathrm{T}}(v - v_0) = 0$ that does *not* go through the origin. One point on the plane is v_0. This is an affine space (or a ***flat***). It is like the solutions to $Av = b$ when the right side is not zero. One particular solution v_0 is added to the nullspace—to produce a flat.

The projection onto the flat has three steps. Translate v_0 to the origin by T_-. Project along the n direction, and translate back along the row vector v_0:

$$\textbf{\textit{Projection onto a flat}} \qquad T_- P T_+ = \begin{bmatrix} I & 0 \\ -v_0 & 1 \end{bmatrix} \begin{bmatrix} I - nn^{\mathrm{T}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v_0 & 1 \end{bmatrix}.$$

I can't help noticing that T_- and T_+ are inverse matrices: translate and translate back. They are like the elementary matrices of Chapter 2.

The exercises will include reflection matrices, also known as *mirror matrices*. These are the fifth type needed in computer graphics. A reflection moves each point twice as far as a projection—***the reflection goes through the plane and out the other side***. So change the projection $I - nn^{\mathrm{T}}$ to $I - 2nn^{\mathrm{T}}$ for a mirror matrix.

The matrix P gave a "*parallel*" projection. All points move parallel to n, until they reach the plane. The other choice in computer graphics is a "*perspective*" projection. This is more popular because it includes foreshortening. With perspective, an object looks larger as it moves closer. Instead of staying parallel to n (and parallel to each other), the lines of projection come *toward the eye*—the center of projection. This is how we perceive depth in a two-dimensional photograph.

The basic problem of computer graphics starts with a scene and a viewing position. Ideally, the image on the screen is what the viewer would see. The simplest image assigns just one bit to every small picture element—called a ***pixel***. It is light or dark. This gives a black and white picture with no shading. You would not approve. In practice, we assign shading levels between 0 and 2^8 for three colors like red, green, and blue. That means $8 \times 3 = 24$ bits for each pixel. Multiply by the number of pixels, and a lot of memory is needed!

Physically, a *raster frame buffer* directs the electron beam. It scans like a television set. The quality is controlled by the number of pixels and the number of bits per pixel. In this area, one standard text is *Computer Graphics: Principles and Practices* by Foley, Van Dam, Feiner, and Hughes (Addison-Wesley, 1995). The newer books still use homogeneous coordinates to handle translations. My best references were notes by Ronald Goldman and by Tony DeRose.

■ REVIEW OF THE KEY IDEAS ■

1. Computer graphics needs shift operations $T(\boldsymbol{v}) = \boldsymbol{v}+\boldsymbol{v}_0$ as well as linear operations $T(\boldsymbol{v}) = A\boldsymbol{v}$.

2. A shift in $\mathbf{R}^n$ can be executed by a matrix of order $n + 1$, using homogeneous coordinates.

3. The extra component 1 in $[x\ y\ z\ 1]$ is preserved when all matrices have the numbers $0, 0, 0, 1$ as last column.

Problem Set 8.7

1 A typical point in $\mathbf{R}^3$ is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. The coordinate vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The coordinates of the point are (x, y, z).

This point in computer graphics is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ + **origin**. Its homogeneous coordinates are (, , ,). Other coordinates for the same point are (, , ,).

2 A linear transformation T is determined when we know $T(\boldsymbol{i}), T(\boldsymbol{j}), T(\boldsymbol{k})$. For an affine transformation we also need $T($_____$)$. The input point $(x, y, z, 1)$ is transformed to $xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) +$ _____.

3 Multiply the 4 by 4 matrix T for translation along $(1, 4, 3)$ and the matrix T_1 for translation along $(0, 2, 5)$. The product TT_1 is translation along _____.

4 Write down the 4 by 4 matrix S that scales by a constant c. Multiply ST and also TS, where T is translation by $(1, 4, 3)$. To blow up the picture around the center point $(1, 4, 3)$, would you use $\boldsymbol{v}ST$ or $\boldsymbol{v}TS$?

5 What scaling matrix S (in homogeneous coordinates, so 3 by 3) would produce a 1 by 1 square page from a standard 8.5 by 11 page?

6 What 4 by 4 matrix would move a corner of a cube to the origin and then multiply all lengths by 2? The corner of the cube is originally at $(1, 1, 2)$.

7 When the three matrices in equation 1 multiply the unit vector $\boldsymbol{a}$, show that they give $(\cos\theta)\boldsymbol{a}$ and $(1 - \cos\theta)\boldsymbol{a}$ and $\mathbf{0}$. Addition gives $\boldsymbol{a}Q = \boldsymbol{a}$ and the rotation axis is not moved.

8 If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$, multiply by the three matrices in 1 to get $(\cos\theta)\boldsymbol{b}$ and $\mathbf{0}$ and a vector perpendicular to $\boldsymbol{b}$. So $Q\boldsymbol{b}$ makes an angle θ with $\boldsymbol{b}$. ***This is rotation.***

9 What is the 3 by 3 projection matrix $I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$ onto the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0$? In homogeneous coordinates add $0, 0, 0, 1$ as an extra row and column in P.

10 With the same 4 by 4 matrix P, multiply T_-PT_+ to find the projection matrix onto the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 1$. The translation T_- moves a point on that plane (choose one) to $(0, 0, 0, 1)$. The inverse matrix T_+ moves it back.

11 Project $(3,3,3)$ onto those planes. Use P in Problem 9 and $T_- P T_+$ in Problem 10.

12 If you project a square onto a plane, what shape do you get?

13 If you project a cube onto a plane, what is the outline of the projection? Make the projection plane perpendicular to a diagonal of the cube.

14 The 3 by 3 mirror matrix that reflects through the plane $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v} = 0$ is $M = I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$. Find the reflection of the point $(3,3,3)$ in the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0$.

15 Find the reflection of $(3,3,3)$ in the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 1$. Take three steps $T_- M T_+$ using 4 by 4 matrices: translate by T_- so the plane goes through the origin, reflect the translated point $(3,3,3,1)T_-$ in that plane, then translate back by T_+.

16 The vector between the origin $(0,0,0,1)$ and the point $(x,y,z,1)$ is the difference $\boldsymbol{v} =$ _____. In homogeneous coordinates, vectors end in _____. So we add a _____ to a point, not a point to a point.

17 If you multiply only the *last* coordinate of each point to get (x,y,z,c), you rescale the whole space by the number _____. This is because the point (x,y,z,c) is the same as $(\ , \ , \ ,1)$.

Chapter 9

Numerical Linear Algebra

9.1 Gaussian Elimination in Practice

Numerical linear algebra is a struggle for *quick* solutions and also *accurate* solutions. We need efficiency but we have to avoid instability. In Gaussian elimination, the main freedom (always available) is to ***exchange equations***. This section explains when to exchange rows for the sake of speed, and when to do it for the sake of accuracy.

The key to accuracy is to avoid unnecessarily large numbers. Often that requires us to avoid small numbers! A small pivot generally means large multipliers (since we divide by the pivot). A good plan is "***partial pivoting***", to choose the *largest candidate* in each new column as the pivot. We will see why this strategy is built into computer programs.

Other row exchanges are done to save elimination steps. In practice, most large matrices are ***sparse***—almost all entries are zeros. Elimination is fastest when the equations are ordered to put those zeros (as far as possible) *outside the band of nonzeros.* Zeros inside the band "fill in" during elimination—the zeros are destroyed and don't help.

Section 9.2 is about instability that can't be avoided. It is built into the problem, and this sensitivity is measured by the "***condition number***". Then Section 9.3 describes how to solve $Ax = b$ by ***iterations***. Instead of direct elimination, the computer solves an easier equation many times. Each answer x_k leads to the next guess x_{k+1}. For good iterations, like **conjugate gradients**, the x_k converge quickly to $x = A^{-1}b$.

The Fastest Supercomputer

A new supercomputing record was announced by IBM and Los Alamos on May 20, 2008. The Roadrunner was the first to achieve a quadrillion (10^{15}) floating-point operations per second: *a petaflop machine*. The benchmark for this world record was a large dense linear system $Ax = b$: linear algebra.

The LINPACK software does elimination with partial pivoting. The biggest difference from this book is to organize the steps to use large submatrices and never single numbers. Roadrunner is a multicore Linux cluster with very remarkable processors, based on the

Cell Broadband Engine from Sony's PlayStation 3. The market for video games dwarfs scientific computing and led to astonishing acceleration in the chips.

This path to petascale is not the approach taken by IBM's BlueGene. A key issue was to count the standard quad-core processors that a petaflop machine would need: 32,000. The new architecture uses much less power, but its hybrid design has a price: a code needs three separate compilers and explicit instructions to move all the data. Please see the excellent article in *SIAM News* (**siam.org**, July 2008) and the details on **www.lanl.gov/roadrunner**.

The TOP500 project ranks the most powerful computer systems in the world. Roadrunner and BlueGene are #1 and #2 as this page is written in 2009.

Our thinking about matrix calculations is reflected in the highly optimized **BLAS** (*Basic Linear Algebra Subroutines*). They come at levels 1, 2, and 3:

1 Linear combinations of vectors $au + v$: $O(n)$ work

2 Matrix-vector multiplications $Au + v$: $O(n^2)$ work

3 Matrix-matrix multiplications $AB + C$: $O(n^3)$ work

Level 1 is a single elimination step (multiply row j by ℓ_{ij} and subtract from row i). Level 2 can eliminate a whole column at once. A high performance solver is rich in Level 3 BLAS (AB has $2n^3$ flops and $2n^2$ data, a good ratio of work to talk).

It is *data passing* and *storage retrieval* that limit the speed of parallel processing. The high-velocity cache between main memory and floating-point computation has to be fully used! Top speed demands a ***block matrix approach*** to elimination.

The big change, coming now, is parallel processing at the chip level.

Roundoff Error and Partial Pivoting

Up to now, any pivot (nonzero of course) was accepted. In practice a small pivot is dangerous. A catastrophe can occur when numbers of different sizes are added. Computers keep a fixed number of significant digits (say three decimals, for a very weak machine). The sum 10,000 + 1 is rounded off to 10,000. The "1" is completely lost. Watch how that changes the solution to this problem:

$$\begin{aligned} .0001u + v &= 1 \\ -u + v &= 0 \end{aligned} \qquad \text{starts with coefficient matrix} \qquad A = \begin{bmatrix} .0001 & 1 \\ -1 & 1 \end{bmatrix}.$$

If we accept .0001 as the pivot, elimination adds 10,000 times row 1 to row 2. Roundoff leaves

$$10{,}000v = 10{,}000 \qquad \text{instead of} \qquad 10{,}001v = 10{,}000.$$

The computed answer $v = 1$ is near the true $v = .9999$. But then back substitution puts the wrong v into the equation for u:

$$.0001\,u + 1 = 1 \qquad \text{instead of} \qquad .0001\,u + .9999 = 1.$$

The first equation gives $u = 0$. The correct answer (look at the second equation) is $u = 1.000$. By losing the "1" in the matrix, we have lost the solution. ***The change from* 10,001 *to* 10,000 *has changed the answer from* $u = 1$ *to* $u = 0$** (100% error!).

If we exchange rows, even this weak computer finds an answer that is correct to three places:

$$\begin{matrix} -u+v=0 \\ .0001u+v=1 \end{matrix} \longrightarrow \begin{matrix} -u+v=0 \\ v=1 \end{matrix} \longrightarrow \begin{matrix} u=1 \\ v=1. \end{matrix}$$

The original pivots were .0001 and 10,000—badly scaled. After a row exchange the exact pivots are -1 and 1.0001—well scaled. The computed pivots -1 and 1 come close to the exact values. Small pivots bring numerical instability, and the remedy is ***partial pivoting***. The kth pivot is decided when we reach and search column k:

Choose the largest number in row k or below. Exchange its row with row k.

The strategy of *complete pivoting* looks also in later columns for the largest pivot. It exchanges columns as well as rows. This expense is seldom justified, and all major codes use partial pivoting. Multiplying a row or column by a scaling constant can also be very worthwhile. *If the first equation above is* $u + 10{,}000v = 10{,}000$ *and we don't rescale, then* 1 *looks like a good pivot and we would miss the essential row exchange.*

For positive definite matrices, row exchanges are *not* required. It is safe to accept the pivots as they appear. Small pivots can occur, but the matrix is not improved by row exchanges. When its condition number is high, the problem is in the matrix and not in the code. In this case the output is unavoidably sensitive to the input.

The reader now understands how a computer actually solves $Ax = b$***—by elimination with partial pivoting.*** Compared with the theoretical description—***find*** A^{-1} ***and multiply*** $A^{-1}b$—the details took time. But in computer time, elimination is much faster. I believe this algorithm is also the best approach to the algebra of row spaces and nullspaces.

Operation Counts: Full Matrices and Band Matrices

Here is a practical question about cost. *How many separate operations are needed to solve* $Ax = b$ *by elimination*? This decides how large a problem we can afford.

Look first at A, which changes gradually into U. When a multiple of row 1 is subtracted from row 2, we do n operations. The first is a division by the pivot, to find the multiplier ℓ. For the other $n-1$ entries along the row, the operation is a "multiply-subtract". For convenience, we count this as a single operation. If you regard multiplying by ℓ and subtracting from the existing entry as two separate operations, *multiply all our counts by* 2.

The matrix A is n by n. The operation count applies to all $n-1$ rows below the first. Thus it requires n times $n-1$ operations, or $n^2 - n$, to produce zeros below the first pivot. *Check*: *All* n^2 *entries are changed, except the* n *entries in the first row.*

When elimination is down to k equations, the rows are shorter. We need only $k^2 - k$ operations (instead of $n^2 - n$) to clear out the column below the pivot. This is true for $1 \le k \le n$. The last step requires no operations ($1^2 - 1 = 0$), since the pivot is set and forward elimination is complete. The total count to reach U is the sum of $k^2 - k$ over all values of k from 1 to n:

$$(1^2 + \cdots + n^2) - (1 + \cdots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}.$$

Those are known formulas for the sum of the first n numbers and the sum of the first n squares. Substituting $n = 1$ into $n^3 - n$ gives zero. Substituting $n = 100$ gives a million minus a hundred—then divide by 3. (That translates into one second on a workstation.) We will ignore the last term n in comparison with the larger term n^3, to reach our main conclusion:

The multiply-subtract count for forward elimination (A to U, producing L) is $\frac{1}{3}n^3$.

That means $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions. Doubling n increases this cost by eight (because n is cubed). 100 equations are easy, 1000 are more expensive, 10000 dense equations are close to impossible. We need a faster computer or a lot of zeros or a new idea.

On the right side of the equations, the steps go much faster. We operate on single numbers, not whole rows. ***Each right side needs exactly n^2 operations.*** Down and back up we are solving two triangular systems, $Lc = b$ forward and $Ux = c$ backward. In back substitution, the last unknown needs only division by the last pivot. The equation above it needs two operations—substituting x_n and dividing by *its* pivot. The kth step needs k multiply-subtract operations, and the total for back substitution is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \tfrac{1}{2}n^2 \quad \text{operations.}$$

The forward part is similar. *The n^2 total exactly equals the count for multiplying $A^{-1}b$!* This leaves Gaussian elimination with two big advantages over $A^{-1}b$:

1 Elimination requires $\frac{1}{3}n^3$ compared to n^3 for A^{-1}.

2 If A is *banded* so are L and U. But A^{-1} is full of nonzeros.

Band Matrices

These counts are improved when A has "*good zeros*". A good zero is an entry that remains zero in L and U. ***The best zeros are at the beginning of a row.*** They require no elimination steps (the multipliers are zero). So we also find those same good zeros in L. That is especially clear for this *tridiagonal matrix* A:

$$\begin{array}{l} \textbf{Tridiagonal} \\ \textbf{Bidiagonal} \\ \textbf{times} \\ \textbf{bidiagonal} \end{array} \quad \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

Rows 3 and 4 of A begin with zeros. No multiplier is needed, so L has the same zeros. Also columns 3 and 4 *start* with zeros. When a multiple of row 1 is subtracted from row 2, no calculation is required beyond the second column. The rows are short. They stay short! Figure 9.1 shows how a band matrix A has band factors L and U.

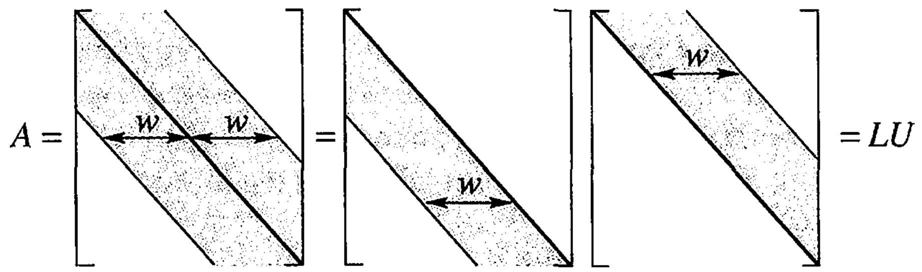


Figure 9.1: $A = L\,U$ for a band matrix. Good zeros in A *stay zero* in L and U.

These zeros lead to a complete change in the operation count, for "half-bandwidth" w:

A band matrix has $a_{ij} = 0$ when $|i - j| > w$.

Thus $w = 1$ for a diagonal matrix, $w = 2$ for tridiagonal, $w = n$ for dense. The length of the pivot row is at most w. There are no more than $w - 1$ nonzeros below any pivot. Each stage of elimination is complete after $w(w-1)$ operations, and *the band structure survives.* There are n columns to clear out. Therefore:

Elimination on a band matrix (A to L and U) needs less than $w^2 n$ operations.

For a band matrix, the count is proportional to n instead of n^3. It is also proportional to w^2. A full matrix has $w = n$ and we are back to n^3. For an exact count, remember that the bandwidth drops below w in the lower right corner (not enough space):

$$\textbf{Band}\quad \frac{w(w-1)(3n-2w+1)}{3} \qquad \textbf{Dense}\quad \frac{n(n-1)(n+1)}{3} = \frac{n^3-n}{3}$$

On the right side, to find $\boldsymbol{x}$ from $\boldsymbol{b}$, the cost is about $2wn$ (compared to the usual n^2). *Main point*: *For a band matrix the operation counts are* ***proportional to*** $\boldsymbol{n}$. This is extremely fast. A tridiagonal matrix of order 10,000 is very cheap, provided *we don't compute* A^{-1}. That inverse matrix has no zeros at all:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We are actually worse off knowing A^{-1} than knowing L and U. Multiplication by A^{-1} needs the full n^2 steps. Solving $Lc = \boldsymbol{b}$ and $Ux = c$ needs only $2wn$. A band structure is very common in practice, when the matrix reflects connections between near neighbors: $a_{13} = 0$ and $a_{14} = 0$ because 1 is not a neighbor of 3 and 4.

We close with counts for Gauss-Jordan and Gram-Schmidt-Householder:

A^{-1} costs n^3 multiply-subtract steps. ***QR costs $\frac{2}{3}n^3$ steps.***

Start with $AA^{-1} = I$. The jth column of A^{-1} solves $Ax_j = j$th column of I. The left side costs $\frac{1}{3}n^3$ as usual. (This is a one-time cost! L and U are not repeated.) The special saving for the jth column of I comes from its first $j-1$ zeros. No work is required on the right side until elimination reaches row j. The forward cost is $\frac{1}{2}(n-j)^2$ instead of $\frac{1}{2}n^2$. Summing over j, the total for forward elimination on the n right sides is $\frac{1}{6}n^3$. The final multiply-subtract count for A^{-1} is n^3 if we actually want the inverse:

$$\textbf{For}\,A^{-1} \qquad \frac{n^3}{3}\ (L \text{ and } U) + \frac{n^3}{6}\ (\text{forward}) \ + n\left(\frac{n^2}{2}\right)(\text{back substitutions}) \ = \boldsymbol{n^3}. \quad (1)$$

Orthogonalization *(A to Q)*: The key difference from elimination is that *each multiplier is decided by a dot product.* That takes n operations, where elimination just divides by the pivot. Then there are n "multiply-subtract" operations to remove from column k its projection along column $j<k$ (see Section 4.4). The combined cost is $2n$ where for elimination it is n. This factor 2 is the price of orthogonality. We are changing a dot product to zero where elimination changes an entry to zero.

Caution To judge a numerical algorithm, it is **not enough** to count the operations. Beyond "flop counting" is a study of stability (Householder wins) and the flow of data.

Reordering Sparse Matrices

In discussing band matrices, we assumed a constant width w. The rows were in an optimal order. But for most sparse matrices in real computations, the width of the band is *not constant* and there are many zeros inside the band. Those zeros can fill in as elimination proceeds—they are lost. We need to ***renumber the equations to reduce fill-in***, and thereby speed up elimination.

Generally speaking, we want to move zeros to early rows and columns. Later rows and columns are shorter anyway. The "approximate minimum degree" algorithm in sparse MATLAB is *greedy*—it chooses the row to eliminate without counting all the consequences. We may reach a nearly full matrix near the end, but the total operation count to reach LU is still much smaller. To renumber for an absolute minimum of nonzeros in L and U is an NP-hard problem, much too expensive, and **amd** is a good compromise.

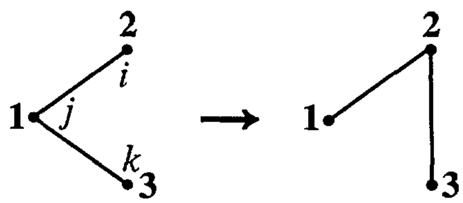
We only need the *positions* of the nonzeros, not their exact values. Think of the n rows as n nodes in a graph. *Node i is connected to node j if $a_{ij} \neq 0$.* Watch to see how elimination can create a new edge from i to k. This means that a zero is filled in, which we are trying to avoid:

When a_{kj} is eliminated, a multiple of the pivot row $j=1$ is subtracted from row $k=3$.

If a_{ji} was nonzero in row j, then a_{ki} becomes nonzero in the new row k. A new edge.

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & \mathbf{0} \\ -2 & \mathbf{0} & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$a_{32} = \mathbf{0}$ $\qquad$ $a_{32} = \mathbf{2}$



$a_{32} = \mathbf{0}$ **before** $\qquad$ $a_{32} \neq \mathbf{0}$ **after**

In this example, the 1's change the 0's into 2's. Those entries fill in.

The graph shows each step—look at the eliminationmovie on **math.mit.edu/18086.** The command **nnz**(L) counts the nonzero multipliers in the lower triangular L, **find** (L) will list them, and **spy**(L) shows them all.

The matrix in the movie is the 2D version of our $-1, 2, -1$ matrix. Instead of second differences along a line, the matrix has x and y differences on a plane grid. Each point is connected to its four nearest neighbors. But it is impossible to number all the points so that neighbors stay together. If we number by rows of the grid, there is a long wait to come around to the gridpoint above.

The goal of **colamd** and **symamd** is a better ordering (permutation P) that reduces fill-in for PA and PAP^{T}—*by choosing the pivot with the fewest nonzeros below it.*

Fast Orthogonalization

There are three ways to reach the important factorization $A = QR$. Gram-Schmidt works to find the orthonormal vectors in Q. Then R is upper triangular because of the order of Gram-Schmidt steps. Now we look at better methods (Householder and Givens), which use a product of specially simple Q's that we *know* are orthogonal.

Elimination gives $A = LU$, orthogonalization gives $A = QR$. We don't want a triangular L, we want an orthogonal Q. L is a product of E's, with 1's on the diagonal and the multiplier ℓ_{ij} below. *Q will be a product of orthogonal matrices.*

There are two simple orthogonal matrices to take the place of the E's. The ***reflection matrices*** $I - 2uu^{\mathrm{T}}$ are named after Householder. The ***plane rotation matrices*** are named after Givens. The simple matrix that rotates the xy plane by θ is Q_{21}:

Givens rotation $$Q_{21} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use Q_{21} the way you used E_{21}, to produce a zero in the $(2, 1)$ position. That determines the angle θ. Bill Hager gives this example in *Applied Numerical Linear Algebra*:

$$Q_{21}A = \begin{bmatrix} .6 & .8 & 0 \\ -.8 & .6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 & -153 & 114 \\ 120 & -79 & -223 \\ 200 & -40 & 395 \end{bmatrix} = \begin{bmatrix} 150 & -155 & -110 \\ 0 & 75 & -225 \\ 200 & -40 & 395 \end{bmatrix}.$$

The zero came from $-.8(90) + .6(120)$. No need to find θ, what we needed was $\cos\theta$:

$$\cos\theta = \frac{90}{\sqrt{90^2 + 120^2}} \quad \text{and} \quad \sin\theta = \frac{-120}{\sqrt{90^2 + 120^2}}. \tag{2}$$

Now we attack the $(3, 1)$ entry. The rotation will be in rows and columns 3 and 1. The numbers $\cos\theta$ and $\sin\theta$ are determined from 150 and 200, instead of 90 and 120.

$$Q_{31}Q_{21}A = \begin{bmatrix} .6 & 0 & .8 \\ 0 & 1 & 0 \\ -.8 & 0 & .6 \end{bmatrix} \begin{bmatrix} 150 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 200 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 75 & -225 \\ 0 & 100 & 325 \end{bmatrix}.$$

One more step to R. The $(3,2)$ entry has to go. The numbers $\cos\theta$ and $\sin\theta$ now come from 75 and 100. The rotation is now in rows and columns 2 and 3:

$$Q_{32}Q_{31}Q_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6 & .8 \\ 0 & -.8 & .6 \end{bmatrix} \begin{bmatrix} 250 & -125 & \cdot \\ 0 & 75 & \cdot \\ 0 & 100 & \cdot \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 125 & 125 \\ 0 & 0 & 375 \end{bmatrix}.$$

We have reached the upper triangular R. What is Q? Move the plane rotations Q_{ij} to the other side to find $A = QR$—just as you moved the elimination matrices E_{ij} to the other side to find $A = LU$:

$$Q_{32}Q_{31}Q_{21}A = R \qquad \text{means} \qquad A = (Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1})R = QR. \tag{3}$$

The inverse of each Q_{ij} is Q_{ij}^{T} (rotation through $-\theta$). The inverse of E_{ij} was not an orthogonal matrix! LU and QR are similar but not the same.

Householder reflections are faster because each one clears out a whole column below the diagonal. Watch how the first column $\boldsymbol{a}_1$ of A becomes column $\boldsymbol{r}_1$ of R:

Reflection by H_1
$H_1 = I - 2\boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{T}}$

$$H_1\,\boldsymbol{a}_1 = \begin{bmatrix} \|\boldsymbol{a}_1\| \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -\|\boldsymbol{a}_1\| \\ 0 \\ \cdot \\ 0 \end{bmatrix} = \boldsymbol{r}_1. \tag{4}$$

The length was not changed, and $\boldsymbol{u}_1$ is in the direction of $\boldsymbol{a}_1 - \boldsymbol{r}_1$. We have $n-1$ entries in the unit vector $\boldsymbol{u}_1$ to get $n-1$ zeros in $\boldsymbol{r}_1$. (Rotations had one angle θ to get one zero.) When we reach column k, $n-k$ available choices in the unit vector $\boldsymbol{u}_k$ lead to $n-k$ zeros in $\boldsymbol{r}_k$. *We just store the $\boldsymbol{u}$'s and $\boldsymbol{r}$'s to know Q and R:*

$$\textbf{Inverse of } H_i \textbf{ is } H_i \quad (H_{n-1}\dots H_1)A = R \ \text{ means } \ A = (H_1\dots H_{n-1})R = QR. \tag{5}$$

This is how LAPACK improves on Gram-Schmidt. Q is *exactly* orthogonal.

Section 9.3 explains how $A = QR$ is used in the other big computation of linear algebra—***the eigenvalue problem***. The factors QR are reversed to give $A_1 = RQ$ which is $Q^{-1}AQ$. Since A_1 is similar to A, the eigenvalues are unchanged. Then A_1 is factored into Q_1R_1, and reversing the factors gives A_2. Amazingly, the entries below the diagonal get smaller in $A_1, A_2, A_3, \dots$ and we can identify the eigenvalues. This is the "QR method" for $A\boldsymbol{x} = \lambda\boldsymbol{x}$, a big success of numerical linear algebra.

Problem Set 9.1

1 Find the two pivots with and without row exchange to maximize the pivot:

$$A = \begin{bmatrix} .001 & 0 \\ 1 & 1000 \end{bmatrix}.$$

With row exchanges to maximize pivots, why are no entries of L larger than 1? Find a 3 by 3 matrix A with all $|a_{ij}| \le 1$ and $|\ell_{ij}| \le 1$ but third pivot $= 4$.

2 Compute the exact inverse of the Hilbert matrix A by elimination. Then compute A^{-1} again by rounding all numbers to three figures:

$$\textbf{Ill-conditioned matrix} \qquad A = \mathsf{hilb}(3) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

3 For the same A compute $\boldsymbol{b} = A\boldsymbol{x}$ for $\boldsymbol{x} = (1,1,1)$ and $\boldsymbol{x} = (0,6,-3.6)$. A small change $\Delta\boldsymbol{b}$ produces a large change $\Delta\boldsymbol{x}$.

4 Find the eigenvalues (by computer) of the 8 by 8 Hilbert matrix $a_{ij} = 1/(i+j-1)$. In the equation $A\boldsymbol{x} = \boldsymbol{b}$ with $\|\boldsymbol{b}\| = 1$, how large can $\|\boldsymbol{x}\|$ be? If $\boldsymbol{b}$ has roundoff error less than 10^{-16}, how large an error can this cause in $\boldsymbol{x}$? See Section 9.2.

5 For back substitution with a band matrix (width w), show that the number of multiplications to solve $U\boldsymbol{x} = \boldsymbol{c}$ is approximately wn.

6 If you know L and U and Q and R, is it faster to solve $LU\boldsymbol{x} = \boldsymbol{b}$ or $QR\boldsymbol{x} = \boldsymbol{b}$?

7 Show that the number of multiplications to invert an upper triangular n by n matrix is about $\frac{1}{6}n^3$. Use back substitution on the columns of I, upward from 1's.

8 Choosing the largest available pivot in each column (partial pivoting), factor each A into $PA = LU$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

9 Put 1's on the three central diagonals of a 4 by 4 tridiagonal matrix. Find the cofactors of the six zero entries. Those entries are nonzero in A^{-1}.

10 (Suggested by C. Van Loan.) Find the LU factorization and solve by elimination when $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$:

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon \\ 2 \end{bmatrix}.$$

The true $\boldsymbol{x}$ is $(1,1)$. Make a table to show the error for each ε. Exchange the two equations and solve again—the errors should almost disappear.

11 (a) Choose $\sin\theta$ and $\cos\theta$ to triangularize A, and find R:

$$\textbf{Givens rotation} \qquad Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = R.$$

(b) Choose $\sin\theta$ and $\cos\theta$ to make QAQ^{-1} triangular. What are the eigenvalues?

12 When A is multiplied by a plane rotation Q_{ij}, which n^2 entries of A are changed? When $Q_{ij}A$ is multiplied on the right by Q_{ij}^{-1}, which entries are changed now?

13 How many multiplications and how many additions are used to compute $Q_{ij}A$? Careful organization of the whole sequence of rotations gives $\frac{2}{3}n^3$ multiplications and $\frac{2}{3}n^3$ additions—the same as for QR by reflectors and twice as many as for LU.

Challenge Problems

14 **(Turning a robot hand)** The robot produces any 3 by 3 rotation A from plane rotations around the x, y, z axes. Then $Q_{32}Q_{31}Q_{21}A = R$, where A is orthogonal so R is I! The three robot turns are in $A = Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1}$. The three angles are "Euler angles" and $\det Q = 1$ to avoid reflection. Start by choosing $\cos\theta$ and $\sin\theta$ so that

$$Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{is zero in the } (2,1) \text{ position.}$$

15 Create the 10 by 10 second difference matrix $K = \textbf{toeplitz}([2\ -1\ \textbf{zeros}(1,8)])$. Permute rows and columns randomly by $KK = K(\textbf{randperm}(10), \textbf{randperm}(10))$. Factor by $[L, U] = \textbf{lu}(K)$ and $[LL, UU] = \textbf{lu}(KK)$, and count nonzeros by $\textbf{nnz}(L)$ and $\textbf{nnz}(LL)$. In this case L is in perfect tridiagonal order, but not LL.

16 Another ordering for this matrix K colors the meshpoints alternately red and black. This permutation P changes the normal $1, \ldots, 10$ to $1, 3, 5, 7, 9, 2, 4, 6, 8, 10$:

$$\textbf{Red-black ordering} \qquad PKP^{\mathrm{T}} = \begin{bmatrix} 2I & D \\ D^{\mathrm{T}} & 2I \end{bmatrix}. \qquad \textit{Find the matrix } D.$$

So many interesting experiments are possible. If you send good ideas they can go on the linear algebra website math.mit.edu/linearalgebra. I also recommend learning the command $B = \textbf{sparse}(A)$, after which $\textbf{find}(B)$ will list the nonzero entries and $\boldsymbol{\ell}\textbf{u}(B)$ will factor B using that sparse format for L and U. Only the nonzeros are computed, where ordinary (dense) MATLAB computes all the zeros too.

17 Jeff Stuart has created a student activity that brilliantly demonstrates ill-conditioning:

$$\begin{bmatrix} 1 & 1.0001 \\ 1 & 1.0000 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.0001 + e \\ 3.0000 + E \end{bmatrix} \quad \begin{array}{l} \textbf{With errors} \\ \boldsymbol{e} \textbf{ and } \boldsymbol{E} \end{array} \quad \begin{array}{l} x = 2 - 10000(e - E) \\ y = 1 + 10000(e - E) \end{array}$$

The algebra shows how errors $\boldsymbol{e}$ and $\boldsymbol{E}$ are amplified by 10000 unless $\boldsymbol{e} = \boldsymbol{E}$.

As always, the solution of a 2 by 2 system is the meeting point of two lines. The neat idea is to replace mathematical lines by ***long sticks held by students***. The sticks for these two equations are almost parallel, and A is almost singular. Perpendicular sticks come from well-conditioned equations.

In Stuart's *Shake a Stick* activity, the students plot where the sticks cross (after multiple shakes). See ***www.plu.edu/~stuartjl*** for the wild movements of that crossing point (x, y), when the sticks are nearly parallel.

9.2 Norms and Condition Numbers

How do we measure the size of a matrix? For a vector, the length is $\|x\|$. For a matrix, ***the norm is*** $\|A\|$. This word "*norm*" is sometimes used for vectors, instead of length. It is always used for matrices, and there are many ways to measure $\|A\|$. We look at the requirements on all "matrix norms" and then choose one.

Frobenius squared all the $|a_{ij}|^2$ and added; his norm $\|A\|_F$ is the square root. This treats A like a long vector with n^2 components: sometimes useful, but not the choice here.

I prefer to start with a vector norm. The triangle inequality says that $\|x + y\|$ is not greater than $\|x\| + \|y\|$. The length of $2x$ or $-2x$ is doubled to $2\|x\|$. The same rules will apply to matrix norms:

$$\|A + B\| \le \|A\| + \|B\| \quad \text{and} \quad \|cA\| = |c|\,\|A\|. \tag{1}$$

The second requirements for a matrix norm are new, because matrices multiply. The norm $\|A\|$ controls the growth from x to Ax, and from B to AB:

$$\textbf{Growth factor } \|A\| \qquad \|Ax\| \le \|A\|\,\|x\| \quad \text{and} \quad \|AB\| \le \|A\|\,\|B\|. \tag{2}$$

This leads to a natural way to define $\|A\|$, the norm of a matrix:

$$\textit{The norm of A is the largest ratio } \|Ax\|/\|x\|: \qquad \|A\| = \max_{x \ne 0} \frac{\|Ax\|}{\|x\|}. \tag{3}$$

$\|Ax\|/\|x\|$ is never larger than $\|A\|$ (its maximum). This says that $\|Ax\| \le \|A\|\,\|x\|$.

Example 1 If A is the identity matrix I, the ratios are $\|x\|/\|x\|$. Therefore $\|I\| = 1$. If A is an orthogonal matrix Q, lengths are again preserved: $\|Qx\| = \|x\|$. The ratios still give $\|Q\| = 1$. An orthogonal Q is good to compute with: errors don't grow.

Example 2 The norm of a diagonal matrix is its largest entry (using absolute values):

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{has norm} \quad \|A\| = 3. \quad \text{The eigenvector} \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{has} \quad Ax = 3x.$$

The eigenvalue is 3. For this A (but not all A), the largest eigenvalue equals the norm.

For a positive definite symmetric matrix the norm is $\|A\| = \lambda_{\max}(A)$.

Choose x to be the eigenvector with maximum eigenvalue. Then $\|Ax\|/\|x\|$ equals $\lambda_{\max}$. The point is that no other x can make the ratio larger. The matrix is $A = Q\Lambda Q^T$, and the orthogonal matrices Q and Q^T leave lengths unchanged. So the ratio to maximize is really $\|\Lambda x\|/\|x\|$. The norm is the largest eigenvalue in the diagonal Λ.

Symmetric matrices Suppose A is symmetric but not positive definite. $A = Q\Lambda Q^T$ is still true. Then the norm is the largest of $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$. We take absolute values, because the norm is only concerned with length. For an eigenvector $\|Ax\| = \|\lambda x\| = |\lambda|$ times $\|x\|$. The x that gives the maximum ratio is the eigenvector for the maximum $|\lambda|$.

Unsymmetric matrices If A is not symmetric, its eigenvalues may not measure its true size. *The norm can be larger than any eigenvalue.* A very unsymmetric example has $\lambda_1 = \lambda_2 = 0$ but its norm is not zero:

$$\|A\| > \lambda_{\max} \quad A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{has norm} \quad \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = 2.$$

The vector $x = (0, 1)$ gives $Ax = (2, 0)$. The ratio of lengths is $2/1$. This is the maximum ratio $\|A\|$, even though x is not an eigenvector.

It is the ***symmetric matrix*** $A^{\mathrm{T}}A$, not the unsymmetric A, that has eigenvector $x = (0, 1)$. The norm is really decided by ***the largest eigenvalue of*** $A^{\mathrm{T}}A$:

The norm of A (symmetric or not) ***is the square root of*** $\lambda_{\max}(A^{\mathrm{T}}A)$:

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \lambda_{\max}(A^{\mathrm{T}}A). \tag{4}$$

The unsymmetric example with $\lambda_{\max}(A) = 0$ has $\lambda_{\max}(A^{\mathrm{T}}A) = 4$:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ leads to } A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \text{ with } \lambda_{\max} = 4. \text{ So the norm is } \|A\| = \sqrt{4}.$$

For any A Choose x to be the eigenvector of $A^{\mathrm{T}}A$ with largest eigenvalue $\lambda_{\max}$. The ratio in equation (4) is $x^{\mathrm{T}}A^{\mathrm{T}}Ax = x^{\mathrm{T}}(\lambda_{\max})x$ divided by $x^{\mathrm{T}}x$. This is $\lambda_{\max}$.

No x can give a larger ratio. The symmetric matrix $A^{\mathrm{T}}A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal eigenvectors $q_1, q_2, \ldots, q_n$. Every x is a combination of those vectors. Try this combination in the ratio and remember that $q_i^{\mathrm{T}}q_j = 0$:

$$\frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \frac{(c_1q_1 + \cdots + c_nq_n)^{\mathrm{T}}(c_1\lambda_1q_1 + \cdots + c_n\lambda_nq_n)}{(c_1q_1 + \cdots + c_nq_n)^{\mathrm{T}}(c_1q_1 + \cdots + c_nq_n)} = \frac{c_1^2\lambda_1 + \cdots + c_n^2\lambda_n}{c_1^2 + \cdots + c_n^2}.$$

The maximum ratio $\lambda_{\max}$ is when all c's are zero, except the one that multiplies $\lambda_{\max}$.

Note 1 The ratio in equation (4) is the ***Rayleigh quotient*** for the symmetric matrix $A^{\mathrm{T}}A$. Its maximum is the largest eigenvalue $\lambda_{\max}(A^{\mathrm{T}}A)$. The minimum ratio is $\lambda_{\min}(A^{\mathrm{T}}A)$. If you substitute any vector x into the Rayleigh quotient $x^{\mathrm{T}}A^{\mathrm{T}}Ax/x^{\mathrm{T}}x$, you are guaranteed to get a number between $\lambda_{\min}(A^{\mathrm{T}}A)$ and $\lambda_{\max}(A^{\mathrm{T}}A)$.

Note 2 The norm $\|A\|$ equals the ***largest singular value*** $\sigma_{\max}$ of A. The singular values $\sigma_1, \ldots, \sigma_r$ are the square roots of the positive eigenvalues of $A^{\mathrm{T}}A$. So certainly $\sigma_{\max} = (\lambda_{\max})^{1/2}$. Since U and V are orthogonal in $A = U\Sigma V^{\mathrm{T}}$, the norm is $\|A\| = \sigma_{\max}$.

The Condition Number of A

Section 9.1 showed that roundoff error can be serious. Some systems are sensitive, others are not so sensitive. The sensitivity to error is measured by the ***condition number***. This is the first chapter in the book which intentionally introduces errors. We want to estimate how much they change $\boldsymbol{x}$.

The original equation is $A\boldsymbol{x} = \boldsymbol{b}$. Suppose the right side is changed to $\boldsymbol{b} + \Delta\boldsymbol{b}$ because of roundoff or measurement error. The solution is then changed to $\boldsymbol{x} + \Delta\boldsymbol{x}$. Our goal is to estimate the change $\Delta\boldsymbol{x}$ in the solution from the change $\Delta\boldsymbol{b}$ in the equation. Subtraction gives the *error equation* $A(\Delta\boldsymbol{x}) = \Delta\boldsymbol{b}$:

$$\text{Subtract } A\boldsymbol{x} = \boldsymbol{b} \text{ from } A(\boldsymbol{x} + \Delta\boldsymbol{x}) = \boldsymbol{b} + \Delta\boldsymbol{b} \quad \text{to find} \quad \boxed{A(\Delta\boldsymbol{x}) = \Delta\boldsymbol{b}.} \tag{5}$$

The error is $\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b}$. It is large when A^{-1} is large (then A is nearly singular). The error $\Delta\boldsymbol{x}$ is especially large when $\Delta\boldsymbol{b}$ points in the worst direction—which is amplified most by A^{-1}. ***The worst error has*** $\|\Delta\boldsymbol{x}\| = \|A^{-1}\|\,\|\Delta\boldsymbol{b}\|$.

This error bound $\|A^{-1}\|$ has one serious drawback. If we multiply A by 1000, then A^{-1} is divided by 1000. The matrix looks a thousand times better. But a simple rescaling cannot change the reality of the problem. It is true that $\Delta\boldsymbol{x}$ will be divided by 1000, but so will the exact solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The ***relative error*** $\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$ will stay the same. It is this relative change in $\boldsymbol{x}$ that should be compared to the relative change in $\boldsymbol{b}$.

Comparing relative errors will now lead to the "condition number" $c = \|A\|\,\|A^{-1}\|$. Multiplying A by 1000 does not change this number, because A^{-1} is divided by 1000 and the condition number c stays the same. It measures the sensitivity of $A\boldsymbol{x} = \boldsymbol{b}$.

The solution error is less than $c = \|A\|\,\|A^{-1}\|$ ***times the problem error:***

$$\textbf{Condition number } c \qquad \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \le c\,\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}. \tag{6}$$

If the problem error is ΔA (error in A instead of $\boldsymbol{b}$), still c controls $\Delta\boldsymbol{x}$:

$$\textbf{Error } \Delta A \textbf{ in } A \qquad \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x} + \Delta\boldsymbol{x}\|} \le c\,\frac{\|\Delta A\|}{\|A\|}. \tag{7}$$

Proof The original equation is $\boldsymbol{b} = A\boldsymbol{x}$. The error equation (5) is $\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b}$. Apply the key property $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$ of matrix norms:

$$\|\boldsymbol{b}\| \le \|A\|\,\|\boldsymbol{x}\| \qquad \text{and} \qquad \|\Delta\boldsymbol{x}\| \le \|A^{-1}\|\,\|\Delta\boldsymbol{b}\|.$$

Multiply the left sides to get $\|\boldsymbol{b}\|\,\|\Delta\boldsymbol{x}\|$, and multiply the right sides to get $c\|\boldsymbol{x}\|\,\|\Delta\boldsymbol{b}\|$. Divide both sides by $\|\boldsymbol{b}\|\,\|\boldsymbol{x}\|$. The left side is now the relative error $\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$. The right side is now the upper bound in equation (6).

The same condition number $c = \|A\|\,\|A^{-1}\|$ appears when the error is in the matrix. We have ΔA instead of $\Delta \boldsymbol{b}$ in the error equation:

Subtract $A\boldsymbol{x} = \boldsymbol{b}$ from $(A + \Delta A)(\boldsymbol{x} + \Delta \boldsymbol{x}) = \boldsymbol{b}$ to find $A(\Delta \boldsymbol{x}) = -(\Delta A)(\boldsymbol{x} + \Delta \boldsymbol{x})$.

Multiply the last equation by A^{-1} and take norms to reach equation (7):

$$\|\Delta \boldsymbol{x}\| \le \|A^{-1}\|\,\|\Delta A\|\,\|\boldsymbol{x} + \Delta \boldsymbol{x}\| \quad \text{or} \quad \frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x} + \Delta \boldsymbol{x}\|} \le \|A\|\,\|A^{-1}\|\frac{\|\Delta A\|}{\|A\|}.$$

Conclusion Errors enter in two ways. They begin with an error ΔA or $\Delta \boldsymbol{b}$—a wrong matrix or a wrong $\boldsymbol{b}$. This problem error is amplified (a lot or a little) into the solution error $\Delta \boldsymbol{x}$. That error is bounded, relative to $\boldsymbol{x}$ itself, by the condition number c.

The error $\Delta \boldsymbol{b}$ depends on computer roundoff and on the original measurements of $\boldsymbol{b}$. The error ΔA also depends on the elimination steps. Small pivots tend to produce large errors in L and U. Then $L + \Delta L$ times $U + \Delta U$ equals $A + \Delta A$. When ΔA or the condition number is very large, the error $\Delta \boldsymbol{x}$ can be unacceptable.

Example 3 When A is symmetric, $c = \|A\|\,\|A^{-1}\|$ comes from the eigenvalues:

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \text{ has norm } 6. \qquad A^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ has norm } \tfrac{1}{2}.$$

This A is symmetric positive definite. Its norm is $\lambda_{\max} = 6$. The norm of A^{-1} is $1/\lambda_{\min} = \frac{1}{2}$. Multiplying norms gives the *condition number* $\|A\|\,\|A^{-1}\| = \lambda_{\max}/\lambda_{\min}$:

$$\textbf{Condition number for positive definite } \boldsymbol{A} \qquad c = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{6}{2} = 3.$$

Example 4 Keep the same A, with eigenvalues 6 and 2. To make $\boldsymbol{x}$ small, choose $\boldsymbol{b}$ along the first eigenvector $(1, 0)$. To make $\Delta \boldsymbol{x}$ large, choose $\Delta \boldsymbol{b}$ along the second eigenvector $(0, 1)$. Then $\boldsymbol{x} = \frac{1}{6}\boldsymbol{b}$ and $\Delta \boldsymbol{x} = \frac{1}{2}\Delta \boldsymbol{b}$. The ratio $\|\Delta \boldsymbol{x}\|/\|\boldsymbol{x}\|$ is exactly $c = 3$ times the ratio $\|\Delta \boldsymbol{b}\|/\|\boldsymbol{b}\|$.

This shows that the worst error allowed by the condition number $\|A\|\,\|A^{-1}\|$ can actually happen. Here is a useful rule of thumb, experimentally verified for Gaussian elimination: *The computer can lose* $\log c$ *decimal places to roundoff error.*

Problem Set 9.2

1 Find the norms $\|A\| = \lambda_{\max}$ and condition numbers $c = \lambda_{\max}/\lambda_{\min}$ of these positive definite matrices:

$$\begin{bmatrix} .5 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

2 Find the norms and condition numbers from the square roots of $\lambda_{\max}(A^{\mathrm{T}}A)$ and $\lambda_{\min}(A^{\mathrm{T}}A)$. Without positive definiteness in A, we go to $A^{\mathrm{T}}A$!

$$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

3 Explain these two inequalities from the definitions (3) of $\|A\|$ and $\|B\|$:

$$\|ABx\| \leq \|A\|\,\|Bx\| \leq \|A\|\,\|B\|\,\|x\|.$$

From the ratio of $\|ABx\|$ to $\|x\|$, deduce that $\|AB\| \leq \|A\|\,\|B\|$. This is the key to using matrix norms. The norm of A^n is never larger than $\|A\|^n$.

4 Use $\|AA^{-1}\| \leq \|A\|\,\|A^{-1}\|$ to prove that the condition number is at least 1.

5 Why is I the only symmetric positive definite matrix that has $\lambda_{\max} = \lambda_{\min} = 1$? Then the only other matrices with $\|A\| = 1$ and $\|A^{-1}\| = 1$ must have $A^{\mathrm{T}}A = I$. Those are _____ matrices: perfectly conditioned.

6 Orthogonal matrices have norm $\|Q\| = 1$. If $A = QR$ show that $\|A\| \leq \|R\|$ and also $\|R\| \leq \|A\|$. Then $\|A\| = \|Q\|\,\|R\|$. Find an example of $A = LU$ with $\|A\| < \|L\|\|U\|$.

7 (a) Which famous inequality gives $\|(A+B)x\| \leq \|Ax\| + \|Bx\|$ for every x?

(b) Why does the definition (3) of matrix norms lead to $\|A+B\| \leq \|A\| + \|B\|$?

8 Show that if λ is any eigenvalue of A, then $|\lambda| \leq \|A\|$. Start from $Ax = \lambda x$.

9 The "*spectral radius*" $\rho(A) = |\lambda_{\max}|$ is the largest absolute value of the eigenvalues. Show with 2 by 2 examples that $\rho(A+B) \leq \rho(A) + \rho(B)$ and $\rho(AB) \leq \rho(A)\rho(B)$ can both be *false*. The spectral radius is not acceptable as a norm.

10 (a) Explain why A and A^{-1} have the same condition number.

(b) Explain why A and A^{T} have the same norm, based on $\lambda(A^{\mathrm{T}}A)$ and $\lambda(AA^{\mathrm{T}})$.

11 Estimate the condition number of the ill-conditioned matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$.

12 Why is the determinant of A no good as a norm? Why is it no good as a condition number?

13 (Suggested by C. Moler and C. Van Loan.) Compute $b - Ay$ and $b - Az$ when

$$b = \begin{bmatrix} .217 \\ .254 \end{bmatrix} \quad A = \begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix} \quad y = \begin{bmatrix} .341 \\ -.087 \end{bmatrix} \quad z = \begin{bmatrix} .999 \\ -1.0 \end{bmatrix}.$$

Is y closer than z to solving $Ax = b$? Answer in two ways: Compare the *residual* $b - Ay$ to $b - Az$. Then compare y and z to the true $x = (1, -1)$. Both answers can be right. Sometimes we want a small residual, sometimes a small Δx.

14 (a) Compute the determinant of A in Problem 13. Compute A^{-1}.

(b) If possible compute $\|A\|$ and $\|A^{-1}\|$ and show that $c > 10^6$.

Problems 15–19 are about vector norms other than the usual $\|x\| = \sqrt{x \cdot x}$.

15 The "ℓ^1 norm" and the "ℓ^∞ norm" of $x = (x_1, \ldots, x_n)$ are

$$\|x\|_1 = |x_1| + \cdots + |x_n| \quad \text{and} \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Compute the norms $\|x\|$ and $\|x\|_1$ and $\|x\|_\infty$ of these two vectors in $\mathbf{R}^5$:

$$x = (1, 1, 1, 1, 1) \qquad x = (.1, .7, .3, .4, .5).$$

16 Prove that $\|x\|_\infty \le \|x\| \le \|x\|_1$. Show from the Schwarz inequality that the ratios $\|x\|/\|x\|_\infty$ and $\|x\|_1/\|x\|$ are never larger than $\sqrt{n}$. Which vector $(x_1, \ldots, x_n)$ gives ratios equal to $\sqrt{n}$?

17 All vector norms must satisfy the *triangle inequality*. Prove that

$$\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty \qquad \text{and} \qquad \|x + y\|_1 \le \|x\|_1 + \|y\|_1.$$

18 Vector norms must also satisfy $\|cx\| = |c|\,\|x\|$. The norm must be positive except when $x = \mathbf{0}$. Which of these are norms for vectors (x_1, x_2) in $\mathbf{R}^2$?

$$\|x\|_A = |x_1| + 2|x_2| \qquad \|x\|_B = \min\,(|x_1|, |x_2|)$$

$$\|x\|_C = \|x\| + \|x\|_\infty \qquad \|x\|_D = \|Ax\| \quad \text{(this answer depends on } A\text{)}.$$

Challenge Problems

19 Show that $x^{\mathrm{T}}y \le \|x\|_1\,\|y\|_\infty$ by choosing components $y_i = \pm 1$ to make $x^{\mathrm{T}}y$ as large as possible.

20 The eigenvalues of the $-1, 2, -1$ difference matrix K are $\lambda = 2 - 2\cos\,(j\pi/n{+}1)$. Estimate $\lambda_{\min}$ and $\lambda_{\max}$ and $c = \mathbf{cond}(K) = \lambda_{\max}/\lambda_{\min}$ as n increases: $c \approx Cn^2$ with what constant C?

Test this estimate with $\mathbf{eig}(K)$ and $\mathbf{cond}(K)$ for $n = 10, 100, 1000$.

21 For unsymmetric matrices, the spectral radius $\rho = \max|\lambda_i|$ is not a norm (Problem 9). But still $\|A^n\|$ grows or decays like ρ^n for large n. Compare those numbers for $A = [1 \quad 1; \quad 0 \quad 1.1]$ using the command **norm**.

In particular $A^n \to 0$ when $\rho < 1$. This is the key to Section 9.3 with $A = S^{-1}T$.

9.3 Iterative Methods and Preconditioners

Up to now, our approach to $Ax = b$ has been direct. We accepted A as it came. We attacked it by elimination with row exchanges. This section is about ***iterative methods***, which replace A by a simpler matrix S. The difference $T = S - A$ is moved over to the right side of the equation. The problem becomes easier to solve, with S instead of A. But there is a price—*the simpler system has to be solved over and over.*

An iterative method is easy to invent. Just split A (carefully) into $S - T$.

$$\textbf{Rewrite } Ax = b \qquad Sx = Tx + b. \tag{1}$$

The novelty is to solve (1) iteratively. Each guess x_k leads to the next x_{k+1}:

$$\textbf{Pure iteration} \qquad Sx_{k+1} = Tx_k + b. \tag{2}$$

Start with any x_0. Then solve $Sx_1 = Tx_0 + b$. Continue to the second iteration $Sx_2 = Tx_1 + b$. A hundred iterations are very common—often more. Stop when (and if!) the new vector x_{k+1} is sufficiently close to x_k—or when the **residual** $r_k = b - Ax_k$ is near zero. We choose the stopping test. Our hope is to get near the true solution, more quickly than by elimination. When the sequence x_k converges, its limit $x = x_\infty$ does solve equation (1). The proof is to let $k \to \infty$ in equation (2).

The two goals of the splitting $A = S - T$ are ***speed per step*** and ***fast convergence***. The speed of each step depends on S and the speed of convergence depends on $S^{-1}T$:

1 Equation (2) should be easy to solve for x_{k+1}. The "***preconditioner***" S could be the diagonal or triangular part of A. A fast way uses $S = L_0U_0$, where those factors have many zeros compared to the exact $A = LU$. This is "*incomplete* LU".

2 The difference $x - x_k$ (this is the error e_k) should go quickly to zero. Subtracting equation (2) from (1) cancels b, and it leaves the ***equation for the error e_k***:

$$\textbf{Error equation} \qquad Se_{k+1} = Te_k \;\; \text{which means} \;\; e_{k+1} = S^{-1}Te_k. \tag{3}$$

At every step the error is multiplied by $S^{-1}T$. If $S^{-1}T$ is small, its powers go quickly to zero. But what is "small"?

The extreme splitting is $S = A$ and $T = 0$. Then the first step of the iteration is the original $Ax = b$. Convergence is perfect and $S^{-1}T$ is zero. But the cost of that step is what we wanted to avoid. The choice of S is a battle between speed per step (a simple S) and fast convergence (S close to A). Here are some popular choices:

J S = diagonal part of A (the iteration is called *Jacobi's method*)

GS S = lower triangular part of A including the diagonal (*Gauss-Seidel method*)

SOR S = combination of Jacobi and Gauss-Seidel (*successive overrelaxation*)

ILU S = approximate L times approximate U (*incomplete L U method*).

Our first question is pure linear algebra: ***When do the x_k's converge to x?*** The answer uncovers the number $|\lambda|_{\max}$ that controls convergence. In examples of **J** and **GS** and **SOR**, we will compute this "*spectral radius*" $|\lambda|_{\max}$. It is the largest eigenvalue of the ***iteration matrix*** $B = S^{-1}T$.

The Spectral Radius $\rho(B)$ Controls Convergence

Equation (3) is $e_{k+1} = S^{-1}Te_k$. Every iteration step multiplies the error by the same matrix $B = S^{-1}T$. The error after k steps is $e_k = B^k e_0$. ***The error approaches zero if the powers of $B = S^{-1}T$ approach zero.*** It is beautiful to see how the eigenvalues of B—the largest eigenvalue in particular—control the matrix powers B^k.

> The powers B^k approach zero if and only if every eigenvalue of B has $|\lambda| < 1$.
> ***The rate of convergence is controlled by the spectral radius of B:*** $\rho = \max|\lambda(B)|$.

The test for convergence is $|\lambda|_{\max} < 1$. Real eigenvalues must lie between -1 and 1. Complex eigenvalues $\lambda = a + ib$ must have $|\lambda|^2 = a^2 + b^2 < 1$. (Chapter 10 will discuss complex numbers.) The spectral radius "*rho*" is the largest distance from 0 to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $B = S^{-1}T$. This is $\rho = |\lambda|_{\max}$.

To see why $|\lambda|_{\max} < 1$ is necessary, suppose the starting error e_0 happens to be an eigenvector of B. After one step the error is $Be_0 = \lambda e_0$. After k steps the error is $B^k e_0 = \lambda^k e_0$. If we start with an eigenvector, we continue with that eigenvector—and it grows or decays with the powers λ^k. *This factor λ^k goes to zero when* $|\lambda| < 1$. Since this condition is required of every eigenvalue, we need $\rho = |\lambda|_{\max} < 1$.

To see why $|\lambda|_{\max} < 1$ is sufficient for the error to approach zero, suppose e_0 is a combination of eigenvectors:

$$e_0 = c_1 x_1 + \cdots + c_n x_n \quad \text{leads to} \quad e_k = c_1(\lambda_1)^k x_1 + \cdots + c_n(\lambda_n)^k x_n. \tag{4}$$

This is the point of eigenvectors! They grow independently, each one controlled by its eigenvalue. When we multiply by B, the eigenvector x_i is multiplied by λ_i. If all $|\lambda_i| < 1$ then equation (4) ensures that e_k goes to zero.

Example 1 $B = \begin{bmatrix} .6 & .5 \\ .6 & .5 \end{bmatrix}$ has $\lambda_{\max} = 1.1$ $\qquad B' = \begin{bmatrix} .6 & 1.1 \\ 0 & .5 \end{bmatrix}$ has $\lambda_{\max} = .6$

B^2 is 1.1 times B. Then B^3 is $(1.1)^2$ times B. The powers of B will blow up. Contrast with the powers of B'. The matrix $(B')^k$ has $(.6)^k$ and $(.5)^k$ on its diagonal. The off-diagonal entries also involve $\rho^k = (.6)^k$, which sets the speed of convergence.

Note There is a technical difficulty when B does not have n independent eigenvectors. (To produce this effect in B', change .5 to .6.) The starting error e_0 may not be a combination of eigenvectors—there are too few for a basis. Then diagonalization is impossible and equation (4) is not correct. We turn to the *Jordan form* when eigenvectors are missing:

$$\textbf{Jordan form } J \qquad B = MJM^{-1} \qquad \text{and} \qquad B^k = MJ^kM^{-1}. \tag{5}$$

Section 6.6 shows how J and J^k are made of "blocks" with one repeated eigenvalue:

$$\text{The powers of a 2 by 2 block in } J \text{ are} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}.$$

If $|\lambda| < 1$ then these powers approach zero. The extra factor k from a double eigenvalue is overwhelmed by the decreasing factor λ^{k-1}. This applies to all Jordan blocks. A block of size $S + 1$ has $k^S \lambda^{k-S}$ in J^k, which also approaches zero when $|\lambda| < 1$.

Diagonalizable or not: Convergence $B^k \to 0$ and its speed depend on $\rho = |\lambda|_{\max} < 1$.

Jacobi versus Gauss-Seidel

We now solve a specific 2 by 2 problem. Watch for that number $|\lambda|_{\max}$.

$$Ax = b \qquad \begin{aligned} 2u - v &= 4 \\ -u + 2v &= -2 \end{aligned} \quad \text{has the solution} \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{6}$$

The first splitting is ***Jacobi's method.*** Keep the *diagonal* of A on the left side (this is S). Move the off-diagonal part of A to the right side (this is T). Then iterate:

$$\textbf{Jacobi iteration} \qquad Sx_{k+1} = Tx_k + b \qquad \begin{aligned} 2u_{k+1} &= v_k + 4 \\ 2v_{k+1} &= u_k - 2. \end{aligned}$$

Start from $u_0 = v_0 = 0$. The first step finds $u_1 = 2$ and $v_1 = -1$. Keep going:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1/4 \end{bmatrix} \begin{bmatrix} 15/8 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1/16 \end{bmatrix} \quad \text{approaches} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This shows convergence. At steps 1, 3, 5 the second component is $-1, -1/4, -1/16$. The error is multiplied by $\frac{1}{4}$ every two steps. The components $0, 3/2, 15/8$ have errors $2, \frac{1}{2}, \frac{1}{8}$. Those also drop by 4 in each two steps. *The error equation is $Se_{k+1} = Te_k$:*

$$\textbf{Error equation} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} e_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_k \quad \text{or} \quad e_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} e_k. \tag{7}$$

That last matrix $S^{-1}T$ has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$. So its spectral radius is $\rho(B) = \frac{1}{2}$:

$$B = S^{-1}T = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad \text{has } |\lambda|_{\max} = \tfrac{1}{2} \quad \text{and} \quad \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Two steps multiply the error by $\frac{1}{4}$ exactly, in this special example. The important message is this: Jacobi's method works well when the main diagonal of A is large compared to the off-diagonal part. The diagonal part is S, the rest is $-T$. We want the diagonal to dominate and $S^{-1}T$ to be small.

The eigenvalue $\lambda = \frac{1}{2}$ is unusually small. Ten iterations reduce the error by $2^{10} = 1024$. More typical and more expensive is $|\lambda|_{\max} = .99$ or $.999$.

The ***Gauss-Seidel method*** keeps the whole lower triangular part of A as S:

$$\textbf{Gauss-Seidel}\qquad \begin{aligned} 2u_{k+1} \phantom{+2v_{k+1}} &= v_k + 4 \\ -u_{k+1} + 2v_{k+1} &= -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} u_{k+1} &= \tfrac{1}{2}v_k + 2 \\ v_{k+1} &= \tfrac{1}{2}u_{k+1} - 1. \end{aligned} \tag{8}$$

Notice the change. The new u_{k+1} from the first equation is used *immediately* in the second equation. With Jacobi, we saved the old u_k until the whole step was complete. With Gauss-Seidel, the new values enter right away and the old u_k is destroyed. This cuts the storage in half. It also speeds up the iteration (usually). And it costs no more than the Jacobi method.

Starting from $(0,0)$, the exact answer $(2,0)$ is reached in one step. That is an accident I did not expect. Test the iteration from another start $u_0 = 0$ and $v_0 = -1$:

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3/2 \\ -1/4 \end{bmatrix} \quad \begin{bmatrix} 15/8 \\ -1/16 \end{bmatrix} \quad \begin{bmatrix} 63/32 \\ -1/64 \end{bmatrix} \quad \text{approaches} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The errors in the first component are 2, 1/2, 1/8, 1/32. The errors in the second component are $-1, -1/4, -1/16, -1/32$. We divide by 4 in *one step* not two steps. ***Gauss-Seidel is twice as fast as Jacobi.*** We have $\rho_{\text{GS}} = (\rho_{\text{J}})^2$.

This double speed is true for every positive definite tridiagonal matrix. Anything is possible when A is strongly nonsymmetric—Jacobi is sometimes better, and both methods might fail. Our example is small and A is positive definite tridiagonal:

$$S = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S^{-1}T = \begin{bmatrix} \mathbf{0} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}.$$

The Gauss-Seidel eigenvalues are 0 and $\frac{1}{4}$. Compare with $\frac{1}{2}$ and $-\frac{1}{2}$ for Jacobi.

With a small push we can explain the ***successive overrelaxation method*** (**SOR**). The new idea is to introduce a parameter ω (omega) into the iteration. Then choose this number ω to make the spectral radius of $S^{-1}T$ as small as possible.

Rewrite $Ax = b$ as $\omega Ax = \omega b$. The matrix S in **SOR** has the diagonal of the original A, but below the diagonal we use ωA. On the right side T is $S - \omega A$:

$$\textbf{SOR}\qquad \begin{aligned} 2u_{k+1} \phantom{+2v_{k+1}} &= (2-2\omega)u_k + \omega v_k + 4\omega \\ -\omega u_{k+1} + 2v_{k+1} &= \phantom{(2-2\omega)u_k +{}} (2-2\omega)v_k - 2\omega. \end{aligned} \tag{9}$$

This looks more complicated to us, but the computer goes as fast as ever. Each new u_{k+1} from the first equation is used immediately to find v_{k+1} in the second equation. This is like Gauss-Seidel, with an adjustable number ω. The key matrix is $S^{-1}T$:

$$\textbf{SOR iteration matrix}\qquad S^{-1}T = \begin{bmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & 1-\omega+\frac{1}{4}\omega^2 \end{bmatrix}. \tag{10}$$

The determinant is $(1-\omega)^2$. At the best ω, both eigenvalues turn out to equal $7 - 4\sqrt{3}$, which is close to $(\frac{1}{4})^2$. Therefore **SOR** is twice as fast as Gauss-Seidel in this example. In other examples **SOR** can converge ten or a hundred times as fast.

I will put on record the most valuable test matrix of order n. It is our favorite -1, 2, -1 tridiagonal matrix K. The diagonal is $2I$. Below and above are -1's. Our example had $n = 2$, which leads to $\cos\frac{\pi}{3} = \frac{1}{2}$ as the Jacobi eigenvalue found above. Notice especially that this eigenvalue is squared for Gauss-Seidel:

The splittings of the $-1, 2, -1$ matrix K of order n yield these eigenvalues of B:

Jacobi ($S = 0, 2, 0$ matrix): $\quad S^{-1}T$ has $|\lambda|_{\max} = \cos\dfrac{\pi}{n+1}$

Gauss-Seidel ($S = -1, 2, 0$ matrix): $\quad S^{-1}T$ has $|\lambda|_{\max} = \left(\cos\dfrac{\pi}{n+1}\right)^2$

SOR (with the best ω): $\quad S^{-1}T$ has $|\lambda|_{\max} = \left(\cos\dfrac{\pi}{n+1}\right)^2 \Big/ \left(1+\sin\dfrac{\pi}{n+1}\right)^2.$

Let me be clear: For the -1, 2, -1 matrix you should not use any of these iterations! Elimination is very fast (exact LU). Iterations are intended for large sparse matrices—when a high percentage of the entries are zero. The not good zeros are inside the band, which is wide. They become nonzero in the exact L and U, which is why elimination becomes expensive.

We mention one more splitting. The idea of "***incomplete*** LU" is to set the small nonzeros in L and U *back to zero*. This leaves triangular matrices L_0 and U_0 which are again sparse. The splitting has $S = L_0U_0$ on the left side. Each step is quick:

Incomplete LU $\qquad L_0U_0\boldsymbol{x}_{k+1} = (L_0U_0 - A)\boldsymbol{x}_k + \boldsymbol{b}.$

On the right side we do sparse matrix-vector multiplications. Don't multiply L_0 times U_0, those are matrices. Multiply $\boldsymbol{x}_k$ by U_0 and then multiply that vector by L_0. On the left side we do forward and back substitutions. If L_0U_0 is close to A, then $|\lambda|_{\max}$ is small. A few iterations will give a close answer.

Multigrid and Conjugate Gradients

I cannot leave the impression that Jacobi and Gauss-Seidel are great methods. Generally the "low-frequency" part of the error decays very slowly, and many iterations are needed. Here are two ideas that bring tremendous improvement. Multigrid can solve problems of size n in $O(n)$ steps. With a good preconditioner, conjugate gradients becomes one of the most popular and powerful algorithms in numerical linear algebra.

Multigrid Solve smaller problems (often coming from coarser grids and doubled stepsizes Δx and Δy). Each iteration will be cheaper and convergence will be faster. Then interpolate between the values computed on the coarse grid to get a quick and close headstart on the full-size problem. Multigrid might go 4 levels down and back.

Conjugate gradients An ordinary iteration like $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - A\boldsymbol{x}_k + \boldsymbol{b}$ involves multiplication by A at each step. If A is sparse, this is not too expensive: $A\boldsymbol{x}_k$ is what we are willing to do. It adds one more basis vector to the growing "Krylov spaces" that contain our approximations. But $\boldsymbol{x}_{k+1}$ is **not the best combination** of $\boldsymbol{x}_0, A\boldsymbol{x}_0, \ldots, A^k\boldsymbol{x}_0$. The ordinary iterations are simple but far from optimal.

The conjugate gradient method chooses **the best combination** $\boldsymbol{x}_k$ at every step. The extra cost (beyond one multiplication by A) is not great. We will give the CG iteration, emphasizing that this method was created for a *symmetric positive definite matrix.* When A is not symmetric, one good choice is GMRES. When $A = A^{\mathrm{T}}$ is not positive definite, there is MINRES. A world of high-powered iterative methods has been created around the idea of making optimal choices of each successive $\boldsymbol{x}_k$.

My textbook *Computational Science and Engineering* describes multigrid and CG in much more detail. Among books on numerical linear algebra, Trefethen-Bau is deservedly popular (others are terrific too). Golub-Van Loan is a level up.

The Problem Set reproduces the five steps in each conjugate gradient cycle from $\boldsymbol{x}_{k-1}$ to $\boldsymbol{x}_k$. We compute that new approximation $\boldsymbol{x}_k$, the new residual $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$, and the new search direction $\boldsymbol{d}_k$ to look for the next $\boldsymbol{x}_{k+1}$.

I wrote those steps for the original matrix A. But a **preconditioner** S can make convergence much faster. Our original equation is $A\boldsymbol{x} = \boldsymbol{b}$. The preconditioned equation is $S^{-1}A\boldsymbol{x} = S^{-1}\boldsymbol{b}$. Small changes in the code give the *preconditioned conjugate gradient method*—the leading iterative method to solve positive definite systems.

The biggest competition is direct elimination, with the equations reordered to take maximum advantage of many zeros in A. It is not easy to outperform Gauss.

Iterative Methods for Eigenvalues

We move from $A\boldsymbol{x} = \boldsymbol{b}$ to $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Iterations are an option for linear equations. They are a necessity for eigenvalue problems. The eigenvalues of an n by n matrix are the roots of an nth degree polynomial. The determinant of $A - \lambda I$ starts with $(-\lambda)^n$. This book must not leave the impression that eigenvalues should be computed that way! Working from $\det(A - \lambda I) = 0$ is a very poor approach—except when n is small.

For $n > 4$ there is no formula to solve $\det(A - \lambda I) = 0$. Worse than that, the λ's can be very unstable and sensitive. It is much better to work with A itself, gradually making it diagonal or triangular. (Then the eigenvalues appear on the diagonal.) Good computer codes are available in the LAPACK library—individual routines are free on **www.netlib.org/lapack**. This library combines the earlier LINPACK and EISPACK, with many improvements (to use matrix-matrix operations in the Level 3 BLAS). It is a collection of Fortran 77 programs for linear algebra on high-performance computers. For your computer and mine, a high quality matrix package is all we need. For supercomputers with parallel processing, move to ScaLAPACK and block elimination.

We will briefly discuss the power method and the QR method (chosen by LAPACK) for computing eigenvalues. It makes no sense to give full details of the codes.

1 Power methods and inverse power methods. Start with any vector $\boldsymbol{u}_0$. Multiply by A to find $\boldsymbol{u}_1$. Multiply by A again to find $\boldsymbol{u}_2$. If $\boldsymbol{u}_0$ is a combination of the eigenvectors, then A multiplies each eigenvector $\boldsymbol{x}_i$ by λ_i. After k steps we have $(\lambda_i)^k$:

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{11}$$

As the power method continues, ***the largest eigenvalue begins to dominate.*** The vectors $\boldsymbol{u}_k$ point toward that dominant eigenvector. We saw this for Markov matrices in Chapter 8:

$$A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix} \quad \text{has} \quad \lambda_{\max} = 1 \quad \text{with eigenvector} \quad \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

Start with $\boldsymbol{u}_0$ and multiply at every step by A:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{u}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}, \ \boldsymbol{u}_2 = \begin{bmatrix} .84 \\ .16 \end{bmatrix} \quad \text{is approaching} \quad \boldsymbol{u}_\infty = \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

The speed of convergence depends on the *ratio* of the second largest eigenvalue λ_2 to the largest λ_1. We don't want λ_1 to be small, we want λ_2/λ_1 to be small. Here $\lambda_2 = .6$ and $\lambda_1 = 1$, giving good speed. For large matrices it often happens that $|\lambda_2/\lambda_1|$ is very close to 1. Then the power method is too slow.

Is there a way to find the *smallest* eigenvalue—which is often the most important in applications? Yes, by the *inverse* power method: Multiply $\boldsymbol{u}_0$ by A^{-1} instead of A. Since we never want to compute A^{-1}, we actually solve $A\boldsymbol{u}_1 = \boldsymbol{u}_0$. By saving the LU factors, the next step $A\boldsymbol{u}_2 = \boldsymbol{u}_1$ is fast. Step k has $A\boldsymbol{u}_k = \boldsymbol{u}_{k-1}$:

$$\textbf{Inverse power method} \qquad \boldsymbol{u}_k = A^{-k}\boldsymbol{u}_0 = \frac{c_1\boldsymbol{x}_1}{(\lambda_1)^k} + \cdots + \frac{c_n\boldsymbol{x}_n}{(\lambda_n)^k}. \tag{12}$$

Now the *smallest* eigenvalue $\lambda_{\min}$ is in control. When it is very small, the factor $1/\lambda_{\min}^k$ is large. For high speed, we make $\boldsymbol{\lambda}_{\min}$ even smaller by shifting the matrix to $A - \lambda^* I$.

That shift doesn't change the eigenvectors. (λ^* might come from the diagonal of A, even better is a Rayleigh quotient $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$). If λ^* is close to $\lambda_{\min}$ then $(A - \lambda^* I)^{-1}$ has the very large eigenvalue $(\lambda_{\min} - \lambda^*)^{-1}$. Each ***shifted inverse power step*** multiplies the eigenvector by this big number, and that eigenvector quickly dominates.

2 The QR Method This is a major achievement in numerical linear algebra. Fifty years ago, eigenvalue computations were slow and inaccurate. We didn't even realize that solving $\det(A - \lambda I) = 0$ was a terrible method. Jacobi had suggested earlier that A should gradually be made triangular—then the eigenvalues appear automatically on the diagonal. He used 2 by 2 rotations to produce off-diagonal zeros. (Unfortunately the previous zeros can become nonzero again. But Jacobi's method made a partial comeback with parallel computers.) At present the ***QR method*** is the leader in eigenvalue computations and we describe it briefly.

The basic step is to factor A, whose eigenvalues we want, into QR. Remember from Gram-Schmidt (Section 4.4) that Q has orthonormal columns and R is triangular. For eigenvalues the key idea is: ***Reverse Q and R.*** The new matrix (same λ's) is $A_1 = RQ$.

The eigenvalues are not changed in RQ *because* $A = QR$ *is similar to* $A_1 = Q^{-1}AQ$:

$$\mathbf{A_1 = RQ \text{ has the same } \lambda} \qquad QRx = \lambda x \quad \text{gives} \quad RQ(Q^{-1}x) = \lambda(Q^{-1}x). \tag{13}$$

This process continues. Factor the new matrix A_1 into Q_1R_1. Then reverse the factors to R_1Q_1. This is the similar matrix A_2 and again no change in the eigenvalues. Amazingly, those eigenvalues begin to show up on the diagonal. Often the last entry of A_4 holds an accurate eigenvalue. In that case we remove the last row and column and continue with a smaller matrix to find the next eigenvalue.

Two extra ideas make this method a success. One is to shift the matrix by a multiple of I, before factoring into QR. Then RQ is shifted back:

$$\text{Factor } A_k - c_kI \text{ into } Q_kR_k. \text{ The next matrix is } A_{k+1} = R_kQ_k + c_kI.$$

A_{k+1} has the same eigenvalues as A_k, and the same as the original $A_0 = A$. A good shift chooses c near an (unknown) eigenvalue. That eigenvalue appears more accurately on the diagonal of A_{k+1}—which tells us a better c for the next step to A_{k+2}.

The other idea is to obtain off-diagonal zeros before the QR method starts. An elimination step E will do it, or a Eivens rotation, but don't forget E^{-1} (to keep λ):

$$EAE^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{1} & 4 & 5 \\ \mathbf{1} & \mathbf{6} & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ \mathbf{1} & 9 & 5 \\ \mathbf{0} & \mathbf{4} & 2 \end{bmatrix}. \ \textbf{Same } \boldsymbol{\lambda}\textbf{'s.}$$

We must leave those nonzeros 1 and 4 along *one subdiagonal*. More E's could remove them, but E^{-1} would fill them in again. This is a "***Hessenberg matrix***" (one nonzero subdiagonal). The zeros in the lower left corner will stay zero through the QR method. The operation count for each QR factorization drops from $O(n^3)$ to $O(n^2)$.

Golub and Van Loan give this example of one shifted QR step on a Hessenberg matrix. The shift is $7I$, taking 7 from all diagonal entries (then shifting back for A_1):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & .001 & 7 \end{bmatrix} \quad \text{leads to} \quad A_1 = \begin{bmatrix} -.54 & 1.69 & 0.835 \\ .31 & 6.53 & -6.656 \\ 0 & .00002 & 7.012 \end{bmatrix}.$$

Factoring $A-7I$ into QR produced $A_1 = RQ+7I$. Notice the very small number .00002. The diagonal entry 7.012 is almost an exact eigenvalue of A_1, and therefore of A. Another QR step on A_1 with shift by $7.012I$ would give terrific accuracy.

For large sparse matrices I would look to ARPACK. Problems 27–29 describe the Arnoldi iteration that orthogonalizes the basis—each step has only three terms when A is symmetric. The matrix becomes tridiagonal and still orthogonally similar to the original A: a wonderful start for computing eigenvalues.

Problem Set 9.3

Problems 1–12 are about iterative methods for $Ax = b$.

1 Change $Ax = b$ to $x = (I - A)x + b$. What are S and T for this splitting? What matrix $S^{-1}T$ controls the convergence of $x_{k+1} = (I - A)x_k + b$?

2 If λ is an eigenvalue of A, then _____ is an eigenvalue of $B = I - A$. The real eigenvalues of B have absolute value less than 1 if the real eigenvalues of A lie between _____ and _____.

3 Show why the iteration $x_{k+1} = (I - A)x_k + b$ does not converge for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

4 Why is the norm of B^k never larger than $\|B\|^k$? Then $\|B\| < 1$ guarantees that the powers B^k approach zero (convergence). No surprise since $|\lambda|_{\max}$ is below $\|B\|$.

5 If A is singular then all splittings $A = S - T$ must fail. From $Ax = 0$ show that $S^{-1}Tx = x$. So this matrix $B = S^{-1}T$ has $\lambda = 1$ and fails.

6 Change the 2's to 3's and find the eigenvalues of $S^{-1}T$ for Jacobi's method:

$$Sx_{k+1} = Tx_k + b \quad \text{is} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_k + b.$$

7 Find the eigenvalues of $S^{-1}T$ for the Gauss-Seidel method applied to Problem 6:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + b.$$

Does $|\lambda|_{\max}$ for Gauss-Seidel equal $|\lambda|^2_{\max}$ for Jacobi?

8 For any 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that $|\lambda|_{\max}$ equals $|bc/ad|$ for Gauss-Seidel and $|bc/ad|^{1/2}$ for Jacobi. We need $ad \neq 0$ for the matrix S to be invertible.

9 The best ω produces two equal eigenvalues for $S^{-1}T$ in the **SOR** method. Those eigenvalues are $\omega - 1$ because the determinant is $(\omega - 1)^2$. Set the trace in equation (10) equal to $(\omega - 1) + (\omega - 1)$ and find this optimal ω.

10 Write a computer code (MATLAB or other) for the Gauss-Seidel method. You can define S and T from A, or set up the iteration loop directly from the entries a_{ij}. Test it on the $-1, 2, -1$ matrices A of order 10, 20, 50 with $b = (1, 0, \ldots, 0)$.

11 The Gauss-Seidel iteration at component i uses earlier parts of x^{new}:

$$\textbf{Gauss-Seidel} \qquad x_i^{\text{new}} = x_i^{\text{old}} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{\text{new}} - \sum_{j=i}^{n} a_{ij}x_j^{\text{old}}\right).$$

If every $x_i^{\text{new}} = x_i^{\text{old}}$ how does this show that the solution x is correct? How does the formula change for Jacobi's method? For **SOR** insert ω outside the parentheses.

12 The **SOR** splitting matrix S is the same as for Gauss-Seidel except that the diagonal is divided by ω. Write a program for **SOR** on an n by n matrix. Apply it with $\omega = 1$, 1.4, 1.8, 2.2 when A is the $-1, 2, -1$ matrix of order $n = 10$.

13 Divide equation (11) by λ_1^k and explain why $|\lambda_2/\lambda_1|$ controls the convergence of the power method. Construct a matrix A for which this method *does not converge*.

14 The Markov matrix $A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}$ has $\lambda = 1$ and .6, and the power method $\boldsymbol{u}_k = A^k \boldsymbol{u}_0$ converges to $\begin{bmatrix} .75 \\ .25 \end{bmatrix}$. Find the eigenvectors of A^{-1}. What does the inverse power method $\boldsymbol{u}_{-k} = A^{-k}\boldsymbol{u}_0$ converge to (after you multiply by $.6^k$)?

15 The tridiagonal matrix of size $n - 1$ with diagonals $-1, 2, -1$ has eigenvalues $\lambda_j = 2 - 2\cos(j\pi/n)$. Why are the smallest eigenvalues approximately $(j\pi/n)^2$? The inverse power method converges at the speed $\lambda_1/\lambda_2 \approx 1/4$.

16 For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ apply the power method $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ three times starting with $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What eigenvector is the power method converging to?

17 In Problem 11 apply the *inverse* power method $\boldsymbol{u}_{k+1} = A^{-1}\boldsymbol{u}_k$ three times with the same $\boldsymbol{u}_0$. What eigenvector are the $\boldsymbol{u}_k$'s approaching?

18 In the QR method for eigenvalues, show that the 2, 1 entry drops from $\sin\theta$ in $A = QR$ to $-\sin^3\theta$ in RQ. (*Compute R and RQ.*) This "cubic convergence" makes the method a success:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = QR = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & ? \\ 0 & ? \end{bmatrix}.$$

19 If A is an orthogonal matrix, its QR factorization has $Q =$ _____ and $R =$ _____. Therefore $RQ =$ _____. These are among the rare examples when the QR method goes nowhere.

20 The shifted QR method factors $A - cI$ into QR. Show that the next matrix $A_1 = RQ + cI$ equals $Q^{-1}AQ$. Therefore A_1 has the _____ eigenvalues as A (but is closer to triangular).

21 When $A = A^{\mathrm{T}}$, the "*Lanczos method*" finds a's and b's and orthonormal $\boldsymbol{q}$'s so that $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ (with $\boldsymbol{q}_0 = \boldsymbol{0}$). Multiply by $\boldsymbol{q}_j^{\mathrm{T}}$ to find a formula for a_j. The equation says that $AQ = QT$ where T is a tridiagonal matrix.

22 The equation in Problem 21 develops from this loop with $b_0 = 1$ and $\boldsymbol{r}_0 = $ any $\boldsymbol{q}_1$:

$$\boldsymbol{q}_{j+1} = \boldsymbol{r}_j/b_j;\ j = j+1;\ a_j = \boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j;\ \boldsymbol{r}_j = A\boldsymbol{q}_j - b_{j-1}\boldsymbol{q}_{j-1} - a_j\boldsymbol{q}_j;\ b_j = \|\boldsymbol{r}_j\|.$$

Write a code and test it on the $-1, 2, -1$ matrix A. $Q^{\mathrm{T}}Q$ should be I.

23 Suppose A is *tridiagonal and symmetric in the QR method.* From $A_1 = Q^{-1}AQ$ show that A_1 is symmetric. Write $A_1 = RAR^{-1}$ to show that A_1 is also tridiagonal. (If the lower part of A_1 is proved tridiagonal then by symmetry the upper part is too.)

Symmetric tridiagonal matrices are the best way to start in the QR method.

Questions 24–26 are about quick ways to estimate the location of the eigenvalues.

24 If the sum of $|a_{ij}|$ along every row is less than 1, explain this proof that $|\lambda| < 1$. Suppose $Ax = \lambda x$ and $|x_i|$ is larger than the other components of x. Then $|\Sigma a_{ij} x_j|$ is less than $|x_i|$. That means $|\lambda x_i| < |x_i|$ so $|\lambda| < 1$.

(Gershgorin circles) Every eigenvalue of A is in one or more of n circles. Each circle is centered at a diagonal entry a_{ii} with radius $r_i = \Sigma_{j \neq i} |a_{ij}|$.

This follows from $(\lambda - a_{ii})x_i = \Sigma_{j \neq i} a_{ij} x_j$. If $|x_i|$ is larger than the other components of x, this sum is at most $r_i |x_i|$. Dividing by $|x_i|$ leaves $|\lambda - a_{ii}| \leq r_i$.

25 What bound on $|\lambda|_{\max}$ does Problem 24 give for these matrices? What are the three Gershgorin circles that contain all the eigenvalues? Those circles show immediately that K is at least positive semidefinite (*actually definite*) and A has $\lambda_{\max} = 1$.

$$A = \begin{bmatrix} .3 & .5 & .2 \\ .3 & .4 & .3 \\ .4 & .1 & .5 \end{bmatrix} \qquad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

26 These matrices are diagonally dominant because each $a_{ii} > r_i$ = absolute sum along the rest of row i. From the Gershgorin circles containing all λ's, show that diagonally dominant matrices are invertible.

$$A = \begin{bmatrix} 1 & .3 & .4 \\ .3 & 1 & .5 \\ .4 & .5 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

Problems 27–30 present two fundamental iterations. Each step involves Aq or Ad.

The key point for large matrices is that matrix-vector multiplication is much faster than matrix-matrix multiplication. A crucial construction starts with a vector b. Repeated multiplication will produce $Ab, A^2b, \ldots$ but those vectors are far from orthogonal. The "**Arnoldi iteration**" creates an orthonormal basis $q_1, q_2, \ldots$ for the same space by the Gram-Schmidt idea: *orthogonalize each new Aq_n against the previous $q_1, \ldots, q_{n-1}$.* The "Krylov space" spanned by $b, Ab, \ldots, A^{n-1}b$ then has a much better basis $q_1, \ldots, q_n$.

Here in pseudocode are two of the most important algorithms in numerical linear algebra: Arnoldi gives a good basis and CG gives a good approximation to $x = A^{-1}b$.

Arnoldi Iteration	**Conjugate Gradient Iteration for Positive Definite A**	
$q_1 = b/\|b\|$	$x_0 = 0, r_0 = b, d_0 = r_0$	
for $n = 1$ **to** $N-1$	**for** $n = 1$ **to** N	
$v = Aq_n$	$\alpha_n = (r_{n-1}^{\mathrm{T}} r_{n-1})/(d_{n-1}^{\mathrm{T}} A d_{n-1})$	step length x_{n-1} to x_n
for $j = 1$ **to** n	$x_n = x_{n-1} + \alpha_n d_{n-1}$	approximate solution
$h_{jn} = q_j^{\mathrm{T}} v$	$r_n = r_{n-1} - \alpha_n A d_{n-1}$	new residual $b - Ax_n$
$v = v - h_{jn} q_j$	$\beta_n = (r_n^{\mathrm{T}} r_n)/(r_{n-1}^{\mathrm{T}} r_{n-1})$	improvement this step
$h_{n+1,n} = \|v\|$	$d_n = r_n + \beta_n d_{n-1}$	next search direction
$q_{n+1} = v/h_{n+1,n}$	*% Notice: only* 1 *matrix-vector multiplication* Aq *and* Ad	

For conjugate gradients, the residuals r_n are orthogonal and the search directions are A-orthogonal: all $d_j^{\mathrm{T}} A d_k = 0$. The iteration solves $Ax = b$ by minimizing the error $e^{\mathrm{T}} A e$ over all vectors in the Krylov subspace. It is a fantastic algorithm.

27 For the diagonal matrix $A = \text{diag}([1 \ \ 2 \ \ 3 \ \ 4])$ and the vector $b = (1, 1, 1, 1)$, go through one Arnoldi step to find the orthonormal vectors q_1 and q_2.

28 Arnoldi's method is finding Q so that $AQ = QH$ (column by column):

$$AQ = \begin{bmatrix} Aq_1 & \cdots & Aq_N \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_N \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \cdot & h_{1N} \\ h_{21} & h_{22} & \cdot & h_{2N} \\ 0 & h_{32} & \cdot & \cdot \\ 0 & 0 & \cdot & h_{NN} \end{bmatrix} = QH$$

H is a "Hessenberg matrix" with one nonzero subdiagonal. Here is the crucial fact when A is symmetric: ***The matrix*** $H = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ ***is symmetric and therefore tridiagonal.*** Explain that sentence.

29 This tridiagonal H (when A is symmetric) gives the **Lanczos iteration**:

$$\textbf{Three terms only} \qquad q_{j+1} = (Aq_j - h_{j,j}q_j - h_{j-1,j}q_{j-1})/h_{j+1,j}$$

From $H = Q^{-1}AQ$, why are the eigenvalues of H the same as the eigenvalues of A? For large matrices, the "Lanczos method" computes the leading eigenvalues by stopping at a smaller tridiagonal matrix H_k. The QR method in the text is applied to compute the eigenvalues of H_k.

30 Apply the conjugate gradient method to solve $Ax = b = \textbf{ones}(100, 1)$, where A is the $-1, 2, -1$ second difference matrix $A = \textbf{toeplitz}([2 \ \ -1 \ \ \textbf{zeros}(1, 98)])$. Graph x_{10} and x_{20} from CG, along with the exact solution x. (Its 100 components are $x_i = (ih - i^2h^2)/2$ with $h = 1/101$. "$\textbf{plot}(i, x(i))$" should produce a parabola.)

Chapter 10

Complex Vectors and Matrices

10.1 Complex Numbers

A complete presentation of linear algebra must include complex numbers. Even when the matrix is real, ***the eigenvalues and eigenvectors are often complex***. Example: A 2 by 2 rotation matrix has no real eigenvectors. Every vector in the plane turns by θ—its direction changes. But the rotation matrix has complex eigenvectors $(1, i)$ and $(1, -i)$.

Notice that those eigenvectors are connected by changing i to $-i$. For a real matrix, the eigenvectors come in "conjugate pairs." The eigenvalues of rotation by θ are also conjugate complex numbers $e^{i\theta}$ and $e^{-i\theta}$. We must move from $\mathbf{R}^n$ to $\mathbf{C}^n$.

The second reason for allowing complex numbers goes beyond λ and x to the matrix A. ***The matrix itself may be complex.*** We will devote a whole section to the most important example—*the Fourier matrix.* Engineering and science and music and economics all use Fourier series. In reality the series is finite, not infinite. Computing the coefficients in $c_1e^{ix} + c_2e^{i2x} + \cdots + c_ne^{inx}$ is a linear algebra problem.

This section gives the main facts about complex numbers. It is a review for some students and a reference for everyone. Everything comes from $i^2 = -1$. The Fast Fourier Transform applies the amazing formula $e^{2\pi i} = 1$. Add angles when $e^{i\theta}$ multiplies $e^{i\theta}$:

The square of $e^{2\pi i/4} = i$ is $e^{4\pi i/4} = -1$. The fourth power of $e^{2\pi i/4}$ is $e^{2\pi i} = 1$.

Adding and Multiplying Complex Numbers

Start with the imaginary number i. Everybody knows that $x^2 = -1$ has no real solution. When you square a real number, the answer is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition and multiplication, with one difference. ***Replace i^2 by -1.***

A complex number (say $3 + 2i$) ***is the sum of a real number*** (3) ***and a pure imaginary number*** ($2i$). Addition keeps the real and imaginary parts separate. Multiplication uses $i^2 = -1$:

$$\textbf{Add:}\quad (3+2i)+(3+2i) = 6+4i$$

$$\textbf{Multiply:}\quad (3+2i)(1-i) = 3+2i-3i-2i^2 = 5-i.$$

If I add $3 + i$ to $1 - i$, the answer is 4. The real numbers $3 + 1$ stay separate from the imaginary numbers $i - i$. We are adding the vectors $(3, 1)$ and $(1, -1)$.

The number $(1 + i)^2$ is $1 + i$ times $1 + i$. The rules give the surprising answer $2i$:

$$(1+i)(1+i) = 1+i+i+i^2 = 2i.$$

In the complex plane, $1+i$ is at an angle of 45°. It is like the vector $(1, 1)$. When we square $1 + i$ to get $2i$, the angle doubles to 90°. If we square again, the answer is $(2i)^2 = -4$. The 90° angle doubled to 180°, the direction of a negative real number.

A real number is just a complex number $z = a + bi$, with zero imaginary part: $b = 0$. A pure imaginary number has $a = 0$:

The ***real part*** is $a = \text{Re}\,(a + bi)$. The ***imaginary part*** is $b = \text{Im}\,(a + bi)$.

The Complex Plane

Complex numbers correspond to points in a plane. Real numbers go along the x axis. Pure imaginary numbers are on the y axis. ***The complex number*** $\mathbf{3 + 2i}$ ***is at the point with coordinates*** $\mathbf{(3, 2)}$. The number zero, which is $0 + 0i$, is at the origin.

Adding and subtracting complex numbers is like adding and subtracting vectors in the plane. The real component stays separate from the imaginary component. The vectors go head-to-tail as usual. The complex plane $\mathbf{C}^1$ is like the ordinary two-dimensional plane $\mathbf{R}^2$, except that we multiply complex numbers and we didn't multiply vectors.

Now comes an important idea. ***The complex conjugate of*** $\mathbf{3 + 2i}$ ***is*** $\mathbf{3 - 2i}$. The complex conjugate of $z = 1 - i$ is $\overline{z} = 1 + i$. In general the conjugate of $z = a + bi$ is $\overline{z} = a - bi$. (**Some writers use a "*bar*" on the number and others use a "*star*":** $\overline{z} = z^*$.) The imaginary parts of z and "z bar" have opposite signs. In the complex plane, $\overline{z}$ is the image of z on the other side of the real axis.

Two useful facts. ***When we multiply conjugates*** $\overline{z}_1$ ***and*** $\overline{z}_2$, ***we get the conjugate of*** $z_1 z_2$. When we add $\overline{z}_1$ and $\overline{z}_2$, we get the conjugate of $z_1 + z_2$:

$$\overline{z}_1 + \overline{z}_2 = (3-2i)+(1+i) = 4-i. \text{ This is the conjugate of } z_1+z_2 = 4+i.$$

$$\overline{z}_1 \times \overline{z}_2 = (3-2i)\times(1+i) = 5+i. \text{ This is the conjugate of } z_1\times z_2 = 5-i.$$

Adding and multiplying is exactly what linear algebra needs. By taking conjugates of $Ax = \lambda x$, when A is real, we have another eigenvalue $\overline{\lambda}$ and its eigenvector $\overline{x}$:

$$\textit{If } Ax = \lambda x \textit{ and } A \textit{ is real then } A\overline{x} = \overline{\lambda}\overline{x}. \tag{1}$$

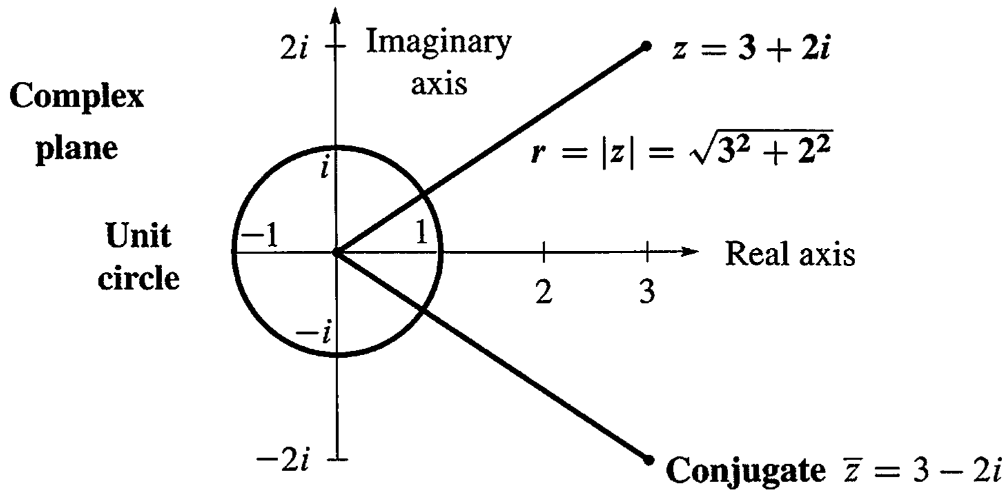


Figure 10.1: The number $z = a + bi$ corresponds to the point (a, b) and the vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.

Something special happens when $z = 3 + 2i$ combines with its own complex conjugate $\bar{z} = 3 - 2i$. The result from adding $z + \bar{z}$ or multiplying $z\bar{z}$ is always real:

$$\begin{aligned} z + \bar{z} &= \textbf{real} \qquad (3 + 2i) + (3 - 2i) = 6 \quad \text{(real)} \\ z\bar{z} &= \textbf{real} \qquad (3 + 2i) \times (3 - 2i) = 9 + 6i - 6i - 4i^2 = 13 \quad \text{(real).} \end{aligned}$$

The sum of $z = a + bi$ and its conjugate $\bar{z} = a - bi$ is the real number $2a$. The product of z times $\bar{z}$ is the real number $a^2 + b^2$:

Multiply z times $\bar{z}$ $$(a + bi)(a - bi) = a^2 + b^2. \tag{2}$$

The next step with complex numbers is $1/z$. How to divide by $a + ib$? The best idea is to multiply by $\bar{z}/\bar{z}$. That produces $z\bar{z}$ in the denominator, which is $a^2 + b^2$:

$$\frac{1}{a + ib} = \frac{1}{a + ib}\,\frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2} \qquad \frac{1}{3 + 2i} = \frac{1}{3 + 2i}\,\frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{13}.$$

In case $a^2 + b^2 = 1$, this says that $(a + ib)^{-1}$ is $a - ib$. ***On the unit circle, $1/z$ equals $\bar{z}$.*** Later we will say: $1/e^{i\theta}$ is $e^{-i\theta}$ (the conjugate). A better way to multiply and divide is to use the polar form with distance r and angle θ.

The Polar Form $re^{i\theta}$

The square root of $a^2 + b^2$ is $|z|$. This is the ***absolute value*** (or ***modulus***) of the number $z = a + ib$. The square root $|z|$ is also written r, because it is the distance from 0 to z. ***The real number r in the polar form gives the size of the complex number z:***

The absolute value of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$. This is called r.

The absolute value of $z = 3 + 2i$ is $|z| = \sqrt{3^2 + 2^2}$. This is $r = \sqrt{13}$.

The other part of the polar form is the angle θ. The angle for $z = 5$ is $\theta = 0$ (because this z is real and positive). The angle for $z = 3i$ is $\pi/2$ radians. The angle for a negative $z = -9$ is π radians. ***The angle doubles when the number is squared.*** The polar form is excellent for multiplying complex numbers (not good for addition).

When the distance is r and the angle is θ, trigonometry gives the other two sides of the triangle. The real part (along the bottom) is $a = r\cos\theta$. The imaginary part (up or down) is $b = r\sin\theta$. Put those together, and the rectangular form becomes the polar form:

$$\textbf{\textit{The number}} \quad \boldsymbol{z = a + ib} \qquad \textbf{\textit{is also}} \quad \boldsymbol{z = r\cos\theta + ir\sin\theta}. \quad \textbf{\textit{This is }}\boldsymbol{re^{i\theta}}$$

Note: $\boldsymbol{\cos\theta + i\sin\theta}$ ***has absolute value*** $\boldsymbol{r = 1}$ ***because*** $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$. Thus $\cos\theta + i\sin\theta$ lies on the circle of radius 1—*the unit circle.*

Example 1 Find r and θ for $z = 1 + i$ and also for the conjugate $\bar{z} = 1 - i$.

Solution The absolute value is the same for z and $\bar{z}$. For $z = 1+i$ it is $r = \sqrt{1+1} = \sqrt{2}$:

$$|z|^2 = 1^2 + 1^2 = 2 \qquad \text{and also} \qquad |\bar{z}|^2 = 1^2 + (-1)^2 = 2.$$

The distance from the center is $\sqrt{2}$. What about the angle? The number $1 + i$ is at the point $(1, 1)$ in the complex plane. The angle to that point is $\pi/4$ radians or $45°$. The cosine is $1/\sqrt{2}$ and the sine is $1/\sqrt{2}$. Combining r and θ brings back $z = 1 + i$:

$$r\cos\theta + ir\sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + i.$$

The angle to the conjugate $1 - i$ can be positive or negative. We can go to $7\pi/4$ radians which is $315°$. Or we can go *backwards through a negative angle*, to $-\pi/4$ radians or $-45°$. ***If*** $\boldsymbol{z}$ ***is at angle*** $\boldsymbol{\theta}$***, its conjugate*** $\boldsymbol{\bar{z}}$ ***is at*** $\boldsymbol{2\pi - \theta}$ ***and also at*** $\boldsymbol{-\theta}$.

We can freely add 2π or 4π or -2π to any angle! Those go full circles so the final point is the same. This explains why there are infinitely many choices of θ. Often we select the angle between zero and 2π radians. But $-\theta$ is very useful for the conjugate $\bar{z}$.

Powers and Products: Polar Form

Computing $(1 + i)^2$ and $(1 + i)^8$ is quickest in polar form. That form has $r = \sqrt{2}$ and $\theta = \pi/4$ (or $45°$). If we square the absolute value to get $r^2 = 2$, and double the angle to get $2\theta = \pi/2$ (or $90°$), we have $(1 + i)^2$. For the eighth power we need r^8 and 8θ:

$$\boldsymbol{(1+i)^8} \qquad r^8 = 2\cdot 2\cdot 2\cdot 2 = 16 \text{ and } 8\theta = 8\cdot\frac{\pi}{4} = 2\pi.$$

This means: $(1 + i)^8$ has absolute value 16 and angle 2π. *The eighth power of* $1 + i$ *is the real number* 16.

Powers are easy in polar form. So is multiplication of complex numbers.

The polar form of z^n has absolute value r^n. The angle is n times θ:

$$\textbf{\textit{The nth power of}} \quad z = r(\cos\theta + i\sin\theta) \quad \textbf{\textit{is}} \quad z^n = r^n(\cos n\theta + i\sin n\theta). \qquad (3)$$

In that case z multiplies itself. In all cases, ***multiply r's and add the angles***:

$$r(\cos\theta + i\sin\theta) \text{ times } r'(\cos\theta' + i\sin\theta') = rr'\big(\cos(\theta+\theta') + i\sin(\theta+\theta')\big). \qquad (4)$$

One way to understand this is by trigonometry. Concentrate on angles. Why do we get the double angle 2θ for z^2?

$$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = \cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta.$$

The real part $\cos^2\theta - \sin^2\theta$ is $\cos 2\theta$. The imaginary part $2\sin\theta\cos\theta$ is $\sin 2\theta$. Those are the "double angle" formulas. They show that θ in z becomes 2θ in z^2.

There is a second way to understand the rule for z^n. It uses the only amazing formula in this section. Remember that $\cos\theta + i\sin\theta$ has absolute value 1. The cosine is made up of even powers, starting with $1 - \frac{1}{2}\theta^2$. The sine is made up of odd powers, starting with $\theta - \frac{1}{6}\theta^3$. The beautiful fact is that $e^{i\theta}$ combines both of those series into $\cos\theta + i\sin\theta$:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \quad \text{becomes} \quad e^{i\theta} = 1 + i\theta + \frac{1}{2}i^2\theta^2 + \frac{1}{6}i^3\theta^3 + \cdots$$

Write -1 for i^2 to see $1 - \frac{1}{2}\theta^2$. ***The complex number*** $e^{i\theta}$ ***is*** $\cos\theta + i\sin\theta$:

$$\textbf{\textit{Euler's Formula}} \quad e^{i\theta} = \cos\theta + i\sin\theta \quad \text{gives} \quad z = r\cos\theta + ir\sin\theta = re^{i\theta} \qquad (5)$$

The special choice $\theta = 2\pi$ gives $\cos 2\pi + i\sin 2\pi$ which is 1. Somehow the infinite series $e^{2\pi i} = 1 + 2\pi i + \frac{1}{2}(2\pi i)^2 + \cdots$ adds up to 1.

Now multiply $e^{i\theta}$ times $e^{i\theta'}$. Angles add for the same reason that exponents add:

$$e^2 \text{ times } e^3 \text{ is } e^5 \qquad e^{i\theta} \text{ times } e^{i\theta} \text{ is } e^{2i\theta} \qquad e^{i\theta} \text{ times } e^{i\theta'} \text{ is } e^{i(\theta+\theta')}$$

The powers $(re^{i\theta})^n$ are equal to $r^n e^{in\theta}$. They stay on the unit circle when $r = 1$ and $r^n = 1$. Then we find n different numbers whose nth powers equal 1:

$$\textbf{\textit{Set}} \;\; w = e^{2\pi i/n}. \;\; \textbf{\textit{The nth powers of}} \; 1, w, w^2, \ldots, w^{n-1} \; \textbf{\textit{all equal 1.}}$$

Those are the "nth roots of 1." They solve the equation $z^n = 1$. They are equally spaced around the unit circle in Figure 10.2b, where the full 2π is divided by n. Multiply their angles by n to take nth powers. That gives $w^n = e^{2\pi i}$ which is 1. Also $(w^2)^n = e^{4\pi i} = 1$. Each of those numbers, to the nth power, comes around the unit circle to 1.

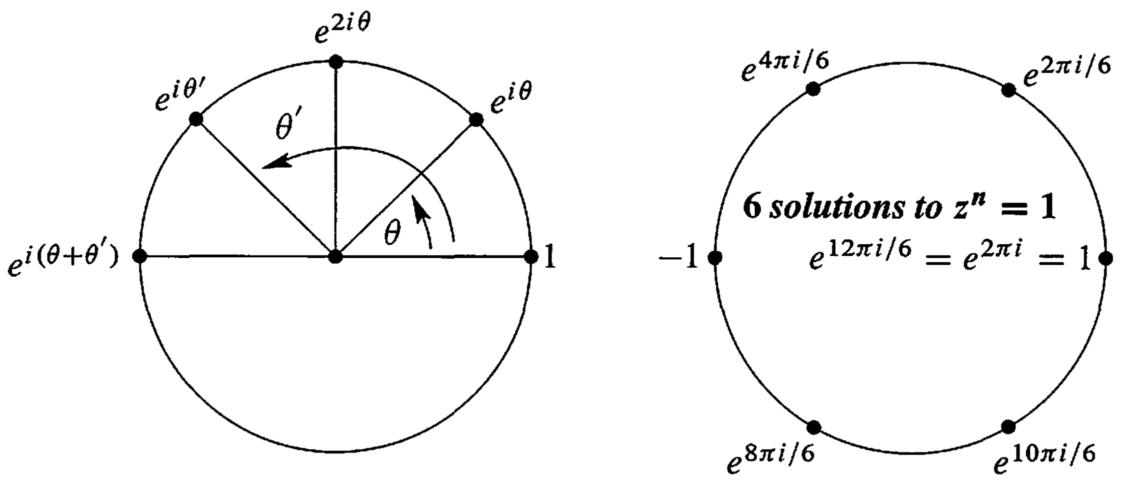


Figure 10.2: (a) Multiplying $e^{i\theta}$ times $e^{i\theta'}$. (b) The nth power of $e^{2\pi i/n}$ is $e^{2\pi i} = 1$.

These n roots of 1 are the key numbers for signal processing. The Discrete Fourier Transform uses w and its powers. Section 10.3 shows how to decompose a vector (a signal) into n frequencies by the Fast Fourier Transform.

■ REVIEW OF THE KEY IDEAS ■

1. Adding $a + ib$ to $c + id$ is like adding $(a, b) + (c, d)$. Use $i^2 = -1$ to multiply.
2. The conjugate of $z = a + bi = re^{i\theta}$ is $\bar{z} = z^* = a - bi = re^{-i\theta}$.
3. z times $\bar{z}$ is $re^{i\theta}$ times $re^{-i\theta}$. This is $r^2 = |z|^2 = a^2 + b^2$ (real).
4. Powers and products are easy in polar form $z = re^{i\theta}$. *Multiply r's and add θ's.*

Problem Set 10.1

Questions 1–8 are about operations on complex numbers.

1 Add and multiply each pair of complex numbers:

(a) $2+i, 2-i$ (b) $-1+i, -1+i$ (c) $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

2 Locate these points on the complex plane. Simplify them if necessary:

(a) $2+i$ (b) $(2+i)^2$ (c) $\frac{1}{2+i}$ (d) $|2+i|$

3 Find the absolute value $r = |z|$ of these four numbers. If θ is the angle for $6-8i$, what are the angles for the other three numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\frac{1}{6-8i}$ (d) $(6+8i)^2$

4 If $|z| = 2$ and $|w| = 3$ then $|z \times w| =$ ____ and $|z + w| \le$ ____ and $|z/w| =$ ____ and $|z - w| \le$ ____.

5 Find $a + ib$ for the numbers at angles $30°, 60°, 90°, 120°$ on the unit circle. If w is the number at $30°$, check that w^2 is at $60°$. What power of w equals 1?

6 If $z = r\cos\theta + ir\sin\theta$ then $1/z$ has absolute value ____ and angle ____. Its polar form is ____. Multiply $z \times 1/z$ to get 1.

7 The complex multiplication $M = (a + bi)(c + di)$ is a 2 by 2 real multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}.$$

The right side contains the real and imaginary parts of M. Test $M = (1+3i)(1-3i)$.

8 $A = A_1 + iA_2$ is a complex n by n matrix and $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ is a complex vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x}_1 + i\boldsymbol{x}_2$. Write $A\boldsymbol{x} = \boldsymbol{b}$ as a real system of size $2n$:

Complex n by n
Real $2n$ by $2n$
$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Questions 9–16 are about the conjugate $\bar{z} = a - ib = re^{-i\theta} = z^*$.

9 Write down the complex conjugate of each number by changing i to $-i$:

(a) $2 - i$ (b) $(2 - i)(1 - i)$ (c) $e^{i\pi/2}$ (which is i)
(d) $e^{i\pi} = -1$ (e) $\frac{1+i}{1-i}$ (which is also i) (f) $i^{103} =$ ____.

10 The sum $z + \bar{z}$ is always ____. The difference $z - \bar{z}$ is always ____. Assume $z \ne 0$. The product $z \times \bar{z}$ is always ____. The ratio $z/\bar{z}$ always has absolute value ____.

11 For a real matrix, the conjugate of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is $A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$. This proves two things: $\bar{\lambda}$ is another eigenvalue and $\bar{\boldsymbol{x}}$ is its eigenvector. Find the eigenvalues $\lambda, \bar{\lambda}$ and eigenvectors $\boldsymbol{x}, \bar{\boldsymbol{x}}$ of $A = [a \;\; b; \;\; -b \;\; a]$.

12 The eigenvalues of a real 2 by 2 matrix come from the quadratic formula:

$$\det\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

gives the two eigenvalues $\lambda = \left[a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}\right]/2$.

(a) If $a = b = d = 1$, the eigenvalues are complex when c is ____.
(b) What are the eigenvalues when $ad = bc$?
(c) The two eigenvalues (plus sign and minus sign) are not always conjugates of each other. Why not?

13 In Problem 12 the eigenvalues are not real when $(\text{trace})^2 = (a+d)^2$ is smaller than ____. Show that the λ's *are* real when $bc > 0$.

14 Find the eigenvalues and eigenvectors of this permutation matrix:

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{has} \quad \det(P_4 - \lambda I) = \text{\_\_\_\_}.$$

15 Extend P_4 above to P_6 (five 1's below the diagonal and one in the corner). Find $\det(P_6 - \lambda I)$ and the six eigenvalues in the complex plane.

16 A real skew-symmetric matrix ($A^{\mathrm{T}} = -A$) has pure imaginary eigenvalues. First proof: If $Ax = \lambda x$ then block multiplication gives

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ ix \end{bmatrix} = i\lambda \begin{bmatrix} x \\ ix \end{bmatrix}.$$

This block matrix is symmetric. Its eigenvalues must be ____! So λ is ____.

Questions 17–24 are about the form $re^{i\theta}$ of the complex number $r\cos\theta + ir\sin\theta$.

17 Write these numbers in Euler's form $re^{i\theta}$. Then square each number:

(a) $1 + \sqrt{3}i$ (b) $\cos 2\theta + i \sin 2\theta$ (c) $-7i$ (d) $5 - 5i$.

18 Find the absolute value and the angle for $z = \sin\theta + i\cos\theta$ (careful). Locate this z in the complex plane. Multiply z by $\cos\theta + i\sin\theta$ to get ____.

19 Draw all eight solutions of $z^8 = 1$ in the complex plane. What is the rectangular form $a + ib$ of the root $z = \overline{w} = \exp(-2\pi i/8)$?

20 Locate the cube roots of 1 in the complex plane. Locate the cube roots of -1. Together these are the sixth roots of ____.

21 By comparing $e^{3i\theta} = \cos 3\theta + i \sin 3\theta$ with $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$, find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

22 Suppose the conjugate $\overline{z}$ is equal to the reciprocal $1/z$. What are all possible z's?

23 (a) Why do e^i and i^e both have absolute value 1?

(b) In the complex plane put stars near the points e^i and i^e.

(c) The number i^e could be $(e^{i\pi/2})^e$ or $(e^{5i\pi/2})^e$. Are those equal?

24 Draw the paths of these numbers from $t = 0$ to $t = 2\pi$ in the complex plane:

(a) e^{it} (b) $e^{(-1+i)t} = e^{-t}e^{it}$ (c) $(-1)^t = e^{t\pi i}$.

10.2 Hermitian and Unitary Matrices

The main message of this section can be presented in one sentence: ***When you transpose a complex vector z or matrix A, take the complex conjugate too.*** Don't stop at z^{T} or A^{T}. Reverse the signs of all imaginary parts. From a column vector with $z_j = a_j + ib_j$, the good row vector is the *conjugate transpose* with components $a_j - ib_j$:

Conjugate transpose $$\bar{z}^{\mathrm{T}} = \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} = \begin{bmatrix} a_1 - ib_1 & \cdots & a_n - ib_n \end{bmatrix}. \qquad (1)$$

Here is one reason to go to $\bar{z}$. The length squared of a real vector is $x_1^2 + \cdots + x_n^2$. The length squared of a complex vector is *not* $z_1^2 + \cdots + z_n^2$. With that wrong definition, the length of $(1, i)$ would be $1^2 + i^2 = 0$. A nonzero vector would have zero length—not good. Other vectors would have complex lengths. Instead of $(a + bi)^2$ we want $a^2 + b^2$, the *absolute value squared.* This is $(a + bi)$ times $(a - bi)$.

For each component we want z_j times $\bar{z}_j$, which is $|z_j|^2 = a_j^2 + b_j^2$. That comes when the components of z multiply the components of $\bar{z}$:

Length squared $$\begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + \cdots + |z_n|^2. \quad \textbf{This is} \quad \bar{z}^{\mathrm{T}} z = \|z\|^2. \qquad (2)$$

Now the squared length of $(1, i)$ is $1^2 + |i|^2 = 2$. The length is $\sqrt{2}$. The squared length of $(1 + i, 1 - i)$ is 4. The only vectors with zero length are zero vectors.

The length $\|z\|$ is the square root of $\bar{z}^{\mathrm{T}} z = z^{\mathrm{H}} z = |z_1|^2 + \cdots + |z_n|^2$

Before going further we replace two symbols by one symbol. Instead of a bar for the conjugate and T for the transpose, we just use a superscript H. Thus $\bar{z}^{\mathrm{T}} = z^{\mathrm{H}}$. This is "$z$ Hermitian," the *conjugate transpose* of z. The new word is pronounced "Hermeeshan." The new symbol applies also to matrices: The conjugate transpose of a matrix A is A^{H}.

Another popular notation is A^*. The MATLAB transpose command $'$ automatically takes complex conjugates (A' is A^{H}).

The vector z^{H} is $\bar{z}^{\mathrm{T}}$. The matrix A^{H} is $\bar{A}^{\mathrm{T}}$, the conjugate transpose of A:

$$A^{\mathrm{H}} = \text{“A Hermitian”} \quad \text{If} \quad A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix} \quad \text{then} \quad A^{\mathrm{H}} = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

Complex Inner Products

For real vectors, the length squared is $x^{\mathrm{T}}x$—*the inner product of x with itself.* For complex vectors, the length squared is $z^{\mathrm{H}}z$. It will be very desirable if $z^{\mathrm{H}}z$ is the inner product of z with itself. To make that happen, the complex inner product should use the conjugate transpose (not just the transpose). The inner product sees no change when the vectors are real, but there is a definite effect from choosing $\bar{u}^{\mathrm{T}}$, when u is complex:

DEFINITION The inner product of real or complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$:

$$\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n. \tag{3}$$

With complex vectors, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ is different from $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$. *The order of the vectors is now important.* In fact $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = \overline{v}_1 u_1 + \cdots + \overline{v}_n u_n$ is the complex conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$. We have to put up with a few inconveniences for the greater good.

Example 1 The inner product of $\boldsymbol{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$.

Example 1 is surprising. Those vectors $(1, i)$ and $(i, 1)$ don't look perpendicular. But they are. ***A zero inner product still means that the*** (complex) ***vectors are orthogonal.*** Similarly the vector $(1, i)$ is orthogonal to the vector $(1, -i)$. Their inner product is $1 - 1 = 0$. We are correctly getting zero for the inner product—where we would be incorrectly getting zero for the length of $(1, i)$ if we forgot to take the conjugate.

Note We have chosen to conjugate the first vector $\boldsymbol{u}$. Some authors choose the second vector $\boldsymbol{v}$. Their complex inner product would be $\boldsymbol{u}^{\mathrm{T}}\overline{\boldsymbol{v}}$. It is a free choice, as long as we stick to it. We wanted to use the single symbol $^{\mathrm{H}}$ in the next formula too:

The inner product of $A\boldsymbol{u}$ with $\boldsymbol{v}$ equals the inner product of $\boldsymbol{u}$ with $A^{\mathrm{H}}\boldsymbol{v}$:

$$A^{\mathrm{H}} = \text{“adjoint” of } A \qquad (A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}). \tag{4}$$

The conjugate of $A\boldsymbol{u}$ is $\overline{A\boldsymbol{u}}$. Transposing it gives $\overline{\boldsymbol{u}}^{\mathrm{T}}\overline{A}^{\mathrm{T}}$ as usual. This is $\boldsymbol{u}^{\mathrm{H}}A^{\mathrm{H}}$. Everything that should work, does work. The rule for $^{\mathrm{H}}$ comes from the rule for $^{\mathrm{T}}$. That applies to products of matrices:

The conjugate transpose of AB is $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.

We constantly use the fact that $(a - ib)(c - id)$ is the conjugate of $(a + ib)(c + id)$.

Hermitian Matrices

Among real matrices, the *symmetric matrices* form the most important special class: $A = A^{\mathrm{T}}$. They have real eigenvalues and a full set of orthogonal eigenvectors. The diagonalizing matrix S is an orthogonal matrix Q. Every symmetric matrix can be written as $A = Q\Lambda Q^{-1}$ and also as $A = Q\Lambda Q^{\mathrm{T}}$ (because $Q^{-1} = Q^{\mathrm{T}}$). All this follows from $a_{ij} = a_{ji}$, when A is real.

Among complex matrices, the special class contains the ***Hermitian matrices***: $A = A^{\mathrm{H}}$. The condition on the entries is $a_{ij} = \overline{a_{ji}}$. In this case we say that "*A is* Hermitian." *Every real symmetric matrix is Hermitian*, because taking its conjugate has no effect. The next matrix is also Hermitian, $A = A^{\mathrm{H}}$:

Example 2 $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ The main diagonal is real since $a_{ii} = \overline{a_{ii}}$. Across it are conjugates $3+3i$ and $3-3i$.

This example will illustrate the three crucial properties of all Hermitian matrices.

If $A = A^{\mathrm{H}}$ and z is any vector, the number $z^{\mathrm{H}}Az$ is real.

Quick proof: $z^{\mathrm{H}}Az$ is certainly 1 by 1. Take its conjugate transpose:

$$(z^{\mathrm{H}}Az)^{\mathrm{H}} = z^{\mathrm{H}}A^{\mathrm{H}}(z^{\mathrm{H}})^{\mathrm{H}} \quad \text{which is } z^{\mathrm{H}}Az \text{ again.}$$

This used $A = A^{\mathrm{H}}$. So the number $z^{\mathrm{H}}Az$ equals its conjugate and must be real. Here is that "energy" $z^{\mathrm{H}}Az$ in our example:

$$\begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{2\bar{z}_1 z_1 + 5\bar{z}_2 z_2}_{\textit{diagonal}} + \underbrace{(3-3i)\bar{z}_1 z_2 + (3+3i) z_1 \bar{z}_2}_{\textit{off-diagonal}}.$$

The terms $2|z_1|^2$ and $5|z_2|^2$ from the diagonal are both real. The off-diagonal terms are conjugates of each other—so their sum is real. (The imaginary parts cancel when we add.) The whole expression $z^{\mathrm{H}}Az$ is real, and this will make λ real.

Every eigenvalue of a Hermitian matrix is real.

Proof Suppose $Az = \lambda z$. *Multiply both sides by z^{H} to get $z^{\mathrm{H}}Az = \lambda z^{\mathrm{H}}z$.* On the left side, $z^{\mathrm{H}}Az$ is real. On the right side, $z^{\mathrm{H}}z$ is the length squared, real and positive. So the ratio $\lambda = z^{\mathrm{H}}Az/z^{\mathrm{H}}z$ is a real number. Q.E.D.

The example above has eigenvalues $\lambda = 8$ and $\lambda = -1$, real because $A = A^{\mathrm{H}}$:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$
$$= \lambda^2 - 7\lambda + 10 - 18 = (\lambda - 8)(\lambda + 1).$$

The eigenvectors of a Hermitian matrix are orthogonal (when they correspond to different eigenvalues). If $Az = \lambda z$ and $Ay = \beta y$ then $y^{\mathrm{H}}z = 0$.

Proof Multiply $Az = \lambda z$ on the left by y^{H}. Multiply $y^{\mathrm{H}}A^{\mathrm{H}} = \beta y^{\mathrm{H}}$ on the right by z:

$$y^{\mathrm{H}}Az = \lambda y^{\mathrm{H}}z \quad \text{and} \quad y^{\mathrm{H}}A^{\mathrm{H}}z = \beta y^{\mathrm{H}}z. \tag{5}$$

The left sides are equal because $A = A^{\mathrm{H}}$. Therefore the right sides are equal. Since β is different from λ, the other factor $y^{\mathrm{H}}z$ must be zero. The eigenvectors are orthogonal, as in our example with $\lambda = 8$ and $\beta = -1$:

$$(A-8I)z = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A+I)y = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}.$$

Take the inner product of those eigenvectors y and z:

Orthogonal eigenvectors $$y^{\mathrm{H}}z = \begin{bmatrix} 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = 0.$$

These eigenvectors have squared length $1^2 + 1^2 + 1^2 = 3$. After division by $\sqrt{3}$ they are unit vectors. They were orthogonal, now they are ***orthonormal.*** They go into the columns of the *eigenvector matrix* S, which diagonalizes A.

When A is real and symmetric, S is Q—an orthogonal matrix. Now A is complex and Hermitian. Its eigenvectors are complex and orthonormal. ***The eigenvector matrix S is like Q, but complex.*** We now assign a new name "unitary" and a new letter U to a complex orthogonal matrix.

Unitary Matrices

A ***unitary matrix*** U is a (complex) square matrix that has ***orthonormal columns.*** U is the complex equivalent of Q. The eigenvectors of A give a perfect example:

Unitary matrix $$U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

This U is also a Hermitian matrix. I didn't expect that! The example is almost too perfect. We will see that the eigenvalues of this U must be 1 and -1.

The matrix test for real orthonormal columns was $Q^{\mathrm{T}}Q = I$. When Q^{T} multiplies Q, the zero inner products appear off the diagonal. In the complex case, Q becomes U. The columns show themselves as orthonormal when U^{H} multiplies U. The inner products of the columns are again 1 and 0. They fill up $\boldsymbol{U^{\mathrm{H}}U = I}$:

Every matrix U with orthonormal columns has $U^{\mathrm{H}}U = I$.

If U is square, it is a unitary matrix. Then $\boxed{U^{\mathrm{H}} = U^{-1}}$.

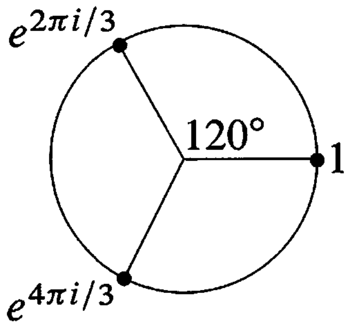
Suppose U (with orthonormal columns) multiplies any z. The vector length stays the same, because $z^{\mathrm{H}}U^{\mathrm{H}}Uz = z^{\mathrm{H}}z$. If z is an eigenvector of U we learn something more: ***The eigenvalues of unitary (and orthogonal) matrices all have absolute value*** $|\lambda| = 1$.

If U is unitary then $\|Uz\| = \|z\|$. Therefore $Uz = \lambda z$ leads to $|\lambda| = 1$.

Our 2 by 2 example is both Hermitian ($U = U^{\mathrm{H}}$) and unitary ($U^{-1} = U^{\mathrm{H}}$). That means real eigenvalues ($\lambda = \overline{\lambda}$), and it means $|\lambda| = 1$. A real number with absolute value 1 has only two possibilities: *The eigenvalues are* 1 *or* -1.

Since the trace is zero for our U, one eigenvalue is $\lambda = 1$ and the other is $\lambda = -1$.

Example 3 The 3 by 3 ***Fourier matrix*** is in Figure 10.3. Is it Hermitian? Is it unitary? F_3 is certainly symmetric. It equals its transpose. But it doesn't equal its conjugate transpose—*it is not Hermitian.* If you change i to $-i$, you get a different matrix.



Fourier matrix $$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 10.3: The cube roots of 1 go into the Fourier matrix $F = F_3$.

Is F unitary? *Yes.* The squared length of every column is $\frac{1}{3}(1 + 1 + 1)$ (unit vector). The first column is orthogonal to the second column because $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. This is the sum of the three numbers marked in Figure 10.3.

Notice the symmetry of the figure. If you rotate it by 120°, the three points are in the same position. Therefore their sum S also stays in the same position! The only possible sum in the same position after 120° rotation is $S = 0$.

Is column 2 of F orthogonal to column 3? Their dot product looks like

$$\tfrac{1}{3}(1 + e^{6\pi i/3} + e^{6\pi i/3}) = \tfrac{1}{3}(1 + 1 + 1).$$

This is not zero. The answer is wrong because we forgot to take complex conjugates. The complex inner product uses $^{\mathrm{H}}$ not $^{\mathrm{T}}$:

$$\begin{aligned}(\text{column } 2)^{\mathrm{H}}(\text{column } 3) &= \tfrac{1}{3}(1 \cdot 1 + e^{-2\pi i/3}e^{4\pi i/3} + e^{-4\pi i/3}e^{2\pi i/3}) \\ &= \tfrac{1}{3}(1 + e^{2\pi i/3} + e^{-2\pi i/3}) = 0.\end{aligned}$$

So we do have orthogonality. ***Conclusion: F is a unitary matrix.***

The next section will study the n by n Fourier matrices. Among all complex unitary matrices, these are the most important. When we multiply a vector by F, we are computing its ***Discrete Fourier Transform.*** When we multiply by F^{-1}, we are computing the *inverse transform.* The special property of unitary matrices is that $F^{-1} = F^{\mathrm{H}}$. The inverse transform only differs by changing i to $-i$:

$$\textbf{Change } i \textbf{ to } -i \qquad F^{-1} = F^{\mathrm{H}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-2\pi i/3} \end{bmatrix}.$$

Everyone who works with F recognizes its value. The last section of the book will bring together Fourier analysis and complex numbers and linear algebra.

This section ends with a table to translate between real and complex—for vectors and for matrices:

Real versus Complex

$\mathbf{R}^n$: vectors with n real components	$\leftrightarrow$	$\mathbf{C}^n$: vectors with n complex components
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|z\|^2 = \lvert z_1\rvert^2 + \cdots + \lvert z_n\rvert^2$
transpose: $(A^{\mathrm{T}})_{ij} = A_{ji}$	$\leftrightarrow$	conjugate transpose: $(A^{\mathrm{H}})_{ij} = \overline{A_{ji}}$
product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \overline{u}_1v_1 + \cdots + \overline{u}_nv_n$
reason for A^{T}: $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$	$\leftrightarrow$	reason for A^{H}: $(A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v})$
orthogonality: $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$	$\leftrightarrow$	orthogonality: $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 0$
symmetric matrices: $A = A^{\mathrm{T}}$	$\leftrightarrow$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}(\text{real } \Lambda)$	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}\ (\text{real } \Lambda)$
skew-symmetric matrices: $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian matrices $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	$\leftrightarrow$	orthonormal columns: $U^{\mathrm{H}}U = I$
$(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ and $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$	$\leftrightarrow$	$(U\boldsymbol{x})^{\mathrm{H}}(U\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$ and $\|U\boldsymbol{z}\| = \|\boldsymbol{z}\|$

The columns and also the eigenvectors of Q and U are orthonormal. Every $|\lambda| = 1$.

Problem Set 10.2

1 Find the lengths of $\boldsymbol{u} = (1+i, 1-i, 1+2i)$ and $\boldsymbol{v} = (i, i, i)$. Also find $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ and $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$.

2 Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

3 Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $\boldsymbol{C}(A^{\mathrm{T}})$. Now it is $\boldsymbol{C}(A^{\mathrm{H}})$.

4 Problem 3 indicates that the four fundamental subspaces are $C(A)$ and $N(A)$ and _____ and _____. Their dimensions are still r and $n-r$ and r and $m-r$. They are still orthogonal subspaces. *The symbol* $^{\mathrm{H}}$ *takes the place of* $^{\mathrm{T}}$.

5 (a) Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix.

(b) If $Az = \mathbf{0}$ then $A^{\mathrm{H}}Az = \mathbf{0}$. If $A^{\mathrm{H}}Az = \mathbf{0}$, multiply by z^{H} to prove that $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are _____. Therefore $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $z = 0$.

6 True or false (give a reason if true or a counterexample if false):

(a) If A is a real matrix then $A + iI$ is invertible.

(b) If A is a Hermitian matrix then $A + iI$ is invertible.

(c) If U is a unitary matrix then $A + iI$ is invertible.

7 When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? Show that iA is skew-Hermitian when A is Hermitian. The 3 by 3 Hermitian matrices are a subspace provided the "scalars" are real numbers.

8 Which classes of matrices does P belong to: invertible, Hermitian, unitary?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

9 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix F. What property of P makes these eigenvectors orthogonal?

10 Write down the 3 by 3 circulant matrix $C = 2I + 5P$. It has the same eigenvectors as P in Problem 8. Find its eigenvalues.

11 If U and V are unitary matrices, show that U^{-1} is unitary and also UV is unitary. Start from $U^{\mathrm{H}}U = I$ and $V^{\mathrm{H}}V = I$.

12 How do you know that the determinant of every Hermitian matrix is real?

13 The matrix $A^{\mathrm{H}}A$ is not only Hermitian but also positive definite, when the columns of A are independent. Proof: $z^{\mathrm{H}}A^{\mathrm{H}}Az$ is positive if z is nonzero because _____.

14 Diagonalize this Hermitian matrix to reach $A = U\Lambda U^{\mathrm{H}}$:

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}.$$

15 Diagonalize this skew-Hermitian matrix to reach $K = U\Lambda U^{\mathrm{H}}$. All λ's are _____:

$$K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}.$$

16 Diagonalize this orthogonal matrix to reach $Q = U\Lambda U^{\mathrm{H}}$. Now all λ's are _____:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

17 Diagonalize this unitary matrix V to reach $V = U\Lambda U^{\mathrm{H}}$. Again all λ's are _____:

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

18 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis for $\mathbf{C}^n$, the matrix with those columns is a _____ matrix. Show that any vector $\boldsymbol{z}$ equals $(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_1 + \cdots + (\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_n$.

19 The functions e^{-ix} and e^{ix} are orthogonal on the interval $0 \le x \le 2\pi$ because their inner product is $\int_0^{2\pi}$ _____ $= 0$.

20 The vectors $\boldsymbol{v} = (1, i, 1)$, $\boldsymbol{w} = (i, 1, 0)$ and $\boldsymbol{z} =$ _____ are an orthogonal basis for _____.

21 If $A = R + iS$ is a Hermitian matrix, are its real and imaginary parts symmetric?

22 The (complex) dimension of $\mathbf{C}^n$ is _____. Find a non-real basis for $\mathbf{C}^n$.

23 Describe all 1 by 1 and 2 by 2 Hermitian matrices and unitary matrices.

24 How are the eigenvalues of A^{H} related to the eigenvalues of the square complex matrix A?

25 If $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u} = 1$ show that $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is the projection onto what line in $\mathbf{C}^n$?

26 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \left[\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right]$ is an orthogonal matrix.

27 If $A + iB$ is Hermitian (A and B are real) show that $\left[\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right]$ is symmetric.

28 Prove that the inverse of a Hermitian matrix is also Hermitian (transpose $A^{-1}A = I$).

29 Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^{\mathrm{H}}.$$

30 A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$. *Prove that* $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. These are exactly the **normal matrices**. Examples are Hermitian, skew-Hermitian, and unitary matrices. Construct a 2 by 2 normal matrix by choosing complex eigenvalues in Λ.

10.3 The Fast Fourier Transform

Many applications of linear algebra take time to develop. It is not easy to explain them in an hour. The teacher and the author must choose between completing the theory and adding new applications. Often the theory wins, but this section is an exception. It explains the most valuable numerical algorithm in the last century.

We want to multiply quickly by F and F^{-1}, the Fourier matrix and its inverse. This is achieved by the Fast Fourier Transform. An ordinary product Fc uses n^2 multiplications (F has n^2 entries). The FFT needs only n times $\frac{1}{2}\log_2 n$. We will see how.

The FFT has revolutionized signal processing. Whole industries are speeded up by this one idea. Electrical engineers are the first to know the difference—they take your Fourier transform as they meet you (if you are a function). Fourier's idea is to represent f as a sum of harmonics $c_k e^{ikx}$. The function is seen in *frequency space* through the coefficients c_k, instead of *physical space* through its values $f(x)$. The passage backward and forward between c's and f's is by the Fourier transform. Fast passage is by the FFT.

Roots of Unity and the Fourier Matrix

Quadratic equations have two roots (or one repeated root). Equations of degree n have n roots (counting repetitions). This is the Fundamental Theorem of Algebra, and to make it true we must allow complex roots. This section is about the very special equation $z^n = 1$. The solutions z are the "nth roots of unity." They are n evenly spaced points around the unit circle in the complex plane.

Figure 10.4 shows the eight solutions to $z^8 = 1$. Their spacing is $\frac{1}{8}(360°) = 45°$. The first root is at $45°$ or $\theta = 2\pi/8$ radians. ***It is the complex number $w = e^{i\theta} = e^{i2\pi/8}$.*** We call this number w_8 to emphasize that it is an 8th root. You could write it in terms of $\cos\frac{2\pi}{8}$ and $\sin\frac{2\pi}{8}$, but don't do it. The seven other 8th roots are $w^2, w^3, \ldots, w^8$, going around the circle. Powers of w are best in polar form, because we work only with the angles $\frac{2\pi}{8}, \frac{4\pi}{8}, \cdots, \frac{16\pi}{8} = 2\pi$.

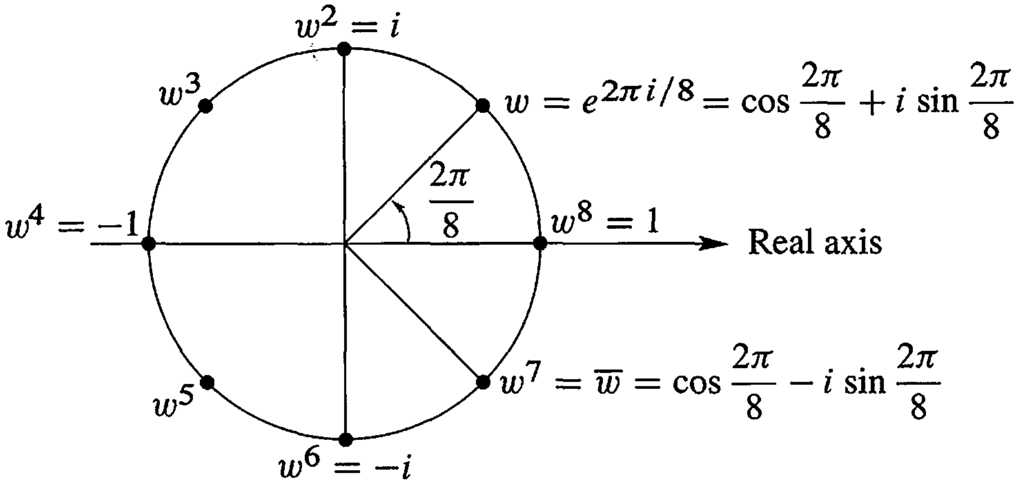


Figure 10.4: The eight solutions to $z^8 = 1$ are $1, w, w^2, \ldots, w^7$ with $w = (1+i)/\sqrt{2}$.

The fourth roots of 1 are also in the figure. They are $i, -1, -i, 1$. The angle is now $2\pi/4$ or $90°$. The first root $w_4 = e^{2\pi i/4}$ is nothing but i. Even the square roots of 1 are seen, with $w_2 = e^{i2\pi/2} = -1$. Do not despise those square roots 1 and -1. The idea behind the FFT is to go from an 8 by 8 Fourier matrix (containing powers of w_8) to the 4 by 4 matrix below (with powers of $w_4 = i$). The same idea goes from 4 to 2. By exploiting the connections of F_8 down to F_4 and up to F_{16} (and beyond), the FFT makes multiplication by F_{1024} very quick.

We describe the *Fourier matrix*, first for $n = 4$. Its rows contain powers of 1 and w and w^2 and w^3. These are the fourth roots of 1, and their powers come in a special order.

Fourier matrix $n = 4$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}.$$

The matrix is symmetric ($F = F^{\mathrm{T}}$). It is *not* Hermitian. Its main diagonal is not real. But $\frac{1}{2}F$ is a ***unitary matrix***, which means that $(\frac{1}{2}F^{\mathrm{H}})(\frac{1}{2}F) = I$:

The columns of F give $\boldsymbol{F^{\mathrm{H}}F = 4I}$. ***Its inverse is*** $\boldsymbol{\frac{1}{4}F^{\mathrm{H}}}$ ***which is*** $\boldsymbol{F^{-1} = \frac{1}{4}\overline{F}}$.

The inverse changes from $w = i$ to $\overline{w} = -i$. That takes us from F to $\overline{F}$. When the Fast Fourier Transform gives a quick way to multiply by F, it does the same for F^{-1}.

The unitary matrix is $U = F/\sqrt{n}$. We avoid that $\sqrt{n}$ and just put $\frac{1}{n}$ outside F^{-1}. The main point is to multiply F times the Fourier coefficients c_0, c_1, c_2, c_3:

4-point Fourier series

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{1}$$

The input is four complex coefficients c_0, c_1, c_2, c_3. The output is four function values y_0, y_1, y_2, y_3. The first output $y_0 = c_0 + c_1 + c_2 + c_3$ is the value of the Fourier series at $x = 0$. *The second output is the value of that series* $\sum c_k e^{ikx}$ *at* $x = 2\pi/4$:

$$y_1 = c_0 + c_1 e^{i2\pi/4} + c_2 e^{i4\pi/4} + c_3 e^{i6\pi/4} = c_0 + c_1 w + c_2 w^2 + c_3 w^3.$$

The third and fourth outputs y_2 and y_3 are the values of $\sum c_k e^{ikx}$ at $x = 4\pi/4$ and $x = 6\pi/4$. These are *finite* Fourier series! They contain $n = 4$ terms and they are evaluated at $n = 4$ points. Those points $x = 0, 2\pi/4, 4\pi/4, 6\pi/4$ are equally spaced.

The next point would be $x = 8\pi/4$ which is 2π. Then the series is back to y_0, because $e^{2\pi i}$ is the same as $e^0 = 1$. Everything cycles around with period 4. In this world $2 + 2$ is 0 because $(w^2)(w^2) = w^0 = 1$. We will follow the convention that j *and* k *go from* 0 *to* $n - 1$ (instead of 1 to n). The "zeroth row" and "zeroth column" of F contain all ones.

The n by n Fourier matrix contains powers of $w = e^{2\pi i/n}$:

$$F_n c = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{n-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{n-1} & w^{2(n-1)} & \cdot & w^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ y_{n-1} \end{bmatrix} = y. \qquad (2)$$

F_n is symmetric but not Hermitian. ***Its columns are orthogonal***, and $F_n \overline{F}_n = nI$. *Then* F_n^{-1} *is* $\overline{F}_n / n$. The inverse contains powers of $\overline{w}_n = e^{-2\pi i/n}$. Look at the pattern in F:

The entry in row j, column k is w^{jk}. Row zero and column zero contain $w^0 = 1$.

When we multiply c by F_n, we sum the series at n points. *When we multiply y by F_n^{-1}, we find the coefficients c from the function values y.* In MATLAB that command is $c = \text{fft}(y)$. The matrix F passes from "frequency space" to "physical space."

Important note. Many authors prefer to work with $\omega = e^{-2\pi i/N}$, which is the *complex conjugate* of our w. (They often use the Greek omega, and I will do that to keep the two options separate.) With this choice, their DFT matrix contains powers of ω not w. It is conj (F) = complex conjugate of our F. This takes us to frequency space.

$\overline{F}$ is a completely reasonable choice! MATLAB uses $\omega = e^{-2\pi i/N}$. The DFT matrix fft(eye(N)) contains powers of this number $\omega = \overline{w}$. **The Fourier matrix with w's reconstructs y from c.** ***The matrix $\overline{F}$ with ω's computes Fourier coefficients as* fft(y).**

Also important. When a function $f(x)$ has period 2π, and we change x to $e^{i\theta}$, the function is defined around the unit circle (where $z = e^{i\theta}$). Then the Discrete Fourier Transform from y to c is matching n values of this $f(z)$ by a polynomial $p(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$.

Interpolation Find $c_0, \ldots, c_{n-1}$ so that $p(z) = f(z)$ at n points $z = 1, \ldots, w^{n-1}$

The Fourier matrix is the Vandermonde matrix for interpolation at those n points.

One Step of the Fast Fourier Transform

We want to multiply F times c as quickly as possible. Normally a matrix times a vector takes n^2 separate multiplications—the matrix has n^2 entries. You might think it is impossible to do better. (If the matrix has zero entries then multiplications can be skipped. But the Fourier matrix has no zeros!) By using the special pattern w^{jk} for its entries, F can be factored in a way that produces many zeros. This is the **FFT**.

The key idea is to connect F_n with the half-size Fourier matrix $F_{n/2}$. Assume that n is a power of 2 (say $n = 2^{10} = 1024$). We will connect F_{1024} to F_{512}—or rather to ***two***

copies of F_{512}. When $n = 4$, the key is in the relation between these matrices:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}.$$

On the left is F_4, with no zeros. On the right is a matrix that is half zero. The work is cut in half. But wait, those matrices are not the same. We need two sparse and simple matrices to complete the FFT factorization:

Factors for FFT
$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{3}$$

The last matrix is a permutation. It puts the even c's (c_0 and c_2) ahead of the odd c's (c_1 and c_3). The middle matrix performs half-size transforms F_2 and F_2 on the evens and odds. The matrix at the left combines the two half-size outputs—in a way that produces the correct full-size output $\boldsymbol{y} = F_4\boldsymbol{c}$.

The same idea applies when $n = 1024$ and $m = \frac{1}{2}n = 512$. The number w is $e^{2\pi i/1024}$. It is at the angle $\theta = 2\pi/1024$ on the unit circle. The Fourier matrix F_{1024} is full of powers of w. The first stage of the FFT is the great factorization discovered by Cooley and Tukey (and foreshadowed in 1805 by Gauss):

$$\boldsymbol{F_{1024}} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} \boldsymbol{F_{512}} & \\ & \boldsymbol{F_{512}} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}. \tag{4}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, w, \ldots, w^{511})$. The two copies of F_{512} are what we expected. Don't forget that they use the 512th root of unity (which is nothing but w^2!!) The permutation matrix separates the incoming vector $\boldsymbol{c}$ into its even and odd parts $\boldsymbol{c}' = (c_0, c_2, \ldots, c_{1022})$ and $\boldsymbol{c}'' = (c_1, c_3, \ldots, c_{1023})$.

Here are the algebra formulas which say the same thing as the factorization of F_{1024}:

(FFT) Set $m = \frac{1}{2}n$. The first m and last m components of $\boldsymbol{y} = F_n\boldsymbol{c}$ combine the half-size transforms $\boldsymbol{y}' = F_m\boldsymbol{c}'$ and $\boldsymbol{y}'' = F_m\boldsymbol{c}''$. Equation (4) shows this step from n to $m = n/2$ as $I\boldsymbol{y}' + D\boldsymbol{y}''$ and $I\boldsymbol{y}' - D\boldsymbol{y}''$:

$$\begin{aligned} y_j &= y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1 \\ y_{j+m} &= y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1. \end{aligned} \tag{5}$$

Split $\boldsymbol{c}$ into $\boldsymbol{c}'$ and $\boldsymbol{c}''$, transform them by F_m into $\boldsymbol{y}'$ and $\boldsymbol{y}''$, and reconstruct $\boldsymbol{y}$.

Those formulas come from separating even c_{2k} from odd c_{2k+1}:

$$y_j = \sum_0^{n-1} w^{jk} c_k = \sum_0^{m-1} w^{2jk} c_{2k} + \sum_0^{m-1} w^{j(2k+1)} c_{2k+1} \text{ with } m = \tfrac{1}{2}n. \tag{6}$$

The even c's go into $c' = (c_0, c_2, \ldots)$ and the odd c's go into $c'' = (c_1, c_3, \ldots)$. Then come the transforms $F_m c'$ and $F_m c''$. The key is $\boldsymbol{w_n^2 = w_m}$. This gives $w_n^{2jk} = w_m^{jk}$.

$$\textbf{Rewrite} \qquad y_j = \sum w_m^{jk} c_k' + (w_n)^j \sum w_m^{jk} c_k'' = y_j' + (w_n)^j y_j''. \tag{7}$$

For $j \geq m$, the minus sign in (5) comes from factoring out $(w_n)^m = -1$.

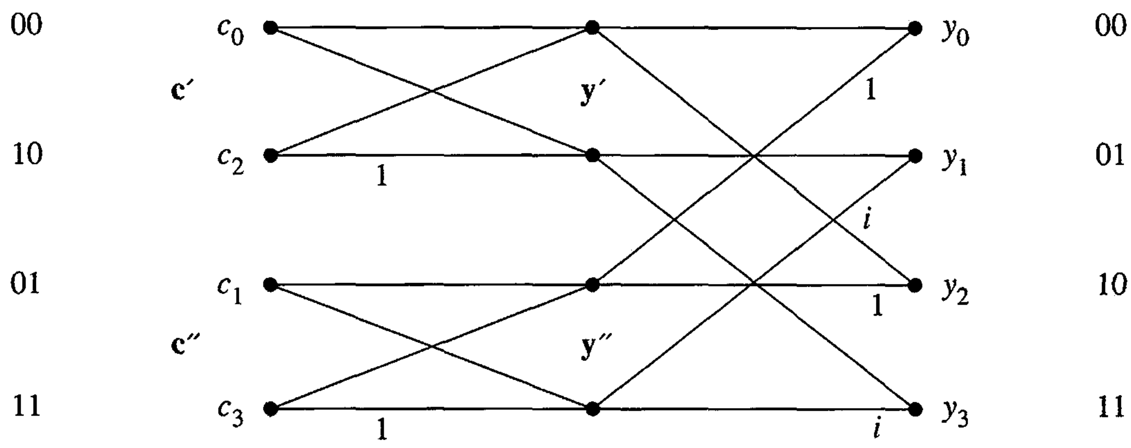
MATLAB easily separates even c's from odd c's and multiplies by w_n^j. We use conj(F) or equivalently MATLAB's inverse transform ifft, because fft is based on $\omega = \overline{w} = e^{-2\pi i/n}$. Problem 17 shows that F and conj(F) are linked by permuting rows.

FFT step from n to $n/2$ in MATLAB

$$\begin{aligned} y' &= \text{ifft}\,(c(0:2:n-2)) * n/2; \\ y'' &= \text{ifft}\,(c(1:2:n-1)) * n/2; \\ d &= w.\hat{}(0:n/2-1)'; \\ y &= [y' + d.* y''; \; y' - d.* y'']; \end{aligned}$$

The flow graph shows c' and c'' going through the half-size F_2. Those steps are called "*butterflies*," from their shape. Then the outputs y' and y'' are combined (multiplying y'' by $1, i$ and also by $-1, -i$) to produce $y = F_4 c$.

This reduction from F_n to two F_m's almost cuts the work in half—you see the zeros in the matrix factorization. That reduction is good but not great. The full idea of the **FFT** is much more powerful. It saves much more than half the time.



The Full FFT by Recursion

If you have read this far, you have probably guessed what comes next. We reduced F_n to $F_{n/2}$. ***Keep going to*** $F_{n/4}$. The matrices F_{512} lead to F_{256} (in four copies). Then 256 leads to 128. *That is recursion.* It is a basic principle of many fast algorithms, and here is the second stage with four copies of $F = F_{256}$ and $D = D_{256}$:

$$\begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick} & 0, 4, 8, \cdots \\ \text{pick} & 2, 6, 10, \cdots \\ \text{pick} & 1, 5, 9, \cdots \\ \text{pick} & 3, 7, 11, \cdots \end{bmatrix}.$$

We will count the individual multiplications, to see how much is saved. Before the **FFT** was invented, the count was the usual $n^2 = (1024)^2$. This is about a million multiplications. I am not saying that they take a long time. The cost becomes large when we have many, many transforms to do—which is typical. Then the saving by the FFT is also large:

$$\textit{The final count for size } n = 2^\ell \textit{ is reduced from } n^2 \textit{ to } \tfrac{1}{2}n\ell.$$

The number 1024 is 2^{10}, so $\ell = 10$. The original count of $(1024)^2$ is reduced to (5)(1024). The saving is a factor of 200. A million is reduced to five thousand. That is why the FFT has revolutionized signal processing.

Here is the reasoning behind $\frac{1}{2}n\ell$. There are ℓ levels, going from $n = 2^\ell$ down to $n = 1$. Each level has $n/2$ multiplications from the diagonal D's, to reassemble the half-size outputs from the lower level. This yields the final count $\frac{1}{2}n\ell$, which is $\frac{1}{2}n \log_2 n$.

One last note about this remarkable algorithm. There is an amazing rule for the order that the c's enter the FFT, after all the even-odd permutations. Write the numbers 0 to $n-1$ in binary (base 2). *Reverse the order of their digits.* The complete picture shows the bit-reversed order at the start, the $\ell = \log_2 n$ steps of the recursion, and the final output $y_0, \ldots, y_{n-1}$ which is F_n times c.

The book ends with that very fundamental idea, a matrix multiplying a vector.

Thank you for studying linear algebra. I hope you enjoyed it, and I very much hope you will use it. It was a pleasure to write about this tremendously useful subject.

Problem Set 10.3

1 Multiply the three matrices in equation (3) and compare with F. In which six entries do you need to know that $i^2 = -1$?

2 Invert the three factors in equation (3) to find a fast factorization of F^{-1}.

3 F is symmetric. So transpose equation (3) to find a new Fast Fourier Transform!

4 All entries in the factorization of F_6 involve powers of $w_6 =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these matrices with $1, w_6, w_6^2$ in D and $w_3 = w_6^2$ in F_3. Multiply!

5 If $\boldsymbol{v} = (1,0,0,0)$ and $\boldsymbol{w} = (1,1,1,1)$, show that $F\boldsymbol{v} = \boldsymbol{w}$ and $F\boldsymbol{w} = 4\boldsymbol{v}$. Therefore $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} =$ _____.

6 What is F^2 and what is F^4 for the 4 by 4 Fourier matrix?

7 Put the vector $\boldsymbol{c} = (1,0,1,0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0,1,0,1)$.

8 Compute $y = F_8c$ by the three FFT steps for $c = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $c = (0, 1, 0, 1, 0, 1, 0, 1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

10 (a) Draw all the sixth roots of 1 on the unit circle. Prove they add to zero.

(b) What are the three cube roots of 1? Do they also add to zero?

11 The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_1 to λ_4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$. The eigenvector matrix (usually S) is F.

12 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. This shows again that the eigenvalue matrix Λ is _____. Which permutation P has eigenvalues = cube roots of 1?

13 (a) Two eigenvectors of C are $(1, 1, 1, 1)$ and $(1, i, i^2, i^3)$. Find the eigenvalues.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2 F^{-1}$ and $P^3 = F\Lambda^3 F^{-1}$. Then $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = F(c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3) F^{-1} = FEF^{-1}$. That matrix E in parentheses is diagonal. It contains the _____ of C.

14 Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ. The -1's in the corners make this matrix periodic:

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1.$$

15 ***Fast convolution.*** To multiply C times a vector x, we can multiply $F(E(F^{-1}x))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

16 Why is row i of $\overline{F}$ the same as row $N - i$ of F (numbered 0 to $N - 1$)?

Solutions to Selected Exercises

Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum = zero vector. Sum = $-$2:00 vector = 8:00 vector. 2:00 is 30° from horizontal $= (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

16 All combinations with $c + d = 1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $V = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v} + c\boldsymbol{w}$ are on the line passing through $(0, 0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v} + \boldsymbol{w}$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0, 0)$.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0, d \geq 0, e \geq 0$, and $c + d + e = \mathbf{1}$.

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

Problem Set 1.2, page 19

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

6 All vectors $w = (c, 2c)$ are perpendicular to v. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1+v_2w_2 = v\cdot w = 0$: perpendicular!

11 $v \cdot w < 0$ means angle $> 90°$; these w's fill half of 3-dimensional space.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $v \cdot (w - cv) = 0$ if $c = v \cdot w/v \cdot v$. Subtracting cv is the key to perpendicular vectors.

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

17 $\cos\alpha = 1/\sqrt{2}, \cos\beta = 0, \cos\gamma = -1/\sqrt{2}$. For any vector v, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|v\|^2 = 1$.

21 $2v\cdot w \leq 2\|v\|\|w\|$ leads to $\|v+w\|^2 = v\cdot v+2v\cdot w+w\cdot w \leq \|v\|^2+2\|v\|\|w\|+\|w\|^2$. This is $(\|v\| + \|w\|)^2$. Taking square roots gives $\|v + w\| \leq \|v\| + \|w\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \leq v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \geq 0$.

23 $\cos\beta = w_1/\|w\|$ and $\sin\beta = w_2/\|w\|$. Then $\cos(\beta-a) = \cos\beta\cos\alpha+\sin\beta\sin\alpha = v_1w_1/\|v\|\|w\| + v_2w_2/\|v\|\|w\| = v \cdot w/\|v\|\|w\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \leq \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \leq \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \leq (.6)(.8) + (.8)(.6) \leq \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

28 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

29 Try $v = (1, 2, -3)$ and $w = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $v \cdot w = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2) = -\frac{1}{2}\|v\|\|w\|$. Then $v \cdot w/\|v\|\|w\| = -\frac{1}{2}$.

Problem Set 1.3, page 29

1 $2s_1 + 3s_2 + 4s_3 = (2, 5, 9)$. The same vector b comes from S times $x = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot x \\ (\text{row } 2) \cdot x \\ (\text{row } 2) \cdot x \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1, y_2 = 0, y_3 = 0$ (right side = column 1) and $y_1 = 1, y_2 = 3, y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

4 The combination $0w_1 + 0w_2 + 0w_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $w_2 = (w_1 + w_3)/2$ so one combination that gives zero is $\frac{1}{2}w_1 - w_2 + \frac{1}{2}w_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $r_2 = \frac{1}{2}(r_1 + r_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

7 All three rows are perpendicular to the solution x (the three equations $r_1 \cdot x = 0$ and $r_2\cdot x = 0$ and $r_3\cdot x = 0$ tell us this). Then the whole plane of the rows is perpendicular to x (the plane is also perpendicular to all multiples cx).

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $Cx = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \phantom{{}- x_1} &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ - x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$ and $\boldsymbol{b} = (2,3,4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution.*

8 Four planes in 4-dimensional space normally meet at a *point.* The solution to $A\boldsymbol{x} = (3,3,3,2)$ is $\boldsymbol{x} = (0,0,1,2)$ if A has columns $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane.*

16 $90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

22 The dot product $Ax = [1 \;\; 4 \;\; 5] \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

25 **ones**$(4, 4) *$ **ones**$(4, 1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four-dimensional* space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* s. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1, 1, 1) = (15, 15, 15)$;
$M_4(1, 1, 1, 1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

14 Subtract 2 times row 1 from row 2 to reach $(d - 10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular — no third pivot. Then if $t=5$ the third equation is $0=0$. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$ row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution.

25 $a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

28 $A(2,:)=A(2,:)-3*A(1,:)$ will subtract 3 times row 1 from row 2.

29 Pivots 2 and 3 can be arbitrarily large. I believe their averages are infinite! *With row exchanges* in MATLAB's lu code, the averages are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1,\ldots,j$ of A. If $A\boldsymbol{x}=\boldsymbol{0}$ then $U\boldsymbol{x}=\boldsymbol{0}$ (not true if $\boldsymbol{b}$ replaces $\boldsymbol{0}$). U is the diagonal of A when A is *lower triangular.*

Problem Set 2.3, page 63

1 $E_{21}=\begin{bmatrix}1&0&0\\-5&1&0\\0&0&1\end{bmatrix}$, $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&7&1\end{bmatrix}$, $P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

3 $\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$ $M=E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\-4&1&0\\10&-2&1\end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot.*

9 $M=\begin{bmatrix}1&0&0\\0&0&1\\-1&1&0\end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$; $E_{31}E_{13}=\begin{bmatrix}2&0&1\\0&1&0\\1&0&1\end{bmatrix}$. Test on the identity matrix!

12 The first product is $\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

18 $EF=\begin{bmatrix}1&0&0\\a&1&0\\b&c&1\end{bmatrix}$, $FE=\begin{bmatrix}1&0&0\\a&1&0\\b+ac&c&1\end{bmatrix}$, $E^2=\begin{bmatrix}1&0&0\\2a&1&0\\2b&0&1\end{bmatrix}$, $F^3=\begin{bmatrix}1&0&0\\0&1&0\\0&3c&1\end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21}-a_{11}$ (c) $a_{21}-2a_{11}$ (d) $(E_{21}Ax)_1 = (Ax)_1 = \sum a_{1j}x_j$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Many solutions if $d = 0 = c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

30 $EM = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ then $FEM = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then $EFEM = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ then $EEFEM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$. So after inverting with $E^{-1} = A$ and $F^{-1} = B$ this is $M = ABAAB$.

Problem Set 2.4, page 75

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF) = (EA)F$: Matrix multiplication is *associative*.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

15 (a) mn (use every entry of A) (b) $mnp = p\times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;
$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

27 (a) (row 3 of A)$\cdot$(column 1 of B) and (row 3 of A)$\cdot$(column 2 of B) are both zero.
(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: **both upper**.

28 A times B with cuts $A\begin{bmatrix} | & | & | & | \end{bmatrix}$, $\begin{bmatrix} \text{—} \end{bmatrix}B$, $\begin{bmatrix} \text{—} \end{bmatrix}\begin{bmatrix} | & | & | & | \end{bmatrix}$, $\begin{bmatrix} | & | \end{bmatrix}\begin{bmatrix} \text{—} \\ \text{—} \end{bmatrix}$

30 In 29, $c = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - cb/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

32 A times $X = [\boldsymbol{x}_1 \ \ \boldsymbol{x}_2 \ \ \boldsymbol{x}_3]$ will be the identity matrix $I = [A\boldsymbol{x}_1 \ \ A\boldsymbol{x}_2 \ \ A\boldsymbol{x}_3]$.

33 $b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $x = 3x_1 + 5x_2 + 8x_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $x_1 = (1,1,1), x_2 = (0,1,1), x_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

35 $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$, **aba, ada** **cba, cda** **bab, bcb** **dab, dcb** **abc, adc** **cbc, cdc** **bad, bcd** **dad, dcd** These show 16 2-step paths in the graph

Problem Set 2.5, page 89

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

7 (a) In $Ax = (1,0,0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $x = (1,1,-1)$ solves $Ax = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

21 Six of the sixteen $0-1$ matrices are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$;

$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$.

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

33 $x = (1,1,\ldots,1)$ has $Px = Qx$ so $(P-Q)x = \mathbf{0}$.

34 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

35 A can be invertible with diagonal zeros. B is singular because each row adds to zero.

38 The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then $\mathrm{inv}(P) = \mathrm{inv}(L^{\mathrm{T}})\mathrm{inv}(L)$.

42 $MM^{-1} = (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V)$ (this is testing formula **3**)
$= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$ (keep simplifying)
$= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$ (formulas **1**, **2**, **4** are similar)

43 4 by 4 still with $T_{11} = 1$ has pivots 1, 1, 1, 1; reversing to $T^* = UL$ makes $T^*_{44} = 1$.

44 Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

Problem Set 2.6, page 102

3 $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse steps to get $A\boldsymbol{u} = \boldsymbol{b}$ from $U\boldsymbol{x} = \boldsymbol{c}$: 1 times $(x+y+z = 5)$+2 times $(y+2z = 2)$+1 times $(z = 2)$ gives $x+3y+6z = 11$.

4 $Lc = \begin{bmatrix}1&&\\1&1&\\1&2&1\end{bmatrix}\begin{bmatrix}5\\2\\2\end{bmatrix} = \begin{bmatrix}5\\7\\11\end{bmatrix}$; $\quad U\boldsymbol{x} = \begin{bmatrix}1&1&1\\&1&2\\&&1\end{bmatrix}\begin{bmatrix}\\ \boldsymbol{x}\\ \\ \end{bmatrix} = \begin{bmatrix}5\\2\\2\end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix}5\\-2\\2\end{bmatrix}$.

6 $\begin{bmatrix}1&&\\0&1&\\0&-2&1\end{bmatrix}\begin{bmatrix}1&&\\-2&1&\\0&0&1\end{bmatrix}A = \begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix} = U$. Then $A = \begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}U$ is the same as $E_{21}^{-1}E_{32}^{-1}U = LU$. The multipliers $\ell_{21}, \ell_{32} = 2$ fall into place in L.

10 $c = 2$ leads to zero in the second pivot position: exchange rows and not singular. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

12 $A = \begin{bmatrix}2&4\\4&11\end{bmatrix} = \begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&4\\0&3\end{bmatrix} = \begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix} = LDU$; U **is** L^{T}

$$\begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&4&0\\0&-4&4\\0&0&4\end{bmatrix} = \begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&&\\&-4&\\&&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix} = LDL^{\mathrm{T}}.$$

14 $\begin{bmatrix}a&r&r&r\\a&b&s&s\\a&b&c&t\\a&b&c&d\end{bmatrix} = \begin{bmatrix}1&&&\\1&1&&\\1&1&1&\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&r&r&r\\&b-r&s-r&s-r\\&&c-s&t-s\\&&&d-t\end{bmatrix}$. Need $\begin{matrix}a\neq 0\\ b\neq r\\ c\neq s\\ d\neq t\end{matrix}$

15 $\begin{bmatrix}1&0\\4&1\end{bmatrix}\boldsymbol{c} = \begin{bmatrix}2\\11\end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix}2\\3\end{bmatrix}$. Then $\begin{bmatrix}2&4\\0&1\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}2\\3\end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix}-5\\3\end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix}2&4\\8&17\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}2\\11\end{bmatrix}$. Forward to $\begin{bmatrix}2&4\\0&1\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}2\\3\end{bmatrix} = \boldsymbol{c}$.

18 (a) Multiply $LDU = L_1D_1U_1$ by inverses to get $L_1^{-1}LD = D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.
(b) L, U, L_1, U_1 have diagonal 1's so $D = D_1$. Then $L_1^{-1}L$ and U_1U^{-1} are both I.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find ℓ and then one for the new pivot!). $T =$ bidiagonal L times bidiagonal U.

23 The 2 by 2 upper submatrix A_2 has the first two pivots 5, 9. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

24 The upper left blocks all factor at the same time as A: A_k is L_kU_k.

25 The i, j entry of L^{-1} is j/i for $i \geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

26 $(K^{-1})_{ij} = j(n-i+1)/(n+1)$ for $i \geq j$ (and symmetric): $(n+1)K^{-1}$ looks good.

Problem Set 2.7, page 115

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. Transpose that to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ A = zero matrix.

6 $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. $(1,2,3,4)$, $(4,3,2,1)$ make 12 even P's.

14 The i,j entry of PAP is the $n-i+1, n-j+1$ entry of A. Diagonal will reverse order.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

20 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

22 $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A = L_1P_1U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

26 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

31 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$; $A^{\mathrm{T}}y = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ $\begin{matrix} \text{1 truck} \\ \text{1 plane} \end{matrix}$

32 $Ax \cdot y$ is the *cost* of inputs while $x \cdot A^{\mathrm{T}}y$ is the *value* of outputs.

33 $P^3 = I$ so three rotations for 360°; P rotates around $(1,1,1)$ by 120°.

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

37 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U$ = upper triangular.

38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix} \text{ is 5 by 5 with } P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^6 = I.$$

Problem Set 3.1, page 127

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(2 + 1) = 8$. The zero vector is the number 1.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$
(b) The plane and line probably intersect in the point $(0, 0, 0)$
(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[\,A \quad \boldsymbol{b}\,] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{(larger column space)} \\ \text{(no solution to } A\boldsymbol{x} = \boldsymbol{b}) \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} (\boldsymbol{b} \text{ is in column space}) \\ (A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A \quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

Problem Set 3.2, page 140

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$
(b) Free variable x_3: solution $(1, -1, 1)$. Special solution for each free variable.

4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

6 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. Total of pivot and free is n.

8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

16 The nullspace contains only $\boldsymbol{x} = \boldsymbol{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1,0,1,0,1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

24 This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

26
30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice rref$(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

32 Any zero rows come after these rows: $R = [\,1 \ \ -2 \ \ -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's!

35 The nullspace of $B = [\,A \ \ A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 If $C\boldsymbol{x} = \boldsymbol{0}$ then $A\boldsymbol{x} = \boldsymbol{0}$ and $B\boldsymbol{x} = \boldsymbol{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection.*

37 *Currents*: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 3.3, page 151

1 (a) and (c) are correct; (d) is false because R might have 1's in nonpivot columns.

3 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = \begin{bmatrix} R_A & R_A \end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero rows go to the bottom

5 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

7 Special solutions in $N = [\,-2 \ \ -4 \ \ 1 \ \ 0;\, -3 \ \ -5 \ \ 0 \ \ 1\,]$ and $[\,1 \ \ 0 \ \ 0;\, 0 \ \ -2 \ \ 1\,]$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

18 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as the R they share.
(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 $A =$ (pivot columns)(nonzero rows of R) $= \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix}\text{columns} \\ \text{times rows}\end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

26 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

27 $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; $\mathbf{rref}(R^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\mathbf{rref}(R^{\mathrm{T}}R) =$ same R

28 The *row-column reduced echelon form* is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is r by r.

Problem Set 3.4, page 163

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - 3\mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - 2\mathbf{b}_1 \end{bmatrix}$ Then $[\,R \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) =$ line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$
(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) Need $b_3 = 2b_2$, because (row 3) $-$ 2(row 2) $= \mathbf{0}$.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}$, $A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists.* The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

18 Rank = 2; rank = 3 unless $q = 2$ (then rank = 2). Transpose has the same rank!

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & \mathbf{-4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 3.5, page 178

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 $= \mathbf{0}$; if $d = 0$ then b(column 1) $-$ a(column 2) $= \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False.* The zero vector is always in the row space.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \ge n)$.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

20 One basis is $(2, 1, 0)$, $(-3, 0, 1)$. A basis for the intersection with the xy plane is $(2, 1, 0)$. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

25 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 The subspace of matrices that have $AS = SA$ has dimension *three*.

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

41 $I = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

42 The dimension of S is (a) zero when $\boldsymbol{x} = \boldsymbol{0}$ (b) one when $\boldsymbol{x} = (1,1,1,1)$ (c) three when $\boldsymbol{x} = (1,1,-1,-1)$ because all rearrangements have $x_1 + \cdots + x_4 = 0$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim S $= 2$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\boldsymbol{0}$. **To be proved**: All coefficients = zero.

Key idea: The part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$, so the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now $\boldsymbol{x} - \boldsymbol{x} = \boldsymbol{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \boldsymbol{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to an m by n matrix fill $\mathbf{R}^n$. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

Problem Set 3.6, page 190

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(N(A^{\mathrm{T}}))$ $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n-r)$ must be 3 (c) $[\,1 \;\; 1\,]$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $N(A)$ and $N(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

6 A: dim $\mathbf{2, 2, 2, 1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $N(A^{\mathrm{T}})$ $(0,1,0)$. B: dim $\mathbf{1, 1, 0, 2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $N(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \boldsymbol{0}$ is in both $N(A)$ and $C(A^{\mathrm{T}})$.

16 If $A\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

18 Row 3−2 row 2+ row 1 = zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A=R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank <2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}+\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

24 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y}=\mathbf{0}$.

26 The rows of $C=AB$ are combinations of the rows of B. So rank $C\le$ rank B. Also rank $C\le$ rank A, because the columns of C are combinations of the columns of A.

29 $a_{11}=1, a_{12}=0, a_{13}=1, a_{22}=0, a_{32}=1, a_{31}=0, a_{23}=1, a_{33}=0, a_{21}=1$.

30 The subspaces for $A=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B=cA$ with $c\neq 0$.

31 (a) $AX=0$ if each column of X is a multiple of $(1,1,1)$; dim(nullspace) = 3. (b) If $AX=B$ then all columns of B add to zero; dimension of the B's = 6. (c) $3+6=\dim(M^{3\times 3})=9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F=G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank = 1).

3 (a) $\begin{bmatrix}1&2&-3\\2&-3&1\\-3&5&-2\end{bmatrix}$ (b) Impossible, $\begin{bmatrix}2\\-3\\5\end{bmatrix}$ not orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$ (c) $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2=0$; take $A=\left[\begin{smallmatrix}1&-1\\1&-1\end{smallmatrix}\right]$ (e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

6 Multiply the equations by $y_1,y_2,y_3=1,1,-1$. Equations add to $0=1$ so no solution: $\boldsymbol{y}=(1,1,-1)$ is in the left nullspace. $A\boldsymbol{x}=\boldsymbol{b}$ would need $0=(\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x}=\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}=1$.

8 $\boldsymbol{x}=\boldsymbol{x}_r+\boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n=\mathbf{0}$ and $A\boldsymbol{x}=A\boldsymbol{x}_r+A\boldsymbol{x}_n=A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $\boldsymbol{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x}=\mathbf{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x}=\mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x}=\mathbf{0}$.

10 (a) With $A^{\mathrm{T}}=A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z}=0$.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r+\boldsymbol{x}_n=(1,-1)+(1,1)=(2,0)$. Notice $\boldsymbol{N}(A^{\mathrm{T}})$ is a plane $(1,0)=(1,1)/2+(1,-1)/2=\boldsymbol{x}_r+\boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W$ = zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $Ax = B\widehat{x}$ means that $\begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} x \\ -\widehat{x} \end{bmatrix} = \mathbf{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $x = (3,1)$ and $\widehat{x} = (1,0)$ and $Ax = B\widehat{x} = (5,6,5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

16 $A^{\mathrm{T}}y = \mathbf{0}$ leads to $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}A^{\mathrm{T}}y = 0$. Then $y \perp Ax$ and $N(A^{\mathrm{T}}) \perp C(A)$.

18 $S^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $S^{\perp}$ is a *subspace* even if S is not.

21 For example $(-5,0,1,1)$ and $(0,1,-1,0)$ span $S^{\perp}$=nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

23 x in $V^{\perp}$ is perpendicular to any vector in V. Since V contains all the vectors in S, x is also perpendicular to any vector in S. So every x in $V^{\perp}$ is also in $S^{\perp}$.

28 (a) $(1,-1,0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $C(B) \leq$ dimension of $N(A)$. This means rank$(B) \leq 4 -$ rank(A).

31 null(N') produces a basis for the *row space* of A (perpendicular to $N(A)$).

32 We need $r^{\mathrm{T}}n = 0$ and $c^{\mathrm{T}}\ell = 0$. All possible examples have the form acr^{T} with $a \neq 0$.

33 Both r's orthogonal to both n's, both c's orthogonal to both ℓ's, each pair independent. All A's with these subspaces have the form $[c_1\ c_2]M[r_1\ r_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = 5/3$; $p = 5a/3$; $e = (-2,1,1)/3$ (b) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = -1$; $p = a$; $e = \mathbf{0}$.

3 $P_1 = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1 b = \dfrac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \dfrac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2 b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

6 $p_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $p_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $p_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $p_1 + p_2 + p_3 = b$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project on all of $\mathbf{R}^2$.

11 (a) $p = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b = (2,3,0)$, $e = (0,0,4)$, $A^{\mathrm{T}}e = \mathbf{0}$ (b) $p = (4,4,6)$, $e = \mathbf{0}$.

15 $2A$ has the same column space as A. $\widehat{x}$ for $2A$ is *half* of $\widehat{x}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So b is in the plane. Projection shows $Pb = b$.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$ (b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

20 $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \dfrac{ee^{\mathrm{T}}}{e^{\mathrm{T}}e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. Pb is in the column space (where P projects). Then its projection $P(Pb)$ is Pb.

24 The nullspace of A^T is *orthogonal* to the column space $C(A)$. So if $A^T\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $C(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^TA)^{-1}A^T\boldsymbol{b} = A(A^TA)^{-1}\mathbf{0}$.

28 $P^2 = P = P^T$ give $P^TP = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of P^TP which is the length squared of column 2.

29 $A = B^T$ has independent columns, so A^TA (which is BB^T) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^T/\boldsymbol{v}^T\boldsymbol{v}$. Always $P_CA = A$ (columns of A project to themselves) and $AP_R = A$. Then $P_CAP_R = A$!

31 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $C(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^T/\boldsymbol{a}^T\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

33 If $P_1P_2 = P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^T is invertible as in Problem 29. Then $(A^TA)(BB^T)$ = product of r by r invertible matrices, so rank r. AB can't have rank $< r$, since A^T and B^T cannot increase the rank. *Conclusion*: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^TA = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^T\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^T = [1\ \ 1\ \ 1\ \ 1]$ and $A^TA = [4]$. $A^T\boldsymbol{b} = [36]$ and $(A^TA)^{-1}A^T\boldsymbol{b} = 9$ = best height C. Errors $\boldsymbol{e} = (-9,-1,-1,11)$.

7 $A = [0\ \ 1\ \ 3\ \ 4]^T$, $A^TA = [26]$ and $A^T\boldsymbol{b} = [112]$. Best $D = 112/26 = 56/13$.

8 $\widehat{x} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^TA\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{b} = 9$ when $\widehat{t} = 2$.

(b) The first equation $Cm + D\sum t_i = \sum b_i$ divided by m gives $C + D\widehat{t} = \widehat{b}$.

13 $(A^TA)^{-1}A^T(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. When $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ averages to $\mathbf{0}$, so does $\widehat{\boldsymbol{x}} - \boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^T$ is $(A^TA)^{-1}A^T(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^TA(A^TA)^{-1}$. When the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^T$ is σ^2I, the average of $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^T$ will be the *output covariance matrix* $(A^TA)^{-1}A^T\sigma^2A(A^TA)^{-1}$ which simplifies to $\sigma^2(A^TA)^{-1}$.

16 $\dfrac{1}{10}b_{10} + \dfrac{9}{10}\widehat{x}_9 = \dfrac{1}{10}(b_1 + \cdots + b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$ = zero vector only.

23 The square of the distance between points on two lines is $E = (y-x)^2 + (3y-x)^2 + (1+x)^2$. Derivatives $\frac{1}{2}\partial E/\partial x = 3x - 4y + 1 = 0$ and $\frac{1}{2}\partial E/\partial y = -4x + 10y = 0$. The solution is $x = -5/7, y = -2/7$; $E = 2/7$, and the minimum distance is $\sqrt{2/7}$.

25 3 points on a line: *Equal slopes* $(b_2-b_1)/(t_2-t_1) = (b_3-b_2)/(t_3-t_2)$. Linear algebra: Orthogonal to $(1, 1, 1)$ and (t_1, t_2, t_3) is $\boldsymbol{y} = (t_2-t_3, t_3-t_1, t_1-t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$.

27 The shortest link connecting two lines in space is *perpendicular to those lines.*

28 Only 1 plane contains $\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

Problem Set 4.4, page 239

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries $1, 4, 9$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1 = \frac{1}{10}(1, 3, 4, 5, 7)$ and $\boldsymbol{q}_2 = \frac{1}{10}(-7, 3, 4, -5, 1)$
(b) Closest in the plane: *project* $QQ^{\mathrm{T}}(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$.

13 The multiple to subtract is $\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$. Then $\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (4, 0) - 2 \cdot (1, 1) = (2, -2)$.

14 $$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}\begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1, 2, -2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2, 1, 2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2, -2, -1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1, 2, 7) = (1, 2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4, 5, 2, 2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1, 4, -4, -4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

18 $A = \boldsymbol{a} = (1, -1, 0, 0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

20 (a) *True* (b) *True*. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1, 1, 1, 1)/2$ and $\boldsymbol{q}_2 = (-5, -1, 1, 5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4, -3, 3, 0)$ projects to $\boldsymbol{p} = (-7, -3, -1, 3)/2$. And $\boldsymbol{b} - \boldsymbol{p} = (-1, -3, 7, -3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1, 1, 2)$, $B = (1, -1, 0)$, $C = (-1, -1, 1)$. These are not yet unit vectors.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The wavelet matrix W has orthonormal columns. Notice $W^{-1} = W^{\mathrm{T}}$ in Section 7.3.

32 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so det can't blow up.

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has det $= 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has det $= 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$ for 3 by 3.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$.
$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b - a)(c - b)$ by doing elimination.

32 Typical determinants of $\mathsf{rand}(n)$ are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. $\mathsf{randn}(n)$ with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

Problem Set 5.2, page 263

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

8 Some term $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and det $= 6$ are impossible five 1's and one -1 will give $AC = (ad - bc)I = (\det A)I$ max(det) $= 4$.

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why?*

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

15 The $1,1$ cofactor of the n by n matrix is E_{n-1}. The $1,2$ cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is $1, 0, -1, -1, 0, 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1,1$ cofactor of the n by n matrix is F_{n-1}. The $1,2$ cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the $1,2$ entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(x) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1, G_3 = 2, G_4 = -3$, and $G_n = (-1)^{n-1}(n-1) =$ (product of the λ's).

24 (a) All L's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ (b) Pivots $5, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+\,(1,1)(2,2)(3,3)(4,4)\ +\ (1,2)(2,1)(3,4)(4,3)\ -\ (1,2)(2,1)(3,3)(4,4)\ -\ (1,1)(2,2)(3,4)(4,3)\ -\ (1,1)(2,3)(3,2)(4,4)$. Total -1.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

Problem Set 5.3, page 278

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0/0}$: *undetermined.*

4 (a) $x_1 = \det\big([\,\boldsymbol{b}\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\,]\big)/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1,3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

17 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces length of cross product $= \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{matrix} -2i - 2j + 8k \\ \text{length} = 6\sqrt{2} \end{matrix}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

21 The maximum volume is $L_1L_2L_3L_4$ reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

23 $A^{\mathrm{T}}A = \begin{bmatrix} a^{\mathrm{T}} \\ b^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^{\mathrm{T}}a & 0 & 0 \\ 0 & b^{\mathrm{T}}b & 0 \\ 0 & 0 & c^{\mathrm{T}}c \end{bmatrix}$ has $\begin{matrix} \det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2 \\ \det A & = & \pm\|a\|\|b\|\|c\| \end{matrix}$

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2 + x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

31 Base area 10, height 2, volume 20.

35 $S = (2, 1, -1)$, area $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners can be $(0, 0, 0)$, $(0, 0, 2)$, $(1, 2, 2)$, $(1, 1, 0)$. The volume of the tilted box is $|\det| = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1}$ is $1/\det A$, construct A^{-1} using the cofactors. *Invert to find* A.

Problem Set 6.1, page 293

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^{∞}. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

8 (a) Multiply Ax to see λx which reveals λ (b) Solve $(A-\lambda I)x = \mathbf{0}$ to find x.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $x_1 = (1,2)$ and $x_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A-\lambda_1 I$ are in the nullspace of $A-\lambda_2 I$ because $M = (A-\lambda_2 I)(A-\lambda_1 I)$ = zero matrix [this is the *Cayley-Hamilton Theorem* in Problem 6.2.32]. Notice that M has *zero eigenvalues* $(\lambda_1-\lambda_2)(\lambda_1-\lambda_1) = 0$ and $(\lambda_2-\lambda_2)(\lambda_2-\lambda_1) = 0$.

13 (a) $Pu = (uu^{\mathrm{T}})u = u(u^{\mathrm{T}}u) = u$ so $\lambda = 1$ (b) $Pv = (uu^{\mathrm{T}})v = u(u^{\mathrm{T}}v) = \mathbf{0}$
(c) $x_1 = (-1,1,0,0)$, $x_2 = (-3,0,1,0)$, $x_3 = (-5,0,0,1)$ all have $Px = 0x = \mathbf{0}$.

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A-\lambda I) = (\lambda_1-\lambda)\dots(\lambda_n-\lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a+d+\sqrt{(a-d)^2+4bc})$ and $\lambda_2 = \frac{1}{2}(a+d-\sqrt{})$ add to $a+d$.
If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A-\lambda I) = (\lambda-3)(\lambda-4) = \lambda^2 - 7\lambda + 12$.

19 (a) rank = 2 (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 Last rows are $-28, 11$ (check trace and det) and $6, -11, 6$ (to match $\det(C-\lambda I)$).

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have det $= -3$.

32 (a) u is a basis for the nullspace, v and w give a basis for the column space
(b) $x = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any cu from the nullspace
(c) If $Ax = u$ had a solution, u would be in the column space: wrong dimension 3.

34 $\det(P-\lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $x_1 = (1,1,1,1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1,-1,1,-1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$.
$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

Problem Set 6.2, page 307

1 $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

3 If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for $A+2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S. $A + 2I = S(\Lambda+2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

6 The columns of S are nonzero multiples of $(2,1)$ and $(0,1)$: either order. Same for A^{-1}.

8 $A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $S\Lambda^k S^{-1} =$
$\dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 (a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $x_1 = (1,1),\ x_2 = (1,-2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $x = (c, -c)$.

15 $A^k = S\Lambda^k S^{-1}$ approaches zero **if and only if every $|\lambda| < 1$**; $A_1^k \to A_1^\infty$, $A_2^k \to 0$.

17 $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,
$A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

19 $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

21 trace $ST = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1}$ = trace of $(\Lambda S^{-1})S = \Lambda$: *sum of the* λ's.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same S. When $S = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

26 Two problems: The nullspace and column space can overlap, so x could be in both. There may not be r independent eigenvectors in the column space.

27 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real.
Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 $A^T = A$ gives $x^TABx = (Ax)^T(Bx) \le \|Ax\|\|Bx\|$ by the Schwarz inequality. $B^T = -B$ gives $-x^TBAx = (Bx)^T(Ax) \le \|Ax\|\|Bx\|$. Add to get Heisenberg's Uncertainty Principle when $AB - BA = I$. Position-momentum, also time-energy.

32 If $A = S\Lambda S^{-1}$ then $(A - \lambda_1 I)\cdots(A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)S^{-1}$. The factor $\Lambda - \lambda_j I$ is zero in row j. *The product is zero in all rows* = ***zero matrix***.

33 $\lambda = 2, -1, 0$ are in Λ and the eigenvectors are in S (below). $A^k = S\Lambda^k S^{-1}$ is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \Lambda^k \frac{1}{6} \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check $k = 4$. The $(2,2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

Problem Set 6.3, page 325

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{u}_2 = e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = (5,-2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$.

4 $d(v+w)/dt = (w-v)+(v-w) = \mathbf{0}$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\ \lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$; $\begin{matrix}v(1) = 20 + 10e^{-2}\\ w(1) = 20 - 10e^{-2}\end{matrix}$ $\begin{matrix}v(\infty) = 20\\ w(\infty) = 20\end{matrix}$

8 $\begin{bmatrix}6 & -2\\ 2 & 1\end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix}2\\1\end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix}1\\2\end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

12 $A = \begin{bmatrix}0 & 1\\ -9 & 6\end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1,3)$.

14 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal.*

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^{t} + 4$; $\quad \boldsymbol{u}_p = \begin{bmatrix}4\\2\end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^{t}\begin{bmatrix}1\\t\end{bmatrix} + c_2e^{t}\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

20 The solution at time $t + T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}$; $\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}e^{t} & 0\\ 0 & \mathbf{1}\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix} = \begin{bmatrix}e^{t} & 4e^{t} - 4\\ 0 & 1\end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}At^2 + \cdots = I + (e^{t} - 1)A = \begin{bmatrix}e^{t} & e^{t} - 1\\ 0 & 1\end{bmatrix}$.

24 $A = \begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\ 0 & \mathbf{3}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{1}{2}\end{bmatrix}$. Then $e^{At} = \begin{bmatrix}e^{t} & \frac{1}{2}(e^{3t} - e^{t})\\ 0 & e^{3t}\end{bmatrix}$.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

27 $(x,y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u} = (y,x)$ is $\begin{bmatrix}2 & -2\\ -4 & 0\end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $\boldsymbol{U}_{n+1} = \begin{bmatrix}1 & \Delta t\\ -\Delta t & 1 - (\Delta t)^2\end{bmatrix}\boldsymbol{U}_n$. At $\Delta t = 1$, $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$ both have $\lambda^6 = 1$ so $A^6 = I$. $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda = \pm i$ and $A^4 = I$
Second A has $\lambda = -1, -1$ and $A^n = (-1)^n\begin{bmatrix}1 - 2n & -2n\\ 2n & 2n + 1\end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $U_{n+1} = \dfrac{1}{1+a^2}\begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} U_n$.

Orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|U_{n+1}\| = \|U_n\|$

31 (a) $(\cos A)x = (\cos\lambda)x$ (b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1, 1$ and $\cos A = I$ (c) $u(t) = 3(\cos 2\pi t)(1,1) + 1(\cos 0t)(1,-1)$ [$u' = Au$ has **exp**, $u'' = Au$ has **cos**]

Problem Set 6.4, page 337

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

5 $Q = \dfrac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A. Each unit eigenvector could be multiplied by -1

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}}$ = zero matrix.

10 If x is not real then $\lambda = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{bmatrix} + 4\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

16 (a) If $Az = \lambda y$ and $A^{\mathrm{T}}y = \lambda z$ then $B[\,y;\ -z\,] = [\,-Az;\ A^{\mathrm{T}}y\,] = -\lambda[\,y;\ -z\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}Az = A^{\mathrm{T}}(\lambda y) = \lambda^2 z$. (c) $\lambda = -1, -1, 1, 1$; $x_1 = (1,0,-1,0)$, $x_2 = (0,1,0,-1)$, $x_3 = (1,0,1,0)$, $x_4 = (0,1,0,1)$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A. Not perpendicular for B since $B^{\mathrm{T}} \neq B$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$ (c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the x's can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $x_1 = (1, i)$ first for A but $x_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or
$$A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

27 The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The 1, 1 entries of $\overline{T}^{\mathrm{T}}\,T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $\left[q_{11} \dots q_{1n}\right]\left[\lambda_1\overline{q}_{11} \dots \lambda_n\overline{q}_{1n}\right]^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \dots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}) = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{x} = -\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$. (b) $\overline{z}^{\mathrm{T}}Az$ is pure imaginary, its real part is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}A\boldsymbol{y} = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

Problem Set 6.5, page 350

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x + 3y)^2$.

8 $A = \begin{bmatrix}3 & 6\\ 6 & 16\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Pivots 3, 4 outside squares, ℓ_{ij} inside. $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 3(x + 2y)^2 + 4y^2$

10 $A = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $\quad B = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $B\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, (c - 1)^2(c + 2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}}A^{-1}\boldsymbol{x} = (A^{-1}\boldsymbol{x})^{\mathrm{T}}A(A^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \ne \boldsymbol{0}$.

17 If a_{jj} were smaller than all λ's, $A - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 16.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}{\sqrt{2}}\begin{bmatrix}\sqrt{9} & \\ & \sqrt{1}\end{bmatrix}\dfrac{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$; $R = Q\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}Q^{\mathrm{T}} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}}C = \begin{bmatrix}9 & 3\\ 3 & 5\end{bmatrix}$; $\begin{bmatrix}4 & 8\\ 8 & 25\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $C = \begin{bmatrix}2 & 4\\ 0 & 3\end{bmatrix}$

29 $H_1 = \begin{bmatrix}6x^2 & 2x\\ 2x & 2\end{bmatrix}$ is positive definite if $x \ne 0$; $F_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix}6x & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ is indefinite, $(0, 1)$ is a saddle point of F_2.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with det $= 1$ are groups. Examples of subgroups are orthogonal matrices with det $= 1$, exponentials e^{An} for integer n.

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3}, 2 - 1, 2, 2 + 1, 2 + \sqrt{3}$: product of eigenvalues $= 6 = \det K$.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}x) = M^{-1}(Ax) = M^{-1}\mathbf{0} = \mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} *is similar to* A.

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $ABx = \lambda x$ *then* $BA(Bx) = \lambda(Bx)$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

22 $A = MJM^{-1}, A^n = MJ^nM^{-1} = 0$ (each J^k has 1's on the kth diagonal). $\det(A - \lambda I) = \lambda^n$ so $J^n = 0$ by the Cayley-Hamilton Theorem.

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3 + \sqrt{5}}{2}, \sigma_2^2 = \dfrac{3 - \sqrt{5}}{2}$. But A is indefinite $\sigma_1 = (1 + \sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5} - 1)/2 = -\lambda_2(A)$; $\boldsymbol{u}_1 = \boldsymbol{v}_1$ but $\boldsymbol{u}_2 = -\boldsymbol{v}_2$.

5 A proof that **eigshow** finds the SVD. When $V_1 = (1, 0), V_2 = (0, 1)$ the demo finds AV_1 and AV_2 at some angle θ. A 90° turn by the mouse to $V_2, -V_1$ finds AV_2 and $-AV_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must produce $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ *at angle* $\pi/2$. Those orthogonal directions give $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

15 The singular values of $A + I$ are *not* $\sigma_j + 1$. Need eigenvalues of $(A + I)^{\mathrm{T}}(A + I)$.

17 $A = U\Sigma V^{\mathrm{T}}$ = [cosines including $\boldsymbol{u}_4$] **diag(sqrt**$(2 - \sqrt{2}, 2, 2 + \sqrt{2})$**)** [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

Problem Set 7.1, page 380

3 $T(v) = (0,1)$ and $T(v) = v_1 v_2$ are not linear.

4 (a) $S(T(v)) = v$ (b) $S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2))$.

5 Choose $v = (1,1)$ and $w = (-1,0)$. $T(v) + T(w) = (0,1)$ but $T(v+w) = (0,0)$.

7 (a) $T(T(v)) = v$ (b) $T(T(v)) = v + (2,2)$ (c) $T(T(v)) = -v$ (d) $T(T(v)) = T(v)$.

10 Not invertible: (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

12 Write v as a combination $c(1,1) + d(2,0)$. Then $T(v) = c(2,2) + d(0,0)$. $T(v) = (4,4)$; $(2,2)$; $(2,2)$; if $v = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(v) = b(2,2) + (0,0)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

Problem Set 7.2, page 395

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

5 $T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $v = c(v_2 - v_3)$ gives $T(v) = \mathbf{0}$; nullspace is $(0,c,-c)$; solutions $(1,0,0) + (0,c,-c)$.

8 For $T^2(v)$ we would need to know $T(w)$. If the w's equal the v's, the matrix is A^2.

12 (c) is wrong because w_1 is not generally in the input space.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 $(a,b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

20 $w_2(x) = 1 - x^2$; $w_3(x) = \frac{1}{2}(x^2 - x)$; $y = 4w_1 + 5w_2 + 6w_3$.

23 The matrix M with these nine entries must be invertible.

27 If T is not invertible, $T(v_1), \ldots, T(v_n)$ is not a basis. We couldn't choose $w_i = T(v_i)$.

30 S takes (x,y) to $(-x,y)$. $S(T(v)) = (\mathbf{-1}, \mathbf{2})$. $S(v) = (-2,1)$ and $T(S(v)) = (\mathbf{1}, \mathbf{-2})$.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$, $(0,0,0,0,1,1,-1,-1)$ and 4 wavelets with a single pair $1,-1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplication by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with this basis is represented by 4 by 4 $A = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$

38 If $\boldsymbol{w}_1 = A\boldsymbol{v}_1$ and $\boldsymbol{w}_2 = A\boldsymbol{v}_2$ then $a_{11} = a_{22} = 1$. All other entries will be zero.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

$A\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sigma_1\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \mathbf{0}$. $\boldsymbol{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $AA^{\mathrm{T}}\boldsymbol{u}_1 = 50\,\boldsymbol{u}_1$.

3 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}\frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

11 $A = [1]\,[5\;\;0\;\;0]V^{\mathrm{T}}$ and $A^+ = V\begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $AA^+ = [1]$

13 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^+) < n$ and $\det A^+ = 0$.

16 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A$ = nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

17 $AA^+\boldsymbol{p} = \boldsymbol{p}$, $AA^+\boldsymbol{e} = \mathbf{0}$, $A^+A\boldsymbol{x}_r = \boldsymbol{x}_r$, $A^+A\boldsymbol{x}_n = \mathbf{0}$.

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v} \\ A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \sigma\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1+f_2+f_3=0$. $\boldsymbol{f}=(-1,0,1)$ gives $c_2u_1-c_2u_2=-1, c_3u_2-c_3u_3=-1, 0=0$. Then $\boldsymbol{u}_{\text{particular}}=(-c_2^{-1}-c_3^{-1},-c_3^{-1},0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}}=(1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx=-\left[c(x)\frac{du}{dx}\right]_0^1=0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx=0$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1=[1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

8 The solution to $-u''=1$ with $u(0)=u(1)=0$ is $u(x)=\frac{1}{2}(x-x^2)$. At $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ this gives $\boldsymbol{u}=2,3,3,2$ (discrete solution in Problem 7) times $(\Delta x)^2=1/25$.

11 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: E = diag(ones(6, 1), 1); K = 64 ∗ (2∗ eye(7) − E − E'); D = 80 ∗ (E− eye(7)); (K + D)\ones(7, 1); % forward; (K − D')\ones(7, 1); % backward; (K + $D/2$ − $D'/2$)\ones(7, 1); % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A=\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}$; nullspace contains $\begin{bmatrix}c\\c\\c\end{bmatrix}$; $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}y=\mathbf{0}$ for $y=(1,-1,1)$; current along edge 1, edge 3, back on edge 2 (full loop).

5 Kirchhoff's Current Law $A^{\mathrm{T}}y=\boldsymbol{f}$ is solvable for $\boldsymbol{f}=(1,-1,0)$ and not solvable for $\boldsymbol{f}=(1,0,0)$; $\boldsymbol{f}$ must be orthogonal to $(1,1,1)$ in the nullspace: $f_1+f_2+f_3=0$.

6 $A^{\mathrm{T}}A\boldsymbol{x}=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}3\\-3\\0\end{bmatrix}=\boldsymbol{f}$ produces $\boldsymbol{x}=\begin{bmatrix}1\\-1\\0\end{bmatrix}+\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $\boldsymbol{x}=1,-1,0$ and currents $-A\boldsymbol{x}=2,1,-1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}}\begin{bmatrix}1&&\\&2&\\&&2\end{bmatrix}A=\begin{bmatrix}3&-1&-2\\-1&3&-2\\-2&-2&4\end{bmatrix}$; $\boldsymbol{f}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ yields $\boldsymbol{x}=\begin{bmatrix}5/4\\1\\7/8\end{bmatrix}+$ any $\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $\boldsymbol{x}=\frac{5}{4},1,\frac{7}{8}$ and currents $-CA\boldsymbol{x}=\frac{1}{4},\frac{3}{4},\frac{1}{4}$.

9 Elimination on $A\boldsymbol{x}=\boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}=0$ in the zero rows of U and R: $-b_1+b_2-b_3=0$ and $b_3-b_4+b_5=0$ (those y's are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

11 $A^{\mathrm{T}}A=\begin{bmatrix}2&-1&-1&0\\-1&3&-1&-1\\-1&-1&3&-1\\0&-1&-1&2\end{bmatrix}$
diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry = -1 if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

13 $A^{\mathrm{T}}CA\boldsymbol{x}=\begin{bmatrix}4&-2&-2&0\\-2&8&-3&-3\\-2&-3&8&-3\\0&-3&-3&6\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$
gives four potentials $\boldsymbol{x}=(\frac{5}{12},\frac{1}{6},\frac{1}{6},0)$
I grounded $x_4=0$ and solved for $\boldsymbol{x}$
currents $y=-CA\boldsymbol{x}=(\frac{2}{3},\frac{2}{3},0,\frac{1}{2},\frac{1}{2})$

17 (a) 8 independent columns (b) f must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

Problem Set 8.3, page 437

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^{\infty} = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $x = (1, 0)$; 1 and $-.8$, $x = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.

6 Add the components of $Ax = \lambda x$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^{\infty}$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$

9 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply on the right by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $x_1 = (b, 1 - a)$.

12 B has $\lambda = 0$ and $-.5$ with $x_1 = (.3, \ .2)$ and $x_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} x_1 = c_1 x_1$.

13 $x = (1, 1, 1)$ is an eigenvector when the row sums are equal; $Ax = (.9, .9, .9)$.

15 The first two A's have $\lambda_{\max} < 1$; $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

19 Λ times $S^{-1} \Delta S$ has the same diagonal as $S^{-1} \Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $Ax = \lambda_{\max}(A)x > 0$ then $Bx > \lambda_{\max}(A)x$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set has corners $(0, 0)$, $(6, 0)$, $(2, 2)$, $(0, 6)$. Minimum cost $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $x = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $x = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 $c = [3 \ \ 5 \ \ 7]$ has minimum cost 12 by the Ph.D. since $x = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \leq 3$, $y \leq 5$, $y \leq 7$. Maximum = 12.

8 $y^{\mathrm{T}} b \leq y^{\mathrm{T}} A x = (A^{\mathrm{T}} y)^{\mathrm{T}} x \leq c^{\mathrm{T}} x$. The first inequality needed $y \geq 0$ and $Ax - b \geq 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$
Notice $j-k \neq 0$ in the denominator. If $j=k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

4 $\int_{-1}^{1}(1)(x^3-cx)\,dx = 0$ and $\int_{-1}^{1}(x^2-\frac{1}{3})(x^3-cx)\,dx = 0$ for all c (odd functions).
Choose c so that $\int_{-1}^{1} x(x^3-cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (*odd* k). The square wave has $\|f\|^2 = 2\pi$.
Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots)$. That infinite series equals $\pi^2/8$.

8 $\|\boldsymbol{v}\|^2 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1+a^2+a^4+\cdots = 1/(1-a^2)$
so $\|\boldsymbol{v}\| = 1/\sqrt{1-a^2}$; $\int_0^{2\pi}(1+2\sin x+\sin^2 x)\,dx = 2\pi+0+\pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1 + \text{square wave})/2$ so the a's are $\frac{1}{2}$, 0, 0, ... and the b's are $2/\pi$, 0, $-2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0, b_k = -2/k$.

11 $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$; $\cos(x+\frac{\pi}{3}) = \cos x\cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

13 $a_0 = \dfrac{1}{2\pi}\int F(x)\,dx = \dfrac{1}{2\pi}, a_k = \dfrac{\sin(kh/2)}{\pi kh/2} \to \dfrac{1}{\pi}$ for delta function; all $b_k = 0$.

Problem Set 8.6, page 458

3 If $\sigma_3 = 0$ the third equation is exact.

4 0, 1, 2 have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\boldsymbol{\Sigma} = \textbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \frac{1}{2}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0; a = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$
recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\Sigma L^{\mathrm{T}} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\Sigma\Sigma^{-1}A(A^{\mathrm{T}}\Sigma^{-1}A)^{-1} = P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$.

9 Row 3 = −row 1 and row 4 = −row 2: A has rank 2.

Problem Set 8.7, page 464

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

4 $S = \text{diag}\,(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\boldsymbol{v}TS$!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

9 $\boldsymbol{n} = \left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}\right)$ has $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \dfrac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. Notice $\|\boldsymbol{n}\| = 1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_-PT_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$. Row vectors!

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I-2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3},-\frac{11}{3},-\frac{1}{3}\right)$.

15 $(3,3,3,1) \to (3,3,0,1) \to (-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1) \to (-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1)$.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}| = |\text{entry/pivot}| \le 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\| = 1/\lambda_{\min}$ since $A^{\mathrm{T}} = A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $< wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So $QR\boldsymbol{x} = \boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x} = \boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2+2^2+\cdots+n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x}-\boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = \frac{1}{\sqrt{10}}$, $\sin\theta = \frac{-3}{\sqrt{10}}$, $R = Q_{21}A = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$ (b) $\begin{matrix} \lambda = 4;\ \text{use} -\theta \\ \boldsymbol{x} = (1,-3)/\sqrt{10} \end{matrix}$

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2, c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1, c = 3$; $\|A\| = 2+\sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

7 The triangle inequality gives $\|A\boldsymbol{x}+B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A+B\| \le \|A\| + \|B\|$.

8 If $Ax = \lambda x$ then $\|Ax\|/\|x\| = |\lambda|$ for that particular vector x. When we maximize the ratio over all vectors we get $\|A\| \geq |\lambda|$.

13 The residual $b - Ay = (10^{-7}, 0)$ is much smaller than $b - Az = (.0013, .0016)$. But z is much closer to the solution than y.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$. $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$ so $\|x\| \leq \sqrt{n}\|x\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|x\|_1 = x \cdot y \leq \|x\|\|y\| = \sqrt{n}\|x\|$. $x = (1, \ldots, 1)$ has $\|x\|_1 = \sqrt{n}\,\|x\|$.

Problem Set 9.3, page 489

2 If $Ax = \lambda x$ then $(I - A)x = (1 - \lambda)x$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^2$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

15 In the jth component of Ax_1, $\lambda_1 \sin \frac{j\pi}{n+1} = 2 \sin \frac{j\pi}{n+1} - \sin \frac{(j-1)\pi}{n+1} - \sin \frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2 \sin \frac{j\pi}{n+1} \cos \frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2\cos \frac{\pi}{n+1}$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $u_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $u_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to u_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta \sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1 + \sin^2\theta) & -\sin^3\theta \\ -\sin^3\theta & -\cos\theta\sin^2\theta \end{bmatrix}$.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $Aq_j = b_{j-1}q_{j-1} + a_j q_j + b_j q_{j+1}$ by q_j^{T} to find $q_j^{\mathrm{T}}Aq_j = a_j$ (because the q's are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

26 If each center a_{ii} is larger than the circle radius r_i (this is diagonal dominance), then 0 is outside all circles: not an eigenvalue so A^{-1} exists.

Problem Set 10.1, page 498

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

4 $|z \times w| = 6$, $|z + w| \leq 5$, $|z/w| = \frac{2}{3}$, $|z - w| \leq 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2-4(ad-bc)$ as $(a-d)^2+4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

21 $\cos 3\theta = \mathrm{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2+2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

3 z = multiple of $(1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^{\mathrm{H}}A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so z (not $\overline{z}$!) is orthogonal to all columns of A^{H} (using complex inner product z^{H} times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{H}}), \boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}Az = \mathbf{0}$ then $(z^{\mathrm{H}}A^{\mathrm{H}})(Az) = 0$. This is $\|Az\|^2 = 0$ so $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^{\mathrm{H}}$.

10 $(1,1,1), (1, e^{2\pi i/3}, e^{4\pi i/3}), (1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$ has the Fourier eigenvector matrix F.

The eigenvalues are $2 + 5 + 4 = 11, 2 + 5e^{2\pi i/3} + 4e^{4\pi i/3}, 2 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$.

13 Determinant = product of the eigenvalues *(all real)*. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

15 $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

18 $V = \frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$.
Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

19 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$: a typical orthonormal expansion.

20 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

21 $R + iS = (R+iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

24 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

27 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A+iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

Problem Set 10.3, page 514

8 $c \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8 c$.
$C \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8 C$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

Conceptual Questions for Review

Chapter 1

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

Chapter 2

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 6 \end{smallmatrix}\right]$ what are its pivots?

2.4 If A is the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 3 \end{smallmatrix}\right]$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \left[\begin{smallmatrix} I & U \\ 0 & I \end{smallmatrix}\right]$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[\,A \quad I\,]$? If you solve the n equations Ax = columns of I then the solutions x are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $Ax = b$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $Ax = b$.

3.9 If $Ax = b$ has exactly one solution for every b, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column $\boldsymbol{u}$ times a row $\boldsymbol{v}^{\mathrm{T}}$ (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^{\mathrm{T}}\boldsymbol{y}$.

3.19 Why do the columns of every invertible matrix yield a basis?

Chapter 4

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If $\boldsymbol{V}$ is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of $\boldsymbol{V}$ and its orthogonal complement add to _____.

4.3 The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector _____.

4.4 The projection matrix onto the line through $\boldsymbol{a}$ is $P =$ _____.

4.5 The key equation to project $\boldsymbol{b}$ onto the column space of A is the *normal equation* _____.

4.6 The matrix $A^{\mathrm{T}}A$ is invertible when the columns of A are _____.

4.7 The least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ minimizes what error function?

4.8 What is the connection between the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection $\boldsymbol{p}$ appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^{\mathrm{T}}Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $\boldsymbol{a} = (2,0)$ and $\boldsymbol{b} = (1,1)$, which two orthonormal vectors does it produce? If we keep $\boldsymbol{a} = (2,0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____.

Chapter 5

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

Chapter 6

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[1\ \ 2\ \ 1]^{\mathrm{T}}[1\ \ 1\ \ 1]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

6.9 Explain why the trace of AB equals the trace of BA.

6.10 How do the eigenvectors of A help to solve $du/dt = Au$?

6.11 How do the eigenvectors of A help to solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$?

6.12 Define the matrix exponential e^A and its inverse and its square.

6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

6.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^{\mathrm{T}}A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^{\mathrm{T}}A$?

Chapter 7

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from $\boldsymbol{v}$ to $S(T(\boldsymbol{v}))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).

GLOSSARY: A DICTIONARY FOR LINEAR ALGEBRA

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ when edges go both ways (undirected).

Affine transformation $Tv = Av + v_0$ = linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC.

Augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the column space of A; then $[\,A \;\; \boldsymbol{b}\,]$ has the same rank as A. Elimination on $[\,A \;\; \boldsymbol{b}\,]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1.

Basis for V. Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ whose linear combinations give each vector in V as $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \ldots + c_d\boldsymbol{v}_d$. V has many bases, each basis gives unique c's. A vector space has many bases!

Big formula for n by n determinants. Det(A) is a sum of $n!$ terms. For each term: Multiply one entry from each row and column of A: rows in order $1, \ldots, n$ and column order given by a permutation P. Each of the $n!$ P's has a $+$ or $-$ sign.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit.

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix.*

Change of basis matrix M. The old basis vectors $\boldsymbol{v}_j$ are combinations $\sum m_{ij}\boldsymbol{w}_i$ of the new basis vectors. The coordinates of $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = d_1\boldsymbol{w}_1 + \cdots + d_n\boldsymbol{w}_n$ are related by $\boldsymbol{d} = M\boldsymbol{c}$. (For $n = 2$ set $\boldsymbol{v}_1 = m_{11}\boldsymbol{w}_1 + m_{21}\boldsymbol{w}_2$, $\boldsymbol{v}_2 = m_{12}\boldsymbol{w}_1 + m_{22}\boldsymbol{w}_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A.

Cholesky factorization $A = C^{\mathrm{T}}C = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C. Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1}S^{n-1}$. $C\boldsymbol{x} =$ **convolution** $\boldsymbol{c} * \boldsymbol{x}$. Eigenvectors in F.

Cofactor C_{ij}. Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ becomes a combination of the columns of A. The system is solvable only when $\boldsymbol{b}$ is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ = space of all combinations of the columns of A.

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put $c_1, \ldots, c_n$ in row n and put $n - 1$ ones just above the main diagonal. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots + c_n\lambda^{n-1} - \lambda^n)$.

Complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ to $A\boldsymbol{x} = \boldsymbol{b}$. (Particular $\boldsymbol{x}_p$) + ($\boldsymbol{x}_n$ in nullspace).

Complex conjugate $\overline{z} = a - ib$ for any complex number $z = a + ib$. Then $z\overline{z} = |z|^2$.

Condition number $cond(A) = c(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $Ax = b$, the relative change $\|\delta x\|/\|x\|$ is less than $cond(A)$ times the relative change $\|\delta b\|/\|b\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $Ax = b$ by minimizing $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ over growing Krylov subspaces.

Covariance matrix Σ. When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\overline{x}_i$, the matrix Σ = mean of $(x - \overline{x})(x - \overline{x})^{\mathrm{T}}$ is positive (semi)definite; Σ is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$. B_j has b replacing column j of A; $x_j = \det B_j / \det A$

Cross product $u \times v$ in $\mathbf{R}^3$. Vector perpendicular to u and v, length $\|u\|\|v\||\sin\theta|$ = area of parallelogram, $u \times v$ = "determinant" of $[\,i \;\; j \;\; k\,;\, u_1 \;\; u_2 \;\; u_3\,;\, v_1 \;\; v_2 \;\; v_3\,]$.

Cyclic shift S. Permutation with $s_{21} = 1, s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are the nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n - 1$, volume of box $= |\det(A)|$.

Diagonal matrix D. $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A. Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. Λ = eigenvalue matrix and S = eigenvector matrix of A. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(V)$ = number of vectors in any basis for V.

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product = Inner product $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$. Complex dot product is $\overline{x}^{\mathrm{T}}y$. Perpendicular vectors have $\overline{x}^{\mathrm{T}}y = 0$. $(AB)_{ij}$ = (row i of A)$^{\mathrm{T}}$(column j of B).

Echelon matrix U. The first nonzero entry (the pivot) in each row comes in a later column than the pivot in the previous row. All zero rows come last.

Eigenvalue λ **and eigenvector** x. $Ax = \lambda x$ with $x \neq \mathbf{0}$ so $\det(A - \lambda I) = 0$.

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = Elementary matrix E_{ij}. The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$. If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n\boldsymbol{x}$ and $F_n^{-1}\boldsymbol{c}$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$ satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1+\sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$.

Four Fundamental Subspaces $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A^{\mathrm{T}})$. Use $\overline{A}^{\mathrm{T}}$ for complex A.

Fourier matrix F. Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}}F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A. Columns without pivots; these are combinations of earlier columns.

Free variable x_i. Column i has no pivot in elimination. We can give the $n - r$ free variables any values, then $A\boldsymbol{x} = \boldsymbol{b}$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $A\boldsymbol{x} = \boldsymbol{b}$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements in $\mathbf{R}^n$(perpendicular from $A\boldsymbol{x} = \boldsymbol{0}$ with dimensions r and $n - r$). Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$ in $\mathbf{R}^m$.

Gauss-Jordan method. Invert A by row operations on $[\,A \;\; I\,]$ to reach $[\,I \;\; A^{-1}\,]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A, orthonormal columns in Q. Each column $\boldsymbol{q}_j$ of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\mathrm{diag}(R) > \boldsymbol{0}$.

Graph G. Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n - 1$ edges and no closed loops.

Hankel matrix H. Constant along each antidiagonal; h_{ij} depends on $i + j$.

Hermitian matrix $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} = A$. Complex analog $\overline{a_{ji}} = a_{ij}$ of a symmetric matrix.

Hessenberg matrix H. Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix hilb(n). Entries $H_{ij} = 1/(i + j - 1) = \int_0^1 x^{i-1}x^{j-1}dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number: H is *ill-conditioned*.

Hypercube matrix P_L^2. Row $n + 1$ counts corners, edges, faces, ... of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**).** Diagonal entries $= 1$, off-diagonal entries $= 0$.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix. A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $v_1, \ldots, v_k$. No combination $c_1 v_1 + \cdots + c_k v_k$ = zero vector unless all $c_i = 0$. If the v's are the columns of A, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$.

Inverse matrix A^{-1}. Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\operatorname{rank}(A) < n$ and $Ax = \mathbf{0}$ for a nonzero vector x. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1's on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector.

Kirchhoff's Laws. *Current Law*: net current (in minus out) is zero at each node. *Voltage Law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, b)$. The subspace spanned by $b, Ab, \ldots, A^{j-1}b$. Numerical methods approximate $A^{-1}b$ by x_j with residual $b - Ax_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\widehat{x}$. The vector $\widehat{x}$ that minimizes the error $\|e\|^2$ solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Then $e = b - A\widehat{x}$ is orthogonal to all columns of A.

Left inverse A^+. If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $N(A^{\mathrm{T}})$. Nullspace of A^{T} = "left nullspace" of A because $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$.

Length $\|x\|$. Square root of $x^{\mathrm{T}}x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ or $\sum c_j v_j$. Vector addition and scalar multiplication.

Linear transformation T. Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = c\,T(v) + d\,T(w)$. Examples: Matrix multiplication Av, differentiation and integration in function space.

Linearly dependent $v_1, \ldots, v_n$. A combination other than all $c_i = 0$ gives $\sum c_i v_i = \mathbf{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \ldots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ from the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$. Compare $L_0 = 2$ with $F_0 = 0$.

Markov matrix M. All $m_{ij} \ge 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $Ms = s > \mathbf{0}$.

Matrix multiplication AB. The i, j entry of AB is (row i of A)·(column j of B) = $\sum a_{ik}b_{kj}$. By columns: Column j of AB = A times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times x equals A times Bx.

Minimal polynomial of A. The lowest degree polynomial with $m(A)$ = zero matrix. This is $p(\lambda) = \det(A - \lambda I)$ if no eigenvalues are repeated; always $m(\lambda)$ divides $p(\lambda)$.

Multiplication $Ax = x_1$(column 1) $+ \cdots + x_n$(column n) = combination of columns.

Multiplicities AM and GM. The algebraic multiplicity AM of λ is the number of times λ appears as a root of $\det(A - \lambda I) = 0$. The geometric multiplicity GM is the number of independent eigenvectors for λ (= dimension of the eigenspace).

Multiplier ℓ_{ij}. The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: ℓ_{ij} = (entry to eliminate) / (jth pivot).

Network. A directed graph that has constants $c_1, \ldots, c_m$ associated with the edges.

Nilpotent matrix N. Some power of N is the zero matrix, $N^k = 0$. The only eigenvalue is $\lambda = 0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$. The "$\boldsymbol{\ell}^2$ norm" of A is the maximum ratio $\|Ax\|/\|x\| = \sigma_{\max}$. Then $\|Ax\| \le \|A\|\|x\|$ and $\|AB\| \le \|A\|\|B\|$ and $\|A + B\| \le \|A\| + \|B\|$. **Frobenius norm** $\|A\|_F^2 = \sum\sum a_{ij}^2$. The $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^\infty$ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Gives the least squares solution to $Ax = b$ if A has full rank n (independent columns). The equation says that (columns of A)$\cdot$($b - A\widehat{x}$) = 0.

Normal matrix. If $NN^{\mathrm{T}} = N^{\mathrm{T}}N$, then N has orthonormal (complex) eigenvectors.

Nullspace $N(A)$ = All solutions to $Ax = \mathbf{0}$. Dimension $n - r$ = (# columns) − rank.

Nullspace matrix N. The columns of N are the $n - r$ special solutions to $As = \mathbf{0}$.

Orthogonal matrix Q. Square matrix with orthonormal columns, so $Q^{\mathrm{T}} = Q^{-1}$. Preserves length and angles, $\|Qx\| = \|x\|$ and $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$. All $|\lambda| = 1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces. Every v in V is orthogonal to every w in W.

Orthonormal vectors $q_1, \ldots, q_n$. Dot products are $q_i^{\mathrm{T}}q_j = 0$ if $i \ne j$ and $q_i^{\mathrm{T}}q_i = 1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q = I$. If $m = n$ then $Q^{\mathrm{T}} = Q^{-1}$ and $q_1, \ldots, q_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $v = \sum(v^{\mathrm{T}}q_j)q_j$.

Outer product uv^{T} = column times row = rank one matrix.

Partial pivoting. In each column, choose the largest available pivot to control roundoff; all multipliers have $|\ell_{ij}| \le 1$. See *condition number*.

Particular solution x_p. Any solution to $Ax = b$; often x_p has free variables = 0.

Pascal matrix P_S = pascal(n) = the symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S = P_L P_U$ all contain Pascal's triangle with det = 1 (see Pascal in the index).

Permutation matrix P. There are $n!$ orders of $1, \ldots, n$. The $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is *even* or *odd* ($\det P = 1$ or -1) based on the number of row exchanges to reach I.

Pivot columns of A. Columns that contain pivots after row reduction. These are *not* combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot. The diagonal entry (*first nonzero*) at the time when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$. Vectors x with $a^{\mathrm{T}}x = 0$. Plane is perpendicular to $a \ne \mathbf{0}$.

Polar decomposition $A = QH$. Orthogonal Q times positive (semi)definite H.

Positive definite matrix A. Symmetric matrix with positive eigenvalues and positive pivots. *Definition*: $x^{\mathrm{T}}Ax > 0$ unless $x = 0$. Then $A = LDL^{\mathrm{T}}$ with $\operatorname{diag}(D) > 0$.

Projection $p = a(a^{\mathrm{T}}b/a^{\mathrm{T}}a)$ onto the line through a. $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ has rank 1.

Projection matrix P onto subspace S. Projection $p = Pb$ is the closest point to b in S, error $e = b - Pb$ is perpendicular to S. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in S or $S^{\perp}$. If columns of A = basis for S then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Pseudoinverse A^+ (Moore-Penrose inverse). The n by m matrix that "inverts" A from column space back to row space, with $N(A^+) = N(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. $\operatorname{Rank}(A^+) = \operatorname{rank}(A)$.

Random matrix rand(n) or randn(n). MATLAB creates a matrix with random entries, uniformly distributed on $[\,0 \quad 1\,]$ for rand and standard normal distribution for randn.

Rank one matrix $A = uv^{\mathrm{T}} \neq 0$. Column and row spaces = lines cu and cv.

Rank $r(A)$ = number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(x) = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ for symmetric A: $\lambda_{\min} \le q(x) \le \lambda_{\max}$. Those extremes are reached at the eigenvectors x for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \operatorname{rref}(A)$. Pivots = 1; zeros above and below pivots; the r nonzero rows of R give a basis for the row space of A.

Reflection matrix (Householder) $Q = I - 2uu^{\mathrm{T}}$. Unit vector u is reflected to $Qu = -u$. All x in the plane mirror $u^{\mathrm{T}}x = 0$ have $Qx = x$. Notice $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+. If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ rotates the plane by θ and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors are $(1, \pm i)$. $c, s = \cos\theta, \sin\theta$.

Row picture of $Ax = b$. Each equation gives a plane in $\mathbf{R}^n$; the planes intersect at x.

Row space $C(A^{\mathrm{T}})$ = all combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j$ = **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Schwarz inequality $|v \cdot w| \le \|v\|\,\|w\|$.Then $|v^{\mathrm{T}}Aw|^2 \le (v^{\mathrm{T}}Av)(w^{\mathrm{T}}Aw)$ for pos def A.

Semidefinite matrix A. (Positive) semidefinite: all $x^{\mathrm{T}}Ax \ge 0$, all $\lambda \ge 0$; A = any $R^{\mathrm{T}}R$.

Similar matrices A **and** B. Every $B = M^{-1}AM$ has the same eigenvalues as A.

Simplex method for linear programming. The minimum cost vector x^* is found by moving from corner to lower cost corner along the edges of the feasible set (where the constraints $Ax = b$ and $x \ge 0$ are satisfied). Minimum cost at a corner!

Singular matrix A. A square matrix that has no inverse: $\det(A) = 0$.

Singular Value Decomposition (SVD) $A = U\Sigma V^{\mathrm{T}}$ = **(orthogonal)(diag)(orthogonal)** First r columns of U and V are orthonormal bases of $C(A)$ and $C(A^{\mathrm{T}})$, $Av_i = \sigma_i u_i$ with singular value $\sigma_i > 0$. Last columns are orthonormal bases of nullspaces.

Skew-symmetric matrix K. The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $Ax = b$. The right side b is in the column space of A.

Spanning set. Combinations of $v_1, \dots, v_m$ fill the space. The columns of A span $C(A)$!

Special solutions to $As = 0$. One free variable is $s_i = 1$, other free variables $= 0$.

Spectral Theorem $A = Q\Lambda Q^{\mathrm{T}}$. Real symmetric A has real λ's and orthonormal q's.

Spectrum of A = the set of eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. **Spectral radius** = max of $|\lambda_i|$.

Standard basis for $\mathbf{R}^n$. Columns of n by n identity matrix (written i, j, k in $\mathbf{R}^3$).

Stiffness matrix If x gives the movements of the nodes, Kx gives the internal forces. $K = A^{\mathrm{T}}CA$ where C has spring constants from Hooke's Law and Ax = stretching.

Subspace S **of** V. Any vector space inside V, including V and Z = {zero vector only}.

Sum $V + W$ **of subspaces.** Space of all (v in V) + (w in W). **Direct sum**: $V \cap W = \{0\}$.

Symmetric factorizations $A = LDL^{\mathrm{T}}$ **and** $A = Q\Lambda Q^{\mathrm{T}}$. Signs in Λ = signs in D.

Symmetric matrix A. The transpose is $A^{\mathrm{T}} = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric.

Toeplitz matrix. Constant down each diagonal = time-invariant (shift-invariant) filter.

Trace of A = sum of diagonal entries = sum of eigenvalues of A. Tr AB = Tr BA.

Transpose matrix A^{T}. Entries $A^{\mathrm{T}}_{ij} = A_{ji}$. A^{T} is n by m, $A^{\mathrm{T}}A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^{\mathrm{T}}A^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$.

Triangle inequality $\|u + v\| \le \|u\| + \|v\|$. For matrix norms $\|A + B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T: $t_{ij} = 0$ if $|i - j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}} = U^{-1}$. Orthonormal columns (complex analog of Q).

Vandermonde matrix V. $Vc = b$ gives coefficients of $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$. $V_{ij} = (x_i)^{j-1}$ and det V = product of $(x_k - x_i)$ for $k > i$.

Vector v **in** $\mathbf{R}^n$. Sequence of n real numbers $v = (v_1, \dots, v_n)$ = point in $\mathbf{R}^n$.

Vector addition. $v + w = (v_1 + w_1, \dots, v_n + w_n)$ = diagonal of parallelogram.

Vector space V. Set of vectors such that all combinations $cv + dw$ remain within V. Eight required rules are given in Section 3.1 for scalars c, d and vectors v, w.

Volume of box. The rows (or the columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$. Stretch and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$.

MATRIX FACTORIZATIONS

1. $A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges. The pivots in D are divided out to leave 1's on the diagonal of U. If A is symmetric then U is L^{T} and $A = LDL^{\mathrm{T}}$.

3. $PA = LU$ (permutation matrix P to avoid zeros in the pivot positions).

   Requirements: A is invertible. Then P, L, U are invertible. P does all of the row exchanges in advance, to allow normal LU. Alternative: $A = L_1 P_1 U_1$.

4. $EA = R$ (m by m invertible E) (any matrix A) = rref(A).

   Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The last $m - r$ rows of E are a basis for the left nullspace of A; they multiply A to give zero rows in R. The first r columns of E^{-1} are a basis for the column space of A.

5. $A = C^{\mathrm{T}}C$ = (lower triangular) (upper triangular) with $\sqrt{D}$ on both diagonals

   Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* C = chol(A) has $C^{\mathrm{T}} = L\sqrt{D}$, so $C^{\mathrm{T}}C = LDL^{\mathrm{T}}$.

6. $A = QR$ = (orthonormal columns in Q) (upper triangular R).

   Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt or Householder process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$.

7. $A = S\Lambda S^{-1}$ = (eigenvectors in S) (eigenvalues in Λ) (left eigenvectors in S^{-1}).

   Requirements: A must have n linearly independent eigenvectors.

8. $A = Q\Lambda Q^{\mathrm{T}}$ = (orthogonal matrix Q) (real eigenvalue matrix Λ) (Q^{T} is Q^{-1}).

   Requirements: A is *real and symmetric*. This is the Spectral Theorem.

9. $A = MJM^{-1}$ = (generalized eigenvectors in M) (Jordan blocks in J) (M^{-1}).

Requirements: A is any square matrix. This *Jordan form* J has a block for each independent eigenvector of A. Every block has only one eigenvalue.

10. $A = U\Sigma V^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m \times n \end{pmatrix} \begin{pmatrix} m \times n \text{ singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n \times n \end{pmatrix}.$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and eigenvectors of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $A^+ = V\Sigma^+U^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ n \times n \end{pmatrix} \begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ m \times m \end{pmatrix}.$

Requirements: None. The *pseudoinverse* A^+ has A^+A = projection onto row space of A and AA^+ = projection onto column space. The shortest least-squares solution to $Ax = b$ is $\widehat{x} = A^+b$. This solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

12. $A = QH$ = (orthogonal matrix Q) (symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $A = U\Lambda U^{-1}$ = (unitary U) (eigenvalue matrix Λ) (U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$: Hermitian case.

14. $A = UTU^{-1}$ = (unitary U) (triangular T with λ's on diagonal) ($U^{-1} = U^{\mathrm{H}}$).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular: Section 6.4.

15. $F_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & \\ & F_{n/2} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$ = one step of the (recursive) **FFT**.

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$: $F_n\overline{F}_n = nI$. D has $1, w, \ldots, w^{n/2-1}$ on its diagonal. For $n = 2^\ell$ the *Fast Fourier Transform* will compute F_nx with only $\frac{1}{2}n\ell = \frac{1}{2}n\log_2 n$ multiplications from ℓ stages of D's.

MATLAB TEACHING CODES

These Teaching Codes are directly available from **web.mit.edu/ 18.06**

cofactor	Compute the n by n matrix of cofactors.
cramer	Solve the system $Ax = b$ by Cramer's Rule.
deter	Matrix determinant computed from the pivots in $PA = LU$.
eigen2	Eigenvalues, eigenvectors, and $\det(A - \lambda I)$ for 2 by 2 matrices.
eigshow	Graphical demonstration of eigenvalues and singular values.
eigval	Eigenvalues and their multiplicity as roots of $\det(A - \lambda I) = 0$.
eigvec	Compute as many linearly independent eigenvectors as possible.
elim	Reduction of A to row echelon form R by an invertible E.
findpiv	Find a pivot for Gaussian elimination (used by **plu**).
fourbase	Construct bases for all four fundamental subspaces.
grams	Gram-Schmidt orthogonalization of the columns of A.
house	2 by 12 matrix giving corner coordinates of a house.
inverse	Matrix inverse (if it exists) by Gauss-Jordan elimination.
leftnull	Compute a basis for the left nullspace.
linefit	Plot the least squares fit to m given points by a line.
lsq	Least squares solution to $Ax = b$ from $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.
normal	Eigenvalues and orthonormal eigenvectors when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$.
nulbasis	Matrix of special solutions to $Ax = 0$ (basis for nullspace).
orthcomp	Find a basis for the orthogonal complement of a subspace.
partic	Particular solution of $Ax = b$, with all free variables zero.
plot2d	Two-dimensional plot for the house figures.
plu	Rectangular $PA = LU$ factorization with row exchanges.
poly2str	Express a polynomial as a string.
project	Project a vector b onto the column space of A.
projmat	Construct the projection matrix onto the column space of A.
randperm	Construct a random permutation.
rowbasis	Compute a basis for the row space from the pivot rows of R.
samespan	Test whether two matrices have the same column space.
signperm	Determinant of the permutation matrix with rows ordered by p.
slu	LU factorization of a square matrix using *no row exchanges.*
slv	Apply **slu** to solve the system $Ax = b$ allowing no row exchanges.
splu	Square $PA = LU$ factorization *with row exchanges.*
splv	The solution to a square, invertible system $Ax = b$.
symmeig	Compute the eigenvalues and eigenvectors of a symmetric matrix.
tridiag	Construct a tridiagonal matrix with constant diagonals a, b, c.

Index

With the single heading* "Matrix" *this page indexes the active life of linear algebra.

S

T

U

Index of Symbols

Linear Algebra Websites

math.mit.edu/linearalgebra Dedicated to help readers and teachers working with this book

ocw.mit.edu MIT's OpenCourseWare site including video lectures in 18.06 and 18.085-6

web.mit.edu/18.06 Current and past exams and homeworks with extra materials

wellesleycambridge.com Ordering information for books by Gilbert Strang

LINEAR ALGEBRA IN A NUTSHELL

((*The matrix* A *is* n *by* n))

Nonsingular	Singular
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x} = \mathbf{0}$ has one solution $\boldsymbol{x} = \mathbf{0}$	$A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions
$A\boldsymbol{x} = \boldsymbol{b}$ has one solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$	$A\boldsymbol{x} = \boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

This book is designed to help students understan[...] central problems of linear algebra:

$Ax = b$	n by n	Chapters 1-2	Linear systems
$Ax = b$	m by n	Chapters 3-4	Least squares
$Ax = \lambda x$	n by n	Chapters 5-6	Eigenvalues
$Av = \sigma u$	m by n	Chapters 6-7	Singular values

The diagram on the front cover shows the four fundamental subspaces for the matrix A. Those subspaces lead to the Fundamental Theorem of Linear Algebra:

1. The dimensions of the four subspaces
2. The orthogonality of the two pairs
3. The best bases for all four subspaces

This is the textbook that accompanies the author's video lectures and the review material on MIT's OpenCourseWare.

ocw.mit.edu* and *web.mit.edu/18.06

Many universities and colleges (and now high schools) use this textbook. Chapters 7-10 are for a second course on linear algebra.

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